J. reine angew. Math. **470** (1996), 51—88

Journal für die reine und angewandte Mathematik

Berlin · New York 1996

PBW-bases of quantum groups

By *Claus Michael Ringel* at Bielefeld

1. Introduction

Let $\Delta = (a_{ij})_{ij}$ be the Cartan matrix of a finite-dimensional semisimple complex Lie algebra of rank n (see [H]), this is a symmetrizable matrix, and we denote by $(\varepsilon_i)_i$ the minimal symmetrization, thus ε_i are positive integers without a proper common divisor such that $\varepsilon_i a_{ij} = \varepsilon_j a_{ji}$.

Let $\mathbb{Q}(v)$ be the function field in one variable v over the field $\mathbb{Q}$ of rational numbers.

We denote by U^+ the $(+)$-part of the quantum group $U = U_q(\Delta)$ over $\mathbb{Q}(v)$ (as defined by Drinfeld [D] and Jimbo [J1] and modified by Lusztig [L2]), it is the free $\mathbb{Q}(v)$-algebra with generators $E_1, \ldots, E_n$ and relations

$$\sum_{t=0}^{n(ij)} (-1)^t \begin{bmatrix} n(ij) \\ t \end{bmatrix}_{\varepsilon_i} E_i^t E_j E_i^{n(ij)-t} = 0,$$

for all $i \neq j$, where $n(ij) = -a_{ij} + 1$; we use the notation

$$[s] = \frac{v^s - v^{-s}}{v - v^{-1}} = v^{s-1} + v^{s-3} + \cdots + v^{-s+1},$$

$$[s]! = \prod_{r=1}^{s} [r], \quad \text{and} \quad \begin{bmatrix} s \\ r \end{bmatrix} = \frac{[s]!}{[r]!\,[s-r]!},$$

here, s, r are non-negative integers, and $r \leqq s$; also, given a polynomial f in the variable v and an integer a, we denote by f_a the polynomial obtained from f by replacing v by v^a, for example, $[s]_2 = \dfrac{v^{2s} - v^{-2s}}{v^2 - v^{-2}}$. The considerations in this paper will be restricted to the $(+)$-part U^+, but the reader should observe that one may use the well-known triangular decomposition $U = U^- \otimes U^0 \otimes U^+$, see [Ro], in order to obtain related results for the Borel part $U^0 \otimes U^+$ or even for U itself.

We denote by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ the standard basis of $\mathbb{Z}^n$. The Cartan matrix defines on $\mathbb{Z}^n$ a symmetric bilinear form by $(\mathbf{e}_i, \mathbf{e}_j) = \varepsilon_i a_{ij}$. We consider U^+ as a $\mathbb{Z}^n$-graded algebra by assigning to E_i the degree $\mathbf{e}_i$. Given a homogeneous element X of U^+, we denote its degree by $\mathbf{dim}\,X$.

We are going to present a sequence $X_1, \ldots, X_m$ of homogeneous elements of U^+ such that the monomials $X_1^{a_1} \cdots X_m^{a_m}$ form a $\mathbb{Q}(v)$-basis of U^+, and such that for all $i<j$

$$X_j X_i = v^{(\mathbf{dim}\,X_i,\, \mathbf{dim}\,X_j)} X_i X_j + \sum_{I(i,j)} c(a_{i+1}, \ldots, a_{j-1}) X_{i+1}^{a_{i+1}} \cdots X_{j-1}^{a_{j-1}}$$

with coefficients $c(a_{i+1}, \ldots, a_{j-1})$ in $\mathbb{Q}(v)$; here, the index set $I(i,j)$ is the set of sequences $(a_{i+1}, a_{i+2}, \ldots, a_{j-1})$ of natural numbers such that $\sum_{t=i+1}^{j-1} a_t\, \mathbf{dim}\,X_t = \mathbf{dim}\,X_i + \mathbf{dim}\,X_j$. We say that the sequence $X_1, \ldots, X_m$ *generates* a PBW-basis of U^+.

It follows that U^+ is an iterated skew polynomial ring over $\mathbb{Q}(v)$. To be more precise: Let U_j be the subalgebra of U^+ generated by $X_1, \ldots, X_j$. Thus $U_0 = \mathbb{Q}(v)$ and for $1 \leqq j \leqq m$, we have $U_j = U_{j-1}[X_j; \iota_j, \delta_j]$, with an automorphism ι_j and an ι_j-1-derivation δ_j of U_{j-1}. Note that the automorphism ι_j is given explicitly by

$$\iota_j(X_i) = v^{(\mathbf{dim}\,X_i,\, \mathbf{dim}\,X_j)} X_i \quad \text{for} \quad i<j,$$

and we will show that

$$\iota_j \delta_j = v^{(\mathbf{dim}\,X_j,\, \mathbf{dim}\,X_j)} \delta_j \iota_j .$$

The last assertion has the following consequence: we can apply a recent result of Goodearl and Letzter [GL] in order to see that all prime ideals of U^+ are completely prime. We are indebted to Alev and Goodearl for drawing our attention to this problem.

There do exist several investigations dealing with the construction of PBW-bases for U^+ (or related algebras), let us mention papers by Khoroshkin and Tolstoy [KT1], [KT2], [KT3], Levendorskii and Soibelman [LS], Lusztig [L1], [L4], Takeuchi [T], Xi [X], and Yamane [Y1], [Y2]. These investigations usually start with the Drinfeld-Jimbo presentation and use direct calculations, often involving a braid group operation. Here we want to show that the Hall algebra approach as introduced in [R3], [R5], [R7] (see also [L3] and especially the new paper by Green [Gr]) is very suitable to deal with the problem. As we will recall below, one may identify U^+ with the socalled twisted generic Hall algebra $\mathscr{H}_*(\vec{\Delta}) \otimes \mathbb{Q}(v)$. Using this identification, we obtain a special $\mathbb{Q}(v)$-basis of U^+, so that the basis elements may be considered not just as elements, but as algebraic objects with a rich structure: as modules over a finite-dimensional hereditary algebra Λ of finite representation type. Since the basis elements may be interpreted as Λ-modules, one can discuss their module theoretical, homological or geometrical properties: whether they are indecomposable, or multiplicityfree and so on. A slight modification of these basis elements will form the PBW-basis of interest; those elements which correspond to indecomposable modules (together with some ordering) will be a generating sequence for the PBW-basis. The Hall algebra approach allows to use the representation theory of finite-dimensional hereditary algebras in order to derive properties of U^+; in particular, the shape of the Auslander-Reiten quiver of Λ will be of importance. The Auslander-Reiten quiver of Λ encodes a lot of information about the PBW-basis which we will present. As examples, we will write down in full detail the rank 2 cases and the case $\mathbb{A}_n$. In the latter case, we provide an explicit comparison with the PBW-basis presented by Yamane [Y1], [Y2].

Before we use the Hall algebra approach, we want to exhibit our results on U^+ without any reference to the representation theory of finite-dimensional hereditary algebras. Here is the recipe for writing down explicitly a PBW-basis: a generating sequence will be indexed by the positive roots for Δ, thus we will consider suitable orderings of the set of positive roots.

We denote by Φ^+ the set of positive roots for Δ, and we assume that Φ^+ is embedded into $\mathbb{Z}^n$ so that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the simple roots and $(-,-)$ is the corresponding bilinear form.

For any positive root $\mathbf{a}$, let us define an element $X(\mathbf{a})$ of U^+. For the simple roots $\mathbf{e}_i$, take $X(\mathbf{e}_i) = E_i$, thus the generators $E_1, \ldots, E_n$ will belong to our basis, the remaining elements will be constructed inductively. Given a ring R, and $x_1, x_2, t_1, t_2 \in R$ with t_1, t_2 central in R; then the element $t_1 x_1 x_2 - t_2 x_2 x_1$ will be called a *skew commutator* of x_1 and x_2. We are going to construct the generating sequence $X_1, \ldots, X_m$ as iterated skew commutators, starting from $E_1, \ldots, E_n$. (This sequence may also be obtained using a braid group operation on U, as we will show at the end of the paper.)

First of all, we choose some orientation of the edges of the graph of Δ. Recall that one attaches a graph to Δ as follows: it has as vertices the integers $1, 2, \ldots, n$ and the edges are the subsets $\{i, j\}$ with $a_{ij} < 0$. To choose an orientation means to select for any edge $\{i, j\}$ one of the pairs (i, j) or (j, i); in case (i, j) is selected, we draw an arrow $i \to j$. Since the graph of Δ is a forest, it is sufficient to choose a total ordering $\prec$ on the set $\{1, 2, \ldots, n\}$ and to write $i \to j$ in case $\{i, j\}$ is an edge and $i \prec j$. Thus, we just may take the natural ordering of the integers $1 < 2 < \cdots < n$; in this case, $i \to j$ means $a_{ij} \neq 0$ and $i < j$. Usually, in Lie theory, constructions using Δ will not depend on the orientation of the edges. Thus, if we want to stress that we use the chosen orientation in an essential way, we will write $\vec{\Delta}$ instead of Δ.

Here is the first such instance: $\vec{\Delta}$ defines a (usually non-symmetric) bilinear form $\langle -, - \rangle$ on $\mathbb{Z}^n$ as follows: let $\langle \mathbf{e}_i, \mathbf{e}_i \rangle = \varepsilon_i$, and for $i \neq j$, let $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \varepsilon_i a_{ij}$ provided $i \to j$, and zero otherwise.

A pair $(\mathbf{b}, \mathbf{a})$ of positive roots will be called *$\vec{\Delta}$-orthogonal*, provided the following two conditions are satisfied:

$$\langle \mathbf{a}, \mathbf{b} \rangle = 0, \quad \text{and} \quad \langle \mathbf{b}, \mathbf{a} \rangle \leqq 0,$$

and we denote $r_{\mathbf{a}}^{\mathbf{b}} = -\langle \mathbf{b}, \mathbf{a} \rangle$. In the simply-laced cases ($\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$) we have $r_{\mathbf{a}}^{\mathbf{b}} \leqq 1$, in the cases $\mathbb{B}_n$, $\mathbb{C}_n$, $\mathbb{F}_4$, we have $r_{\mathbf{a}}^{\mathbf{b}} \leqq 2$, whereas for $\mathbb{G}_2$, we have $r_{\mathbf{a}}^{\mathbf{b}} \leqq 3$.

Suppose that $(\mathbf{b}, \mathbf{a})$ is a $\vec{\Delta}$-orthogonal pair, and that $X(\mathbf{a})$, $X(\mathbf{b})$ are already defined. If $r = r_{\mathbf{a}}^{\mathbf{b}} \geqq 1$, then also $\mathbf{a} + \mathbf{b}$ is a positive root and we define

$$X(\mathbf{a} + \mathbf{b}) = X(\mathbf{b})\, X(\mathbf{a}) - v^{-r} X(\mathbf{a})\, X(\mathbf{b}).$$

If $r = 2$, then one of $\mathbf{a}, \mathbf{b}$ is a short root, the other one is a long root. Consider first the case that $\mathbf{a}$ is a short root, thus $2\mathbf{a} + \mathbf{b}$ is a root and we define

$$X(2\mathbf{a}+\mathbf{b}) = \frac{1}{[2]}\left(X(\mathbf{a}+\mathbf{b})\,X(\mathbf{a}) - X(\mathbf{a})\,X(\mathbf{a}+\mathbf{b})\right).$$

Second, in case $\mathbf{a}$ is a long root, then $\mathbf{a}+2\mathbf{b}$ is a root and we define

$$X(\mathbf{a}+2\mathbf{b}) = \frac{1}{[2]}\left(X(\mathbf{b})\,X(\mathbf{a}+\mathbf{b}) - X(\mathbf{a}+\mathbf{b})\,X(\mathbf{b})\right).$$

It remains to consider the case $r=3$, thus we deal with $\mathbb{G}_2$, and we assume that $\mathbf{a}$ is a short root and $\mathbf{b}$ is a long root. Then also $2\mathbf{a}+\mathbf{b}$, $3\mathbf{a}+\mathbf{b}$ and $3\mathbf{a}+2\mathbf{b}$ are positive roots, and we define

$$X(2\mathbf{a}+\mathbf{b}) = \frac{1}{[2]}\left(X(\mathbf{a}+\mathbf{b})\,X(\mathbf{a}) - v^{-1}\,X(\mathbf{a})\,X(\mathbf{a}+\mathbf{b})\right),$$

$$X(3\mathbf{a}+\mathbf{b}) = \frac{1}{[3]}\left(X(2\mathbf{a}+\mathbf{b})\,X(\mathbf{a}) - v\cdot X(\mathbf{a})\,X(2\mathbf{a}+\mathbf{b})\right),$$

$$X(3\mathbf{a}+2\mathbf{b}) = \frac{1}{[3]}\left(X(\mathbf{a}+\mathbf{b})\,X(2\mathbf{a}+\mathbf{b}) - v\cdot X(2\mathbf{a}+\mathbf{b})\,X(\mathbf{a}+\mathbf{b})\right).$$

We will show that these definitions are well-defined and we obtain in this way a set of elements $X(\mathbf{a})$ labelled by the positive roots $\mathbf{a}$.

Also, we may index the positive roots $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ such that $\langle \mathbf{a}_i, \mathbf{a}_j\rangle > 0$ implies $i \leqq j$. (We will see in Lemma 1 that such an ordering has the following additional property: If $\langle \mathbf{a}_i, \mathbf{a}_j\rangle < 0$, then $i > j$.) Then $X(\mathbf{a}_1), X(\mathbf{a}_2), \ldots, X(\mathbf{a}_m)$ generates a desired PBW-basis.

For any positive root $\mathbf{a}$, let $\varepsilon(\mathbf{a}) = \frac{1}{2}(\mathbf{a}, \mathbf{a})$; it is well-known that $\varepsilon(\mathbf{a})$ is a non-negative integer (it is equal to 1 in the simply-laced cases, it is 1 or 2 for $\mathbb{B}_n$, $\mathbb{C}_n$, $\mathbb{F}_4$, and 1, 2, or 3 in the case $\mathbb{G}_2$); of course, we have $\varepsilon(\mathbf{e}_i) = \varepsilon_i$. Consider the element

$$X(\mathbf{a})^{(t)} = \frac{1}{[t]!_{\varepsilon(a)}}\,X(\mathbf{a})^t,$$

these elements are called *divided powers.*

Let $\mathscr{A} = \mathbb{Z}[v, v^{-1}]$, and let $U^+_{\mathscr{A}}$ be the $\mathscr{A}$-subalgebra of U^+ generated by the elements $E_i^{(t)}$. We will see that $U^+_{\mathscr{A}}$ contains all the divided powers $X(\mathbf{a})^{(t)}$, thus $U^+_{\mathscr{A}}$ may be considered as an analogue of the Kostant $\mathbb{Z}$-form in classical Lie-theory.

The author is endebted to the referee for very useful comments concerning the presentation of the results.

2. Hall algebras

It has been shown in [R7] that U^+ may be identified with the twisted generic Hall algebra $\mathscr{H} = \mathscr{H}_*(\vec{\Delta}) \otimes \mathbb{Q}(v)$, where $\vec{\Delta}$ is obtained from Δ by choosing some orientation of the edges of the graph of Δ. (For a new, and much better proof we refer to Green [Gr].) For the convenience of the reader, we recall the main definitions.

Choose a k-species $\mathscr{S} = (F_i, {}_iM_j)$ of type $\vec{\Delta}$; we say that $\mathscr{S}$ is *reduced* provided $\varepsilon_i = \dim_k F_i$ for all i. (Readers not familiar with k-species may restrict their attention to the simply-laced cases $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$. In these cases, we may consider instead of $\mathscr{S}$ a quiver of type Δ and its representations over k.) A reduced k-species $\mathscr{S}$ is given by field extensions F_i over k of degree ε_i, and, for $i \to j$ an F_i-F_j-bimodule ${}_iM_j$ of k-dimension $-\varepsilon_i a_{ij}$. Actually, in case k is a finite field, there is (up to isomorphism) just one reduced k-species of type $\vec{\Delta}$.

We consider representations of $\mathscr{S}$, a representation being of the form (V_i, f_{ij}), with V_i a finite-dimensional right F_i-vector space, and f_{ij}: $V_i \otimes {}_iM_j \to V_j$ an F_j-linear map, for any $i \to j$. Note that the representations of $\mathscr{S}$ are just the finite-dimensional right modules over the tensor algebra of $\mathscr{S}$.

Given a representation M of $\mathscr{S}$, we denote its isomorphism class by $[M]$ and by $\mathbf{dim}\, M$ its dimension vector, it is an element in the Grothendieck group $K_0(\mathscr{S})$ of all representations modulo exact sequences. We identify $\mathbb{Z}^n$ with $K_0(\mathscr{S})$ so that $\mathbf{dim}\, S_i = \mathbf{e}_i$, where S_i is the simple representation of $\mathscr{S}$ corresponding to the vertex i.

It is known [Ga], [DR1] that $\mathbf{dim}$ furnishes a bijection between the isomorphism classes of the indecomposable representations of $\mathscr{S}$ and the positive roots. For a positive root $\mathbf{a}$, we denote by $M(\mathbf{a}) = M_{\mathscr{S}}(\mathbf{a})$ an indecomposable representation of $\mathscr{S}$ with $\mathbf{dim}\, M(\mathbf{a}) = \mathbf{a}$. In particular, $S_i = M(\mathbf{e}_i)$ is the simple representation corresponding to the vertex i. Given a map α: $\Phi^+ \to \mathbb{N}_0$, we set

$$M(\alpha) = M_{\mathscr{S}}(\alpha) = \bigoplus_{\mathbf{a}} \alpha(\mathbf{a})\, M(\mathbf{a})$$

(for $t \in \mathbb{N}_0$, and any representation N, we denote by tN the direct sum of t copies of N). The theorem of Krull-Remak-Schmidt asserts that in this way, we obtain a bijection between the set of isomorphism classes of representations of $\mathscr{S}$ and the set $\mathscr{B}$ of maps α: $\Phi^+ \to \mathbb{N}_0$. For any $\alpha \in \mathscr{B}$, let $\mathbf{dim}\, \alpha = \sum_{\mathbf{a}} \alpha(\mathbf{a})\mathbf{a}$, thus $\mathbf{dim}\, M(\alpha) = \mathbf{dim}\, \alpha$.

Note that we will identify $\mathbf{a} \in \Phi^+$ with the corresponding characteristic function $\Phi^+ \to \mathbb{N}_0$; in particular, the simple roots $\mathbf{e}_i$ are considered as elements of $\mathscr{B}$. Given two functions $\Phi^+ \to \mathbb{N}_0$, we may add them, and we can take multiples by non-negative integers. However, since we identify positive roots with the corresponding characteristic function, we have to be careful about the addition: the addition inside $\mathbb{Z}^n$ will be denoted by $+$, that inside the set $\mathscr{B}$ will be denoted by $\oplus$. For $t \in \mathbb{N}_0$, t-fold multiple of $\alpha \in \mathscr{B}$ will be denoted by $t\alpha$ (for $t \geqq 2$, this uses the addition in $\mathscr{B}$; note that there are no proper multiples in Φ^+).

In order to define the multiplication of $\mathscr{H}$, we need Hall polynomials and the Euler form. Recall that we have defined a bilinear form $\langle -, - \rangle$ on $\mathbb{Z}^n$. This form is called the Euler form, since for representations M, N of $\mathscr{S}$, we have

$$\begin{aligned} \langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle &= \sum_{t \geqq 0} (-1)^t \dim_k \operatorname{Ext}^t(M, N) \\ &= \dim_k \operatorname{Hom}(M, N) - \dim_k \operatorname{Ext}^1(M, N), \end{aligned}$$

see [R1]. Note that $(-, -)$ is the symmetrization of $\langle -, - \rangle$.

We will have to consider polynomials in one variable q, with integral coefficients; we usually will consider the corresponding polynomial ring $\mathbb{Z}[q]$ as a subring of $\mathbb{Q}(v)$, where $q = v^2$.

We recall from [R3] that given three elements $\alpha, \beta, \gamma \in \mathscr{B}$, there exists a polynomial $\phi^{\beta}_{\alpha\gamma}$ in $\mathbb{Z}[q]$ such that for k a finite field, and $\mathscr{S}$ a reduced k-species of type $\vec{\Delta}$, the number $\phi^{\beta}_{\alpha\gamma}(|k|)$ is equal to the number of subrepresentations U of $M_{\mathscr{S}}(\beta)$ which are isomorphic to $M_{\mathscr{S}}(\gamma)$ such that $M_{\mathscr{S}}(\beta)/U$ is isomorphic to $M_{\mathscr{S}}(\alpha)$. The polynomials $\phi^{\beta}_{\alpha\gamma}$ are called Hall polynomials (some have been calculated explicitly, see [R4]). Sometimes, it will be convenient to write $\phi^{M(\beta)}_{M(\alpha)M(\gamma)}$ instead of $\phi^{\beta}_{\alpha\gamma}$.

We note the following: If the polynomial $\phi^{\beta}_{\alpha\gamma}$ is non-zero, then $\phi^{\beta}_{\alpha\gamma}(|k|) \neq 0$ for some finite field k. Consider now a reduced k-species $\mathscr{S}$ of type $\vec{\Delta}$, then $M_{\mathscr{S}}(\beta)$ has a submodule U which is isomorphic to $M_{\mathscr{S}}(\gamma)$ with $M_{\mathscr{S}}(\beta)/U$ isomorphic to $M_{\mathscr{S}}(\alpha)$. If U is a direct summand, then $M_{\mathscr{S}}(\beta) \cong M_{\mathscr{S}}(\alpha \oplus \gamma)$. If U is not a direct summand, then $\mathrm{Ext}^1(M_{\mathscr{S}}(\alpha), M_{\mathscr{S}}(\gamma)) \neq 0$.

More generally, we also will need the Hall polynomials $\phi^{\beta}_{\alpha_1,\ldots,\alpha_n}$, where $\alpha_1, \ldots, \alpha_n, \beta$ are elements of $\mathscr{B}$; again, these are polynomials in a variable q with integer coefficients and with the following property: for any finite field k, and $\mathscr{S}$ a reduced k-species of type $\vec{\Delta}$, the number $\phi^{\beta}_{\alpha_1,\ldots,\alpha_n}(|k|)$ is the number of filtrations $M_{\mathscr{S}}(\beta) = N_0 \supseteqq N_1 \supseteqq \cdots \supseteqq N_t = 0$ such that N_{i-1}/N_i is isomorphic to $M_{\mathscr{S}}(\alpha_i)$, for $1 \leqq i \leqq t$.

By definition, $\mathscr{H}$ is the free $\mathbb{Q}(v)$-module with basis the set $\mathscr{B}$ of functions $\Phi^+ \to \mathbb{N}_0$ (or, equivalently, the set of isomorphism classes of representations of some fixed reduced k-species $\mathscr{S}$ of type $\vec{\Delta}$), with multiplication

$$\alpha * \gamma = v^{\langle \mathbf{dim}\,\alpha,\, \mathbf{dim}\,\gamma\rangle} \sum_{\beta \in \mathscr{B}} \phi^{\beta}_{\alpha\gamma}\,\beta,$$

for $\alpha, \gamma \in \mathscr{B}$.

Theorem. *There exists an isomorphism $\eta\colon U^+ \to \mathscr{H}$ of $\mathbb{Z}^n$-graded $\mathbb{Q}(v)$-algebras such that $\eta(E_i) = \mathbf{e}_i$.*

For a proof, see [R7], or, better [Gr].

In the last sections, we also will consider $\mathscr{H}_*(\vec{\Delta})$ itself; by definition, it is the $\mathscr{A}$-subalgebra generated by $\mathscr{B}$. Note that $\mathscr{B}$ is a free $\mathscr{A}$-basis of $\mathscr{H}_*(\vec{\Delta})$.

3. The basic formula

We will work with $\mathscr{H}$ instead of U^+. As we have seen, $\mathscr{H}$ is the free $\mathbb{Q}(v)$-module with basis the set $\mathscr{B} = \{\Phi^+ \to \mathbb{N}_0\}$, but sometimes it will be more convenient to work with a slightly modified basis:

We denote by dim: $\mathbb{Z}^n \to \mathbb{Z}$ the linear form given by $\dim \mathbf{e}_i = \varepsilon_i$. For $\alpha \in \mathscr{B}$, let $\dim \alpha = \dim(\mathbf{dim}\,\alpha)$; thus $\dim \alpha = \dim_k M_{\mathscr{S}}(\alpha)$, for any reduced k-species $\mathscr{S}$.

Also, let us recall from [R3] that given $\alpha \in \mathscr{B}$ and any reduced k-species $\mathscr{S}$ of type $\vec{\Delta}$, the k-dimension of the endomorphism ring of $M_{\mathscr{S}}(\alpha)$ is independent of $\mathscr{S}$ and is denoted by $\varepsilon(\alpha)$. Also observe that for a positive root $\mathbf{a}$, the k-dimension of the endomorphism ring of $M_{\mathscr{S}}(\mathbf{a})$ is just $\frac{1}{2}(\mathbf{a}, \mathbf{a})$, thus the two definitions of $\varepsilon(\mathbf{a})$ coincide.

Let

$$\langle\alpha\rangle = v^{-\dim\alpha + \varepsilon(\alpha)}\alpha,$$

the set of these elements $\langle\alpha\rangle$ with $\alpha \in \mathscr{B}$ is again a $\mathbb{Q}(v)$-basis of $\mathscr{H}$; this is the basis we are mainly interested in. Note that $\langle\mathbf{e}_i\rangle = \mathbf{e}_i$. The multiplication formula may be rewritten in this basis as follows:

$$\langle\alpha\rangle * \langle\gamma\rangle = v^{\varepsilon(\alpha)+\varepsilon(\gamma)+\langle\mathbf{dim}\,\alpha,\,\mathbf{dim}\,\gamma\rangle} \sum_{\beta\in\mathscr{B}} v^{-\varepsilon(\beta)} \phi^{\beta}_{\alpha\gamma}\langle\beta\rangle.$$

We have noted that for any reduced k-species $\mathscr{S}$, the k-dimension of the endomorphism ring of $M_{\mathscr{S}}(\alpha)$ is independent of $\mathscr{S}$. Similarly [R3], also the k-dimension of $\mathrm{Hom}(M_{\mathscr{S}}(\alpha), M_{\mathscr{S}}(\beta))$ is independen of $\mathscr{S}$ and will be denoted by $\varepsilon(\alpha, \beta)$. In the same way, the k-dimension of $\mathrm{Ext}^1(M_{\mathscr{S}}(\alpha), M_{\mathscr{S}}(\beta))$ is independent of $\mathscr{S}$ and will be denoted by $\zeta(\alpha, \beta)$. Note that $\langle\mathbf{dim}\,\alpha, \mathbf{dim}\,\beta\rangle = \varepsilon(\alpha, \beta) - \zeta(\alpha, \beta)$.

Proposition 1. *Let $\alpha_1, \ldots, \alpha_t \in \mathscr{B}$ and let us assume that for $i<j$, we have both $\zeta(\alpha_i, \alpha_j) = 0$ and $\varepsilon(\alpha_j, \alpha_i) = 0$. Then*

$$\left\langle \bigoplus_{i=1}^{t} \alpha_i \right\rangle = \langle\alpha_1\rangle * \cdots * \langle\alpha_t\rangle.$$

Proof. It is sufficient to prove the assertion for $t = 2$, the general case follows by induction. Thus, let $\alpha = \alpha_1$, and $\gamma = \alpha_2$. Since $\zeta(\alpha, \gamma) = 0$, the only β with $\phi^{\beta}_{\alpha\gamma} \neq 0$ is $\beta = \alpha \oplus \gamma$. And $\phi^{\alpha\oplus\gamma}_{\alpha\gamma} = 1$, since $\varepsilon(\gamma, \alpha) = 0$. Also,

$$\langle\mathbf{dim}\,\alpha, \mathbf{dim}\,\gamma\rangle = \varepsilon(\alpha, \gamma) - \zeta(\alpha, \gamma) = \varepsilon(\alpha, \gamma).$$

On the other hand, we have

$$\varepsilon(\alpha \oplus \gamma) = \varepsilon(\alpha) + \varepsilon(\gamma) + \varepsilon(\alpha, \gamma) + \varepsilon(\gamma, \alpha) = \varepsilon(\alpha) + \varepsilon(\gamma) + \varepsilon(\alpha, \gamma).$$

Altogether:

$$\begin{aligned}
\langle\alpha\rangle * \langle\gamma\rangle &= v^{-\dim\alpha + \varepsilon(\alpha) - \dim\gamma + \varepsilon(\gamma)}\alpha * \gamma \\
&= v^{-\dim\alpha\oplus\gamma + \varepsilon(\alpha) + \varepsilon(\gamma) + \langle\mathbf{dim}\,\alpha,\,\mathbf{dim}\,\gamma\rangle}\phi^{\alpha\oplus\gamma}_{\alpha\gamma}\alpha \oplus \gamma \\
&= v^{-\dim\alpha\oplus\gamma + \varepsilon(\alpha\oplus\gamma)}\alpha \oplus \gamma \\
&= \langle\alpha \oplus \gamma\rangle.
\end{aligned}$$

Theorem 1. *Let $\alpha, \gamma \in \mathscr{B}$ with $\varepsilon(\gamma, \alpha) = 0$ and $\zeta(\alpha, \gamma) = 0$. Then we have*

$$\langle\gamma\rangle * \langle\alpha\rangle = v^{(\mathbf{dim}\,\alpha,\,\mathbf{dim}\,\gamma)}\langle\alpha\rangle * \langle\gamma\rangle + \sum_{\beta\in J(\alpha,\gamma)} c_\beta\langle\beta\rangle,$$

with coefficients c_β in $\mathbb{Z}[v, v^{-1}]$, where $J(\alpha, \gamma)$ is the set of maps $\beta \in \mathscr{B}$ different from $\alpha \oplus \gamma$ such that $\phi^{\beta}_{\gamma,\alpha} \neq 0$.

Proof. The previous proposition shows that

$$\langle\alpha\rangle * \langle\gamma\rangle = \langle\alpha \oplus \gamma\rangle .$$

Thus, let us consider $\langle\gamma\rangle * \langle\alpha\rangle$, it can be written in the form $\sum_\beta c'_\beta \beta$, where β ranges over all elements from $\mathscr{B}$ such that $\phi^{\beta}_{\gamma,\alpha} \neq 0$, and clearly the coefficients c'_β belong to $\mathbb{Z}[v, v^{-1}]$. Of course, we have

$$c'_\beta = c_\beta v^{-\dim\beta + \varepsilon(\beta)} .$$

It remains to calculate the coefficient $c'_{\alpha\oplus\gamma}$. Let $\mathscr{S}$ be a reduced k-species. Let U be a submodule of $M_{\mathscr{S}}(\alpha \oplus \gamma)$ isomorphic to $M_{\mathscr{S}}(\alpha)$, with factor module isomorphic to $M_{\mathscr{S}}(\gamma)$. Clearly, U has to be a direct summand (see [R2], Lemma 2.3.1). Since $\operatorname{Hom}(M_{\mathscr{S}}(\gamma), M_{\mathscr{S}}(\alpha)) = 0$, the theorem of Krull-Remak-Schmidt asserts that U is the image of a homomorphism of the form $(1, f)\colon M_{\mathscr{S}}(\alpha) \to M_{\mathscr{S}}(\alpha) \oplus M_{\mathscr{S}}(\gamma)$, where f is a homomorphism $M_{\mathscr{S}}(\alpha) \to M_{\mathscr{S}}(\gamma)$. In this way, we obtain a bijection between the considered submodules U and the elements of $\operatorname{Hom}(M_{\mathscr{S}}(\alpha), M_{\mathscr{S}}(\gamma))$. Thus, we see that

$$\begin{aligned} \phi^{\alpha\oplus\gamma}_{\gamma\alpha} &= q^{\varepsilon(\alpha,\gamma)} \\ &= v^{2\cdot\varepsilon(\alpha,\gamma)} . \end{aligned}$$

Also,

$$\varepsilon(\alpha \oplus \gamma) = \varepsilon(\alpha) + \varepsilon(\gamma) + \varepsilon(\alpha, \gamma) .$$

Finally, we note that

$$\begin{aligned} (\mathbf{dim}\,\alpha, \mathbf{dim}\,\gamma) &= \langle \mathbf{dim}\,\alpha, \mathbf{dim}\,\gamma\rangle + \langle \mathbf{dim}\,\gamma, \mathbf{dim}\,\alpha\rangle \\ &= \varepsilon(\alpha, \gamma) - \zeta(\gamma, \alpha) . \end{aligned}$$

Altogether, we see that

$$\begin{aligned} c'_{\alpha\oplus\gamma} &= v^{-\dim\gamma + \varepsilon(\gamma) - \dim\alpha + \varepsilon(\alpha)} v^{\langle \mathbf{dim}\,\gamma, \mathbf{dim}\,\alpha\rangle} \phi^{\alpha\oplus\gamma}_{\gamma\alpha} , \\ &= v^{-\dim\alpha\oplus\gamma + \varepsilon(\alpha\oplus\gamma) - \varepsilon(\alpha,\gamma)} v^{-\zeta(\gamma,\alpha)} v^{2\cdot\varepsilon(\alpha,\gamma)} \\ &= v^{-\dim\alpha\oplus\gamma + \varepsilon(\alpha\oplus\gamma)} v^{(\dim\alpha, \dim\gamma)} . \end{aligned}$$

This shows that

$$c'_{\alpha\oplus\gamma} \alpha \oplus \gamma = v^{(\mathbf{dim}\,\alpha, \mathbf{dim}\,\gamma)} \langle\alpha \oplus \gamma\rangle .$$

4. Ordering the positive roots

It is well-known that $\mathscr{S}$ is representation-directed, see [BGP], [DR2]; this means that there exists a total ordering of the positive roots, say $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ such that

$\operatorname{Hom}(M_{\mathscr{S}}(\mathbf{a}_i), M_{\mathscr{S}}(\mathbf{a}_j)) \neq 0$ (or equivalently, $\varepsilon(\mathbf{a}_i, \mathbf{a}_j) \neq 0$) implies that $i \leqq j$: such an ordering will be called *$\vec{\Delta}$-admissible*. The usual visualization using the Auslander-Reiten quiver will be recalled in Section 6.

Lemma 1. *A total ordering $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ of the positive roots is $\vec{\Delta}$-admissible if and only if the following property is satisfied: $\langle \mathbf{a}_i, \mathbf{a}_j \rangle > 0$ implies $i \leqq j$.*

Such an ordering has the additional property: $\langle \mathbf{a}_i, \mathbf{a}_j \rangle < 0$ implies $i > j$.

Proof. We write $M(\mathbf{a}_i)$ instead of $M_{\mathscr{S}}(\mathbf{a}_i)$.

First, let us observe that $\operatorname{Ext}^1(M(\mathbf{a}_i), M(\mathbf{a}_j)) \neq 0$ implies $i > j$. For, assume that there exists an exact sequence

$$0 \to M(\mathbf{a}_j) \to N \to M(\mathbf{a}_i) \to 0$$

which does not split. Let $M(\mathbf{a}_t)$ be an indecomposable direct summand of N. Then $\operatorname{Hom}(M(\mathbf{a}_j), M(\mathbf{a}_t)) \neq 0$ and $\operatorname{Hom}(M(\mathbf{a}_t), M(\mathbf{a}_i)) \neq 0$, thus $j \leqq t \leqq i$. We cannot have $\mathbf{a}_j = \mathbf{a}_i$, since $\operatorname{Ext}^1(M(\mathbf{a}_i), M(\mathbf{a}_i)) = 0$. Thus $i > j$. Of course, $i > j$ implies that $\operatorname{Hom}(M(\mathbf{a}_i), M(\mathbf{a}_j)) = 0$. In particular, the groups $\operatorname{Ext}^1(M(\mathbf{a}_i), M(\mathbf{a}_j))$ and $\operatorname{Hom}(M(\mathbf{a}_i), M(\mathbf{a}_j))$ cannot be non-zero at the same time.

As a consequence, we see: if $\langle \mathbf{a}_i, \mathbf{a}_j \rangle > 0$, then

$$\langle \mathbf{a}_i, \mathbf{a}_j \rangle = \dim_k \operatorname{Hom}(M(\mathbf{a}_i), M(\mathbf{a}_j)),$$

whereas, if $\langle \mathbf{a}_i, \mathbf{a}_j \rangle < 0$, then

$$\langle \mathbf{a}_i, \mathbf{a}_j \rangle = -\dim_k \operatorname{Ext}^1(M(\mathbf{a}_i), M(\mathbf{a}_j)).$$

Assume that there is given an ordering $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ with the property that $\langle \mathbf{a}_i, \mathbf{a}_j \rangle > 0$ implies $i \leqq j$. If $\operatorname{Hom}(M(\mathbf{a}_i), M(\mathbf{a}_j)) \neq 0$, then $\operatorname{Ext}^1(M(\mathbf{a}_i), M(\mathbf{a}_j)) = 0$ and therefore $\langle \mathbf{a}_i, \mathbf{a}_j \rangle > 0$, thus, by assumption $i \leqq j$.

For the converse, let us assume that the ordering $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ is $\vec{\Delta}$-admissible. Let $\langle \mathbf{a}_i, \mathbf{a}_j \rangle > 0$. Then $\operatorname{Hom}(M(\mathbf{a}_i), M(\mathbf{a}_j)) = \langle \mathbf{a}_i, \mathbf{a}_j \rangle > 0$, and therefore $i \leqq j$. On the other hand, let $\langle \mathbf{a}_i, \mathbf{a}_j \rangle < 0$. Then $\operatorname{Ext}^1(M(\mathbf{a}_i), M(\mathbf{a}_j)) = -\langle \mathbf{a}_i, \mathbf{a}_j \rangle < 0$. As we have seen above, we have $i > j$. This completes the proof.

Let us fix some $\vec{\Delta}$-ordering $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$.

Proposition 1'. *For any $\alpha \in \mathscr{B}$, we have*

$$\langle \alpha \rangle = \langle \alpha(\mathbf{a}_1)\mathbf{a}_1 \rangle * \cdots * \langle \alpha(\mathbf{a}_m)\mathbf{a}_m \rangle.$$

Proof. For $i < j$, we have $\varepsilon(\mathbf{a}_j, \mathbf{a}_i) = 0$, by assumption, and we also have $\zeta(\mathbf{a}_i, \mathbf{a}_j) = 0$. Thus, the assertion is a direct consequence of Proposition 1.

Given an element w in $\mathcal{H}$, and $t \geqq 0$, then w^{*t} denotes its t-th power. For a positive root $\mathbf{a}$, we also consider a corresponding divided power of $\langle \mathbf{a} \rangle$:

$$\langle \mathbf{a} \rangle^{(*t)} = \frac{1}{[t]!_{\varepsilon(\mathbf{a})}} \langle \mathbf{a} \rangle^{*t}.$$

Lemma 2. *Let* $\mathbf{a}$ *be a positive root, and* $t \geqq 0$. *Then*

$$\langle t\mathbf{a} \rangle = \langle \mathbf{a} \rangle^{(*t)}.$$

Proof. Let $\mathcal{S}$ be a k-species, where k is a finite field. The number of filtrations

$$tM_{\mathcal{S}}(\mathbf{a}) = M_0 \supset M_1 \supset \cdots \supset M_t = 0$$

with factors isomorphic to $M_{\mathcal{S}}(\mathbf{a})$ is given by evaluating the following polynomial at $|k|$:

$$\frac{(v^{2\varepsilon(\mathbf{a})t} - 1)(v^{2\varepsilon(\mathbf{a})(t-1)} - 1) \cdots (v^{2\varepsilon(\mathbf{a})} - 1)}{(v^{2\varepsilon(\mathbf{a})} - 1)^t} = \left(v^{\binom{t}{2}}[t]!\right)_{\varepsilon(\mathbf{a})} = v^{\varepsilon(\mathbf{a})\binom{t}{2}} \cdot [t]!_{\varepsilon(\mathbf{a})}.$$

Let us express $\mathbf{a}^{*t}$ in the basis $\mathcal{B}$. Since $\zeta(\mathbf{a}, \mathbf{a}) = 0$, we see that $\mathbf{a}^{*t}$ is a multiple of $t\mathbf{a}$, namely

$$\mathbf{a}^{*t} = v^{\varepsilon(\mathbf{a})\binom{t}{2}} v^{\varepsilon(\mathbf{a})\binom{t}{2}} \cdot [t]!_{\varepsilon(\mathbf{a})} \cdot t\mathbf{a}.$$

It follows that

$$\begin{aligned}
\langle \mathbf{a} \rangle^{(*t)} &= \frac{1}{[t]!_{\varepsilon(\mathbf{a})}} \langle \mathbf{a} \rangle^{*t} \\
&= \frac{1}{[t]!_{\varepsilon(\mathbf{a})}} v^{-t \dim \mathbf{a} + t\varepsilon(\mathbf{a})} \mathbf{a}^{*t} \\
&= v^{-t \dim \mathbf{a} + t\varepsilon(\mathbf{a})} v^{\varepsilon(\mathbf{a})\binom{t}{2}} v^{\varepsilon(\mathbf{a})\binom{t}{2}} \cdot t\mathbf{a} \\
&= \langle t\mathbf{a} \rangle,
\end{aligned}$$

since $\varepsilon(t\mathbf{a}) = t^2 \varepsilon(\mathbf{a}) = \left(t + 2\binom{t}{2}\right) \varepsilon(\mathbf{a})$. This completes the proof.

We define

$$X_i = \langle \mathbf{a}_i \rangle.$$

Proposition 2. *Let* $\alpha \in \mathcal{B}$, *and set* $\alpha(i) = \alpha(\mathbf{a}_i)$. *Then*

$$\langle \alpha \rangle = X_1^{(*\alpha(1))} * \cdots * X_m^{(*\alpha(m))} = \left(\prod_{i=1}^{m} \frac{1}{[\alpha(i)]!_{\varepsilon(\mathbf{a}_i)}} \right) X_1^{*\alpha(1)} * \cdots * X_m^{*\alpha(m)}.$$

Proof. According to Proposition 1′, we have

$$\langle \alpha \rangle = \langle \alpha(1)\mathbf{a}_1 \rangle * \cdots * \langle \alpha(m)\mathbf{a}_m \rangle.$$

Lemma 2 asserts that

$$\langle \alpha(i)\,\mathbf{a}_i \rangle = X_i^{(*\alpha(i))}.$$

This shows the first equality. The second uses just the definition of divided powers.

Theorem 2. *The monomials* $X_1^{*\alpha(1)} * \cdots * X_m^{*\alpha(m)}$ *with* $\alpha(1), \ldots, \alpha(m) \in \mathbb{N}_0$ *form a* $\mathbb{Q}(v)$*-basis of* $\mathscr{H}$*, and for all* $i < j$

$$X_j * X_i = v^{(\dim X_i, \dim X_j)} X_i * X_j + \sum_{I(i,j)} c(a_{i+1}, \ldots, a_{j-1}) X_{i+1}^{*a_{i+1}} * \cdots * X_{j-1}^{*a_{j-1}}$$

with coefficients $c(a_{i+1}, \ldots, a_{j-1})$ *in* $\mathbb{Q}(v)$*. Here, the index set* $I(i,j)$ *is the set of sequences* $(a_{i+1}, a_{i+2}, \ldots, a_{j-1})$ *of natural numbers such that* $\sum_{t=i+1}^{j-1} a_t \mathbf{a}_t = \mathbf{a}_i + \mathbf{a}_j$.

Proof. Given $\alpha(1), \ldots, \alpha(m) \in \mathbb{N}_0$, define $\alpha \in \mathscr{B}$ by $\alpha(\mathbf{a}_i) = \alpha(i)$. According to Proposition 2, we have

$$X_1^{*\alpha(1)} * \cdots * X_m^{*\alpha(m)} = \left(\prod_{i=1}^{m} [\alpha(i)]!_{\varepsilon(\mathbf{a}_i)} \right) \langle \alpha \rangle,$$

thus the given monomials are non-zero scalar multiples of the elements of $\mathscr{B}$, and therefore form a $\mathbb{Q}(v)$-basis of $\mathscr{H}$.

Let $i < j$. We apply Theorem 1 to $\mathbf{a}_i$, $\mathbf{a}_j$. We have to show that for $\beta \in J(\mathbf{a}_i, \mathbf{a}_j)$, the element $\langle \beta \rangle$ is a scalar multiple of some monomial $X_{i+1}^{*a_{i+1}} * \cdots * X_{j-1}^{*a_{j-1}}$ with $\sum_{t=i+1}^{j-1} a_t \mathbf{a}_t = \mathbf{a}_i + \mathbf{a}_j$.

Let $\beta \in J(\mathbf{a}_i, \mathbf{a}_j)$, and let $\beta(t) = \beta(\mathbf{a}_t)$. Since $\phi^{\beta}_{\mathbf{a}_j \mathbf{a}_i} \neq 0$, there is an exact sequence

$$0 \longrightarrow M(\mathbf{a}_i) \xrightarrow{\ f\ } \bigoplus_{t=1}^{m} \beta(t) M(\mathbf{a}_t) \xrightarrow{\ g\ } M(\mathbf{a}_j) \longrightarrow 0,$$

and we write $f = (f_t)_t$ with $f_t \colon M(\mathbf{a}_i) \to \beta(t) M(\mathbf{a}_t)$. Note that the sequence does not split, since otherwise $\beta = \mathbf{a}_i \oplus \mathbf{a}_j$, contrary to the assumption $\beta \in J(\mathbf{a}_i, \mathbf{a}_j)$. Consider some t with $\beta(t) > 0$. We claim that then $f_t \neq 0$. Otherwise, the cokernel of f would split off $\beta(t)$ copies of $M(\mathbf{a}_t)$, and since the cokernel of f is indecomposable, this would mean that the sequence splits. Since $\operatorname{Hom}(M(\mathbf{a}_i), M(\mathbf{a}_t)) \neq 0$, it follows that $i \leqq t$. Also, we can exclude the case $i = t$, since in this case f_t, and therefore also f would be a split monomorphism. Altogether, we see that $i < t$. The dual arguments (applied to g) show that also $t < j$. According to Proposition 2, $\langle \beta \rangle$ is a scalar multiple of $X_{i+1}^{*a_{i+1}} * \cdots * X_{j-1}^{*a_{j-1}}$. Also, the exact sequence exhibited above shows that $\sum_{t=i+1}^{j-1} a_t \mathbf{a}_t = \mathbf{a}_i + \mathbf{a}_j$. This completes the proof.

5. The automorphisms and skew derivations

For any element $\mathbf{d} \in \mathbb{Z}^n$, there exists an automorphism $\iota_{\mathbf{d}}$ of $\mathscr{H}$ given by

$$\iota_{\mathbf{d}}(w) = v^{(\mathbf{dim}\, w, \mathbf{d})} w$$

for w a homogeneous element of $\mathscr{H}$ (of degree $\mathbf{dim}\, w$).

We also note the following rather obvious assertion:

Lemma 3. *Let R be a ring, let ι be an endomorphism of R. For $r \in R$, we define a map $\delta_r\colon R \to R$ by*

$$\delta_r(x) = rx - \iota(x)r \quad \text{for} \quad x \in R.$$

Then δ_r is a ι-1-derivation.

Proof. The map δ_r clearly is additive. Also, for $x, y \in R$,

$$\begin{aligned}\delta_r(xy) &= rxy - \iota(xy)r \\ &= rxy - \iota(x)ry + \iota(x)ry - \iota(x)\iota(y)r \\ &= \big(rx - \iota(x)r\big)y + \iota(x)\big(ry - \iota(y)r\big) \\ &= \delta_r(x)\cdot y - \iota(x)\cdot\delta_r(y).\end{aligned}$$

One may call δ_r an *inner* ι-1-derivation.

Let K be a commutative ring, and assume that R is a K-algebra and that ι is K-linear. Then also δ_r is K-linear, for any $r \in R$. If R is generated as a K-algebra by $r_1, \ldots, r_n$, and δ is a ι-1-derivation, the values $\delta(r_i)$, with $1 \leqq i \leqq n$, determine δ uniquely.

Let $\mathscr{H}_j$ be the subalgebra of $\mathscr{H}$ generated by $X_1, \ldots, X_j$. Thus $\mathscr{H}_0 = \mathbb{Q}(v)$ and for $1 \leqq j \leqq m$, we have $\mathscr{H}_j = \mathscr{H}_{j-1}[X_j; \iota_j, \delta_j]$, with an automorphism ι_j and a ι_j-1-derivation δ_j of $\mathscr{H}_{j-1}$. Note that the automorphism ι_j of $\mathscr{H}_{j-1}$ is given explicitly by

$$\iota_j(X_i) = v^{(\mathbf{dim}\, X_i,\, \mathbf{dim}\, X_j)} X_i \quad \text{for} \quad i < j;$$

it is just the restriction of $\iota_{\mathbf{a}_j}$ to $\mathscr{H}_{j-1}$.

The ι_j-1-derivation δ_j is given by the formula

$$\begin{aligned}\delta_j(X_i) &= X_j * X_i - \iota_j(X_i) * X_j \\ &= \sum_{I(i,j)} c(a_{i+1}, \ldots, a_{j-1}) X_{i+1}^{*a_{i+1}} * \cdots * X_{j-1}^{*a_{j-1}}\end{aligned}$$

for $i < j$; in particular, δ_j is the restriction of the inner ι_j-1-derivation δ_{X_j} to $\mathscr{H}_{j-1}$.

Altogether, we see that $\mathscr{H}$ is an iterated skew polynomial ring over $\mathbb{Q}(v)$.

Theorem 3. *The automorphism ι_j and the ι_j-1-derivation δ_j of $\mathscr{H}_{j-1}$ satisfy the following relation*:

$$\iota_j \delta_j = v^{(\mathbf{a}_j, \mathbf{a}_j)} \delta_j \iota_j .$$

Proof. Let $i<j$. First of all, we have $\iota_j(X_i) = v^{(\mathbf{a}_i, \mathbf{a}_j)} X_i$, and therefore

$$\delta_j \iota_j (X_i) = v^{(\mathbf{a}_i, \mathbf{a}_j)} \delta_j (X_i) .$$

Let us denote $\mathbf{d} = \mathbf{a}_i + \mathbf{a}_j$. We know that $\delta_j(X_i)$ is a linear combination of monomials of the form $X_{i+1}^{*a_{i+1}} * \cdots * X_{j-1}^{*a_{j-1}}$ where $\sum_{t=i+1}^{j-1} a_t \mathbf{a}_t = \mathbf{a}_i + \mathbf{a}_j = \mathbf{d}$, thus $\delta_j(X_i)$ belongs to $\mathscr{H}_{\mathbf{d}}$. It follows that

$$\iota_j \delta_j (X_j) = v^{(\mathbf{d}, \mathbf{a}_j)} \delta_j (X_i) = v^{(\mathbf{a}_j, \mathbf{a}_j) + (\mathbf{a}_i, \mathbf{a}_j)} \delta_j (X_i) = v^{(\mathbf{a}_j, \mathbf{a}_j)} \delta_j \iota_j (X_i) ,$$

since $(\mathbf{d}, \mathbf{a}_j) = (\mathbf{a}_i + \mathbf{a}_j, \mathbf{a}_j) = (\mathbf{a}_i, \mathbf{a}_j) + (\mathbf{a}_j, \mathbf{a}_j)$.

As an application we obtain:

Corollary. *Any prime ideal of U^+ is completely prime.*

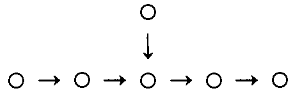
Proof. The corresponding assertion for $\mathscr{H}$ is a direct consequence of Theorem 2, Theorem 3 and a recent result of Goodearl and Letzter [GL], Theorem 2.3, see also [Go].

6. The Auslander-Reiten quiver

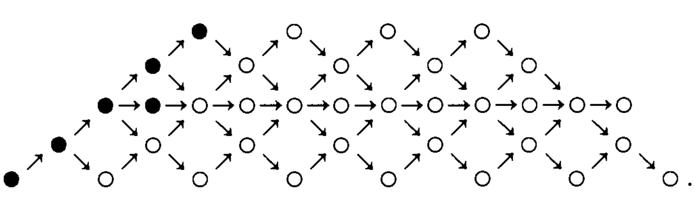
The Auslander-Reiten quiver is a convenient tool to visualize the module category of a finite-dimensional algebra. We want to point out that the Auslander-Reiten quiver is also extremely useful for dealing with the corresponding Hall algebras. Let us formulate a combinatorial version of those definitions which are needed.

As vertices of the Auslander-Reiten quiver Γ take the positive roots. If $\mathbf{a}$, $\mathbf{b}$ are positive roots, write $\mathbf{a} \to \mathbf{b}$ provided the following conditions are satisfied: first, $\mathbf{a} \neq \mathbf{b}$, second $\langle \mathbf{a}, \mathbf{b} \rangle > 0$, and third, if $\mathbf{c}$ is a positive root with $\langle \mathbf{a}, \mathbf{c} \rangle > 0$ and $\langle \mathbf{c}, \mathbf{b} \rangle > 0$, then $\mathbf{a} = \mathbf{c}$ or $\mathbf{c} = \mathbf{b}$. In general, if $\langle \mathbf{a}, \mathbf{b} \rangle > 0$, then there exists a path $\mathbf{a} = \mathbf{a}_0 \to \mathbf{a}_1 \to \cdots \to \mathbf{a}_t = \mathbf{b}$ (of length $t \geqq 0$) from $\mathbf{a}$ to $\mathbf{b}$.

For example, the Auslander-Reiten quiver for $\mathbb{E}_6$ with the following orientation



is of the form



We may endow Γ with a valuation by associating to any arrow $\mathbf{a} \to \mathbf{b}$ a pair of positive integers, namely $\left(\frac{\langle \mathbf{a}, \mathbf{b}\rangle}{\varepsilon(b)}, \frac{\langle \mathbf{a}, \mathbf{b}\rangle}{\varepsilon(a)}\right)$. In case the valuation of an arrow is (1,1), one usually drops these numbers, this happens for all the arrows in the simply-laced cases; in the cases $\mathbb{B}_n$, $\mathbb{C}_n$, $\mathbb{F}_4$, some arrows will carry a valuation (1, 2) or (2, 1); for $\mathbb{G}_2$, all the arrows will carry a valuation (1, 3) or (3, 1). Note that the valuation of Γ allows to recover $\varepsilon(\mathbf{a})$ for any vertex $\mathbf{a}$.

The Auslander-Reiten quiver is usually considered as a translation quiver: some of the vertices are called projective, for any of the remaining vertices, say $\mathbf{a}$, there is defined a vertex $\tau\mathbf{a}$, such that there exists an arrow $\tau\mathbf{a} \to \mathbf{b}$ if and only if there exists an arrow $\mathbf{b} \to \mathbf{a}$. Here is the combinatorial recipe:

For any vertex i of Δ, let $\bar{\sigma}_i$ be the reflection in $\mathbb{Z}^n$ at $\mathbf{e}_i$ with respect to the symmetric bilinear form $(-,-)$. The orientation $\vec{\Delta}$ determines a unique Coxeter element C; for example, if we start with the natural ordering $1 < 2 < \cdots < n$ of the vertex set, then $C = \bar{\sigma}_1 \bar{\sigma}_2 \cdots \bar{\sigma}_n$. A positive root $\mathbf{a}$ will be said to be *projective*, provided $C(\mathbf{a})$ is no longer positive; for the remaining positive roots, let $\tau\mathbf{a} = C(\mathbf{a})$. Note that for any non-projective positive root $\mathbf{a}$, there are paths from $\tau\mathbf{a}$ to $\mathbf{a}$, and all are of length 2. (In the drawing of an Auslander-Reiten quiver, the translation τ usually will correspond to a shift from right to left; sometimes one connects $\mathbf{a}$ and $\tau\mathbf{a}$ by a dotted line; in the drawing above, the projective vertices have been denoted by $\bullet$.) The translation τ is very useful. For example, given positive roots $\mathbf{a}, \mathbf{b}$, then either $\mathbf{a}$ is projective and then $\zeta(\mathbf{a}, \mathbf{b}) = 0$, or else $\zeta(\mathbf{a}, \mathbf{b}) = \varepsilon(\mathbf{b}, \tau\mathbf{a})$. In particular, if $\zeta(\mathbf{a}, \mathbf{b}) \neq 0$, then there is a path from $\mathbf{b}$ to $\mathbf{a}$.

A path $\mathbf{a}_1 \to \mathbf{a}_2 \to \cdots \to \mathbf{a}_t$ in the Auslander-Reiten quiver is said to be *sectional*, provided we have $\mathbf{a}_{i-1} \neq \tau\mathbf{a}_{i+1}$, for all $1 < i < t$. Finally, two vertices $\mathbf{a}, \mathbf{b}$ are said to be *incomparable* provided there is no path from $\mathbf{a}$ to $\mathbf{b}$, and no path from $\mathbf{b}$ to $\mathbf{a}$.

What kind of information can be read off from the Auslander-Reiten quiver? First of all, the $\vec{\Delta}$-admissible orderings of the positive roots are just the refinements of the arrow relation: in order to deal with a PBW-basis, we may start with any total ordering $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ of the positive roots such that $\mathbf{a}_i \to \mathbf{a}_j$ implies $i < j$. Given positive roots $\mathbf{a}, \mathbf{b}$, there exists a $\vec{\Delta}$-admissible ordering $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ such that $\mathbf{a} = \mathbf{a}_i$, $\mathbf{b} = \mathbf{a}_j$ with $i < j$, if and only if there is no path from $\mathbf{b}$ to $\mathbf{a}$.

Also, *in case there is an arrow* $\mathbf{a} \to \mathbf{b}$ *in the Auslander-Reiten quiver, then*

$$\langle \mathbf{b}\rangle * \langle \mathbf{a}\rangle = v^{\max(\varepsilon(\mathbf{a}), \varepsilon(\mathbf{b}))} \langle \mathbf{a}\rangle * \langle \mathbf{b}\rangle .$$

Actually, the same formula is true in the general case that there exists a sectional path $\mathbf{a} = \mathbf{a}_1 \to \mathbf{a}_2 \cdots \to \mathbf{a}_t = \mathbf{b}$ (note that the existence of such a sectional path implies that we have $\operatorname{Ext}^1(M_{\mathscr{S}}(\mathbf{a}) \oplus M_{\mathscr{S}}(\mathbf{b}), M_{\mathscr{S}}(\mathbf{a}) \oplus M_{\mathscr{S}}(\mathbf{b})) = 0$). Of course, *in case the vertices* $\mathbf{a}, \mathbf{b}$ *are incomparable, then*

$$\langle \mathbf{a}\rangle * \langle \mathbf{b}\rangle = \langle \mathbf{b}\rangle * \langle \mathbf{a}\rangle .$$

As we have seen above, the main problem to be solved is an effective procedure for calculating the various skew derivations.

Lemma 4. *Let* $\mathbf{a}, \mathbf{b}$ *be positive roots, and assume that there is no path from* $\mathbf{b}$ *to* $\mathbf{a}$. *If* $\delta_{\langle \mathbf{b}\rangle}(\langle \mathbf{a}\rangle) \neq 0$, *then* $\zeta(\mathbf{b}, \mathbf{a}) \neq 0$, *and therefore, there is a path from* $\mathbf{a}$ *to* $\mathbf{b}$.

Proof. Since there is no path from $\mathbf{b}$ to $\mathbf{a}$, we know that $\zeta(\mathbf{a}, \mathbf{b}) = 0$, thus $\langle \mathbf{a}\rangle * \langle \mathbf{b}\rangle$ is a multiple of $\langle \mathbf{a} \oplus \mathbf{b}\rangle$. If we assume that $\zeta(\mathbf{b}, \mathbf{a}) = 0$, then also $\langle \mathbf{b}\rangle * \langle \mathbf{a}\rangle$ is a multiple of $\langle \mathbf{a} \oplus \mathbf{b}\rangle$, therefore $\delta_{\langle \mathbf{b}\rangle}(\langle \mathbf{a}\rangle) = 0$.

Thus, we have to consider pairs $\mathbf{a}, \mathbf{b}$ with a path from $\mathbf{a}$ to $\mathbf{b}$. If we express $\delta_{\langle \mathbf{b}\rangle}(\langle \mathbf{a}\rangle)$ as a linear combination

$$\sum_{\beta \in J(\mathbf{a}, \mathbf{b})} c_\beta \langle \beta \rangle$$

as in Theorem 1, the index set $J(\mathbf{a}, \mathbf{b})$ will involve only $\beta \in \mathscr{B}$ such that $\beta(\mathbf{c}) \neq 0$ implies that there exists paths of length at least 1 from $\mathbf{a}$ to $\mathbf{c}$ and from $\mathbf{c}$ to $\mathbf{b}$.

7. The rank 2 cases

Assume that we deal with a Cartan matrix of the form

$$\begin{pmatrix} 2 & a_{12} \\ a_{21} & 2 \end{pmatrix};$$

where one of the numbers $-a_{12}, -a_{21}$ is equal to 1, whereas the other is $r = -\varepsilon_1 a_{12} = -\varepsilon_2 a_{21} = 1, 2$, or 3. We work with the orientation $1 \to 2$. Note that $\mathbf{d} = \mathbf{e}_1 + \mathbf{e}_2$ is a positive root and let us stress that $\mathbf{e}_i = \langle \mathbf{e}_i \rangle$.

We have

$$\mathbf{e}_2 * \mathbf{e}_1 = \mathbf{e}_1 \oplus \mathbf{e}_2,$$

whereas

$$\mathbf{e}_1 * \mathbf{e}_2 = v^{-r}(\mathbf{d} + \mathbf{e}_1 \oplus \mathbf{e}_2).$$

Also note that $\dim \mathbf{d} = \varepsilon_1 + \varepsilon_2$, and $\varepsilon(\mathbf{d}) = \varepsilon_1 + \varepsilon_2 - r$, thus $\langle \mathbf{d}\rangle = v^{-r}\mathbf{d}$. It follows that

$$\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle = \langle \mathbf{d} \rangle = v^{-r}\mathbf{d} = \mathbf{e}_1 * \mathbf{e}_2 - v^{-r}\mathbf{e}_2 * \mathbf{e}_1.$$

Consider now the cases $r \geqq 2$, and let us assume that $\varepsilon_2 = 1$, thus $\mathbf{a}_2$ is a short root, whereas $\mathbf{a}_1$ is a long one. On the one hand, we have

$$\mathbf{e}_2 * \langle \mathbf{d} \rangle = \langle \mathbf{d} \oplus \mathbf{e}_2 \rangle = v^{-r+1}\mathbf{d} \oplus \mathbf{e}_2,$$

where we use that $\dim \mathbf{d} \oplus \mathbf{e}_2 = r + 2$ and $\varepsilon(\mathbf{d} \oplus \mathbf{e}_2) = 3$. On the other hand, we use that $\zeta(\mathbf{d}, \mathbf{e}_2) = r - 1$, and a rather easy calculation of Hall polynomials, in order to see that

$$\begin{aligned} \langle \mathbf{d} \rangle * \mathbf{e}_2 &= v^{-r}\mathbf{d} * \mathbf{e}_2 \\ &= v^{-2r+1}((q+1)(\mathbf{e}_1 + 2\mathbf{e}_2) + q(\mathbf{d} \oplus \mathbf{e}_2)) \\ &= v^{-2r+2}(v + v^{-1})(\mathbf{e}_1 + 2\mathbf{e}_2) + v^{-2r+3}\mathbf{d} \oplus \mathbf{e}_2. \end{aligned}$$

It follows that

$$\mathbf{e}_1 + 2\mathbf{e}_2 = \frac{v^{2r-2}}{[2]} \left(\langle \mathbf{d} \rangle * \mathbf{e}_2 - v^{-r+2} \cdot \mathbf{e}_2 * \langle \mathbf{d} \rangle\right).$$

After these preparations, we may consider in detail the different cases:

Case $\mathbb{A}_2$. In this case, we have $r = 1$, thus

$$\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle = \mathbf{e}_1 * \mathbf{e}_2 - v^{-1} \cdot \mathbf{e}_2 * \mathbf{e}_1 .$$

The corresponding Auslander-Reiten quiver is of the form

$$\begin{tikzcd}[row sep=small, column sep=small] & \mathbf{e}_1 + \mathbf{e}_2 \arrow[dr] & \\ \mathbf{e}_2 \arrow[ur] & \cdots & \mathbf{e}_1 . \end{tikzcd}$$

Case $\mathbb{B}_2$, with $\varepsilon_1 = 2$. The considerations above show that

$$\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle = \mathbf{e}_1 * \mathbf{e}_2 - v^{-2} \cdot \mathbf{e}_2 * \mathbf{e}_1 ,$$

$$\langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle = \frac{1}{[2]} \left(\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle * \mathbf{e}_2 - \mathbf{e}_2 * \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle\right);$$

here, we have used that $\dim \mathbf{e}_1 + 2\mathbf{e}_2 = 4$, whereas $\varepsilon(\mathbf{e}_1 + 2\mathbf{e}_2) = 2$, so that

$$\langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle = v^{-4+2}(\mathbf{e}_1 + 2\mathbf{e}_2) = \frac{1}{[2]} \left(\langle \mathbf{d} \rangle * \mathbf{e}_2 - \mathbf{e}_2 * \langle \mathbf{d} \rangle\right).$$

In this case, we deal with the Auslander-Reiten quiver

$$\begin{tikzcd}[row sep=small, column sep=small] & \mathbf{e}_1 + 2\mathbf{e}_2 \arrow[dr] & \cdots & \mathbf{e}_1 \\ \mathbf{e}_2 \arrow[ur] & \cdots & \mathbf{e}_1 + \mathbf{e}_2 . \arrow[ur] & \end{tikzcd}$$

The arrows $\nearrow$ carry the valuation $(1, 2)$, the arrows $\searrow$ carry the valuation $(2, 1)$.

In Appendix 1, we will present the multiplication table for the elements

$$X_1 = \langle \mathbf{e}_2 \rangle, \quad X_2 = \langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle, \quad X_3 = \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle, \quad X_4 = \langle \mathbf{e}_1 \rangle .$$

Case $\mathbb{B}_2$, with $\varepsilon_1 = 1$. Here, a similar calculation shows that

$$\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle = \mathbf{e}_1 * \mathbf{e}_2 - v^{-2} \cdot \mathbf{e}_2 * \mathbf{e}_1 ,$$

$$\langle 2\mathbf{e}_1 + \mathbf{e}_2 \rangle = \frac{1}{[2]} \left(\mathbf{e}_1 * \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle - \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle * \mathbf{e}_1\right).$$

The corresponding Auslander-Reiten quiver is of the form

$$\begin{tikzcd}[row sep=small, column sep=small] & \mathbf{e}_1 + \mathbf{e}_2 \arrow[dr] & \cdots & \mathbf{e}_1 \\ \mathbf{e}_2 \arrow[ur] & \cdots & 2\mathbf{e}_1 + \mathbf{e}_2 . \arrow[ur] & \end{tikzcd}$$

Here, the arrows $\nearrow$ carry the valuation (2, 1), the arrows $\searrow$ carry the valuation (1, 2).

Case $\mathbb{G}_2$, with $\varepsilon_1 = 3$. The Auslander-Reiten quiver looks as follows:

$$\begin{array}{ccccccccccc} & & \mathbf{e}_1+3\mathbf{e}_2 & & \cdots & & 2\mathbf{e}_1+3\mathbf{e}_2 & & \cdots & & \mathbf{e}_1 \\ & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\ \mathbf{e}_2 & & \cdots & & \mathbf{e}_1+2\mathbf{e}_2 & & \cdots & & \mathbf{e}_1+\mathbf{e}_2\,. & & \end{array}$$

The arrows $\nearrow$ carry the valuation (1, 3), the arrows $\searrow$ carry the valuation (3, 1).

For any positive root $\mathbf{a}$, we can express $\langle \mathbf{a} \rangle$ as a skew commutator as follows:

$$\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle = \mathbf{e}_1 * \mathbf{e}_2 - v^{-3}\mathbf{e}_2 * \mathbf{e}_1,$$

$$\langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle = \frac{1}{[2]}\left(\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle * \mathbf{e}_2 - v^{-1}\mathbf{e}_2 * \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle\right),$$

$$\langle \mathbf{e}_1 + 3\mathbf{e}_2 \rangle = \frac{1}{[3]}\left(\langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle * \mathbf{e}_2 - v \cdot \mathbf{e}_2 * \langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle\right),$$

$$\langle 2\mathbf{e}_1 + 3\mathbf{e}_2 \rangle = \frac{1}{[3]}\left(\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle * \langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle - v \cdot \langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle * \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle\right).$$

The first equality has been shown above. For the second equality, we only have to add to previous considerations that $\langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle = v^{-5+1}(\mathbf{e}_1 + 2\mathbf{e}_2)$, since in this case, $\varepsilon(\mathbf{e}_1 + 2\mathbf{e}_2) = 1$.

The calculation of $\langle \mathbf{e}_1 + 3\mathbf{e}_2 \rangle$ proceeds as follows: We have

$$\mathbf{e}_2 * \langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle = \langle (\mathbf{e}_1 + 2\mathbf{e}_2) \oplus \mathbf{e}_2 \rangle = v^{-2}\big((\mathbf{e}_1 + 2\mathbf{e}_2) \oplus \mathbf{e}_2\big),$$

and, on the other hand,

$$\begin{aligned} \langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle * \mathbf{e}_2 &= v^{-5}\big((q^2+q+1)(\mathbf{e}_1 + 3\mathbf{e}_2) + q^2\big((\mathbf{e}_1 + 2\mathbf{e}_2) \oplus \mathbf{e}_2\big)\big) \\ &= v^{-3}\big([3](\mathbf{e}_1 + 3\mathbf{e}_2)\big) + v^{-1}\big((\mathbf{e}_1 + 2\mathbf{e}_2) \oplus \mathbf{e}_2\big). \end{aligned}$$

Since $\mathbf{e}_1 + 3\mathbf{e}_2$ has dimension 6, and the dimension of its endomorphism ring is 3, we see that $\langle \mathbf{e}_1 + 3\mathbf{e}_2 \rangle = v^{-3}(\mathbf{e}_1 + 3\mathbf{e}_2)$.

A similar proof shows the last equality.

In Appendix 1, we will present the multiplication table for the elements

$$\begin{array}{lll} X_1 = \langle \mathbf{e}_2 \rangle, & X_2 = \langle \mathbf{e}_1 + 3\mathbf{e}_2 \rangle, & X_3 = \langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle, \\ X_4 = \langle 2\mathbf{e}_1 + 3\mathbf{e}_2 \rangle, & X_5 = \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle, & X_6 = \langle \mathbf{e}_1 \rangle. \end{array}$$

Case $\mathbb{G}_2$, with $\varepsilon_1 = 1$. The Auslander-Reiten quiver looks as follows:

$$\begin{array}{ccccccccccc} & \nearrow & \mathbf{e}_1 + \mathbf{e}_2 & \searrow & \cdots & \nearrow & 2\mathbf{e}_1 + \mathbf{e}_2 & \searrow & \cdots & \nearrow & \mathbf{e}_1 \\ \mathbf{e}_2 & & \cdots & & 3\mathbf{e}_1 + 2\mathbf{e}_2 & & \cdots & & 3\mathbf{e}_1 + \mathbf{e}_2\,. & & \end{array}$$

This time, the arrows $\nearrow$ carry the valuation $(3,1)$, the arrows $\searrow$ carry the valuation $(1,3)$.

For any positive root $\mathbf{a}$, we can express $\langle \mathbf{a} \rangle$ as a skew commutator as follows:

$$\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle = \mathbf{e}_1 * \mathbf{e}_2 - v^{-3} \mathbf{e}_2 * \mathbf{e}_1 ,$$

$$\langle 2\mathbf{e}_1 + \mathbf{e}_2 \rangle = \frac{1}{[2]} (\mathbf{e}_1 * \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle - v^{-1} \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle * \mathbf{e}_1),$$

$$\langle 3\mathbf{e}_1 + \mathbf{e}_2 \rangle = \frac{1}{[3]} (\mathbf{e}_1 * \langle 2\mathbf{e}_1 + \mathbf{e}_2 \rangle - v \cdot \langle 2\mathbf{e}_1 + \mathbf{e}_2 \rangle * \mathbf{e}_1),$$

$$\langle 3\mathbf{e}_1 + 2\mathbf{e}_2 \rangle = \frac{1}{[3]} (\langle 2\mathbf{e}_1 + \mathbf{e}_2 \rangle * \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle - v \cdot \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle * \langle 2\mathbf{e}_1 + \mathbf{e}_2 \rangle).$$

We are going to present the elements of the form $\langle \mathbf{e}_1 + t\mathbf{e}_2 \rangle$ with $1 \leqq t \leqq a_{12}$ as linear combinations of monomials; this will be needed at the end of the paper.

Case $\mathbb{A}_2$, thus $\varepsilon_1 = 1$, $\varepsilon_2 = 1$:

$$\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle = \mathbf{e}_1 * \mathbf{e}_2 - v^{-1} \mathbf{e}_2 * \mathbf{e}_1 .$$

Case $\mathbb{B}_2$, with $\varepsilon_1 = 2$, $\varepsilon_2 = 1$:

$$\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle = \mathbf{e}_1 * \mathbf{e}_2 - v^{-2} \mathbf{e}_2 * \mathbf{e}_1 .$$

Case $\mathbb{G}_2$, with $\varepsilon_1 = 3$, $\varepsilon_2 = 1$:

$$\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle = \mathbf{e}_1 * \mathbf{e}_2 - v^{-3} \mathbf{e}_2 * \mathbf{e}_1 .$$

Case $\mathbb{B}_2$, with $\varepsilon_1 = 1$, $\varepsilon_2 = 2$:

$$\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle = \mathbf{e}_1 * \mathbf{e}_2 - v^{-2} \mathbf{e}_2 * \mathbf{e}_1 ,$$

$$\langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle = \mathbf{e}_1 * \mathbf{e}_2^{(*2)} - v^{-1} \mathbf{e}_2 * \mathbf{e}_1 * \mathbf{e}_2 + v^{-2} \mathbf{e}_2^{(*2)} * \mathbf{e}_1 .$$

Case $\mathbb{G}_2$, with $\varepsilon_1 = 1$, $\varepsilon_2 = 3$:

$$\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle = \mathbf{e}_1 * \mathbf{e}_2 - v^{-3} \mathbf{e}_2 * \mathbf{e}_1 ,$$

$$\langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle = \mathbf{e}_1 * \mathbf{e}_2^{(*2)} - v^{-2} \mathbf{e}_2 * \mathbf{e}_1 * \mathbf{e}_2 + v^{-4} \mathbf{e}_2^{(*2)} * \mathbf{e}_1 ,$$

$$\langle \mathbf{e}_1 + 3\mathbf{e}_2 \rangle = \mathbf{e}_1 * \mathbf{e}_2^{(*3)} - v^{-1} \mathbf{e}_2 * \mathbf{e}_1 * \mathbf{e}_2^{(*2)} + v^{-2} \mathbf{e}_2^{(*2)} * \mathbf{e}_1 * \mathbf{e}_2 - v^{-3} \mathbf{e}_2^{(*2)} * \mathbf{e}_1 .$$

For arbitrary $\vec{\Delta}$, we will be interested in the elements of the form $\langle \mathbf{e}_j + t\mathbf{e}_i \rangle$ with $j \neq i$ and $0 \leqq t \leqq a_{ij}$. Here is the general rule for expressing them as linear combinations of monomials:

Proposition 3. *Let $0 \leqq t \leqq -a_{ij}$. If i is a sink for $\vec{\Delta}$, then*

$$\langle \mathbf{e}_j + t\mathbf{e}_i \rangle = \sum_{r+s=t} (-1)^r v^{-\varepsilon_i r(-a_{ij}-t+1)} E_i^{(r)} E_j E_i^{(s)};$$

if i is a source, then

$$\langle \mathbf{e}_j + t\mathbf{e}_i \rangle = \sum_{r+s=t} (-1)^r v^{-\varepsilon_i r(-a_{ij}-t+1)} E_i^{(s)} E_j E_i^{(r)}.$$

Proof. We may restrict to the case where i, j are the only vertices. If we assume that i is a sink, then we relabel the vertices as follows: $j = 1$, $i = 2$, and we use the previous consideration. The case when i is a source follows by duality.

A direct proof of Proposition 3 may be given along the line of the proof of the fundamental relations, see [R5].

8. The inductive construction of exceptional modules

Let Λ be a finite-dimensional hereditary k-algebra (for example, the tensor algebra of a k-species as above), let n be the number of isomorphism classes of simple Λ-modules.

Recall that a (finite-dimensional) Λ-module M is said to be *exceptional* provided its endomorphism ring is a division ring and $\mathrm{Ext}^1(M, M) = 0$. A set M_i $(i \in I)$ of modules is said to be *orthogonal* provided $\mathrm{Hom}(M_i, M_j) = 0$ for all $i \neq j$.

Proposition 4. *Let M be a non-simple exceptional Λ-module. Then there exist orthogonal exceptional modules M_1, M_2, and an exact sequence*

$$0 \to a_2 M_2 \to M \to a_1 M_1 \to 0,$$

with $a_1, a_2 \geqq 1$.

Remark. In the case of an algebraically closed field, the result is due to Schofield [S].

Crawley-Boevey recently has shown that a braid group operates transitively on the set of complete exceptional sequences. Ideas from [CB] and [R8], which have been developed in order to establish this result, will be used for a proof of Proposition 4.

We recall the following: A sequence $(M_1, \ldots, M_s)$ of exceptional modules is called *exceptional*, provided $\mathrm{Hom}(M_j, M_i) = 0 = \mathrm{Ext}^1(M_j, M_i)$ for any pair $i < j$. An exceptional sequence $(M_1, \ldots, M_s)$ is said to be *complete*, if $s = n$. (Note that a pair $(\mathbf{b}, \mathbf{a})$ of positive roots is $\vec{\Delta}$-orthogonal if and only if $(M(\mathbf{b}), M(\mathbf{a}))$ is an orthogonal exceptional sequence; this explains the order of the roots.) In our case of a finite-dimensional hereditary k-algebra, the indecomposable projective modules, the simple modules, as well as the indecomposable injective modules, always taken in a suitable order, yield examples of complete exceptional sequences.

Given a module N, we denote by $\mathscr{C}(N)$ the smallest subcategory containing N and being closed under extensions, kernels of epimorphisms and cokernels of monomorphisms. In case $(M_1, \ldots, M_s)$ is an exceptional sequence, then Crawley-Boevey [CB] has shown that $\mathscr{C}(M_1 \oplus \cdots \oplus M_s)$ is equivalent to the module category of a finite-dimensional hereditary algebra with precisely s isomorphism classes of simple modules.

Lemma 5. *Let M be a non-simple exceptional Λ-module. Then there exists a module N such that (M, N) or (N, M) is an exceptional sequence, and M considered as an object of $\mathscr{C}(M \oplus N)$ is not simple.*

Proof. Take a complete exceptional sequence $(N_1, \ldots, N_n)$ with $M = N_j$ for some j, and such that the length of $\bigoplus_{i=1}^{n} N_i$ is minimal.

Let M be an exceptional Λ-module with the following property: If (M, N) or (N, M) is an exceptional sequence, then M is simple in $\mathscr{C}(M \oplus N)$. Under this assumption, the reduction process as exhibited in [R8] shows that the sequence is orthogonal. Namely, in case we use transpositions, the modules N_i are not changed, only their indices are. We cannot use a proper reduction which does not involve M, since this would contradict our minimality assumption. Thus, assume that we make a proper reduction involving M. Up to duality, we have $\operatorname{Hom}(N_j, N_{j+1}) \neq 0$. By assumption we know that $M = N_j$ is a simple object in $\mathscr{C}(M \oplus N_{j+1})$, thus there is an exact sequence

$$0 \to tM \to N_{j+1} \to N' \to 0$$

with $\operatorname{Hom}(M, N') = 0$. The proper reduction replaces the pair (M, N_{j+1}) by the pair (N', M), thus we obtain a new exceptional sequence

$$(N_1, \ldots, N_{j-1}, N', M, N_{j+2}, \ldots, N_n)$$

of smaller length. This contradiction shows that our given sequence was orthogonal.

Note that given an orthogonal, complete exceptional sequence $(N_1, \ldots, N_n)$, all the modules N_i are simple [R8]. Thus, we see that M is simple.

Proof of Proposition 4. Assume that M is non-simple, and exceptional. According to Lemma 5, there exists an exceptional sequence (M, N) or (N, M) such that M is not simple in $\mathscr{C}(M \oplus N)$. Let M_1, M_2 be the two simple objects in this subcategory $\mathscr{C}(M \oplus N)$, with $\operatorname{Ext}^1(M_1, M_2) \neq 0$. Since M is not simple, in $\mathscr{C}(M \oplus N)$, there exists an exact sequence

$$0 \to a_2 M_2 \to M \to a_1 M_1 \to 0,$$

with $a_1, a_2 \geqq 1$. This completes the proof.

Consider now the special case when Λ is the tensor algebra of a k-species $\mathscr{S}$ of type $\vec{\Delta}$. Assume that there is given an exact sequence

$$0 \to a_2 M_2 \to M \to a_1 M_1 \to 0,$$

with indecomposable modules M, M_1, M_2, and a_1, $a_2 \geqq 1$. Let $M = M(\mathbf{a})$, $M_1 = M(\mathbf{a}_1)$, $M_2 = M(\mathbf{a}_2)$ with positive roots $\mathbf{a}, \mathbf{a}_1, \mathbf{a}_2$. We see that $\mathbf{a} = a_1 \mathbf{a}_1 + a_2 \mathbf{a}_2$. Note that the linear combinations of $\mathbf{a}_1, \mathbf{a}_2$ which are roots form a root system of rank 2, thus it is of type $\mathbb{A}_2$, $\mathbb{B}_2$, or $\mathbb{G}_2$; in particular, we see that $a_1 a_2 \leqq 3$. We are looking for conditions in order to have $a_1 = 1 = a_2$.

Recall that a root $\mathbf{a}$ is called *sincere*, provided $\mathbf{a} = \sum_{i=1}^{n} c_i \mathbf{e}_i$ with $c_i \neq 0$ for all i.

Proposition 5. *Let $\mathscr{S}$ be a k-species of type $\vec{\Delta}$, with Δ not of the form $\mathbb{A}_1$ or $\mathbb{G}_2$. Let $\mathbf{a}$ be a sincere positive root. In case Δ is of the form $\mathbb{C}_n$ for some $n \geqq 2$, assume in addition that $\mathbf{a}$ is short. Then there is an exact sequence*

$$0 \to M(\mathbf{a}_2) \to M(\mathbf{a}) \to M(\mathbf{a}_1) \to 0$$

with $\vec{\Delta}$-orthogonal positive roots $\mathbf{a}_1, \mathbf{a}_2$.

Proof. According to Proposition 4, there is an exact sequence

$$0 \to a_2 M(\mathbf{a}_2) \to M(\mathbf{a}) \to a_1 M(\mathbf{a}_1) \to 0,$$

with $\vec{\Delta}$-orthogonal positive roots $\mathbf{a}_1, \mathbf{a}_2$ and $a_1, a_2 \geqq 1$. In case the roots $\mathbf{a}_1, \mathbf{a}_2$ have the same length, the root system generated by $\mathbf{a}_1, \mathbf{a}_2$ is of type $\mathbb{A}_2$, thus $a_1 = 1 = a_2$. We assume that $\mathbf{a}_1$, $\mathbf{a}_2$ have different length. The existence of a sincere root implies that Δ is connected, thus Δ is of the form $\mathbb{B}_n$, $\mathbb{C}_n$ with $n \geqq 2$, or $\mathbb{F}_4$. If $\mathbf{a}$ is a short root, then $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$, thus again $a_1 = 1 = a_2$. Thus, we can assume that $\mathbf{a}$ is a long root, and therefore only the cases $\mathbb{B}_n$, with $n \geqq 3$, and $\mathbb{F}_4$ remain.

In drawing the graph of Δ, we use the usual conventions. In the case $\mathbb{B}_n$, we label the vertices of the graph of Δ as follows:

$$\underset{1}{\circ} \Leftarrow \underset{2}{\circ} — \underset{3}{\circ} — \cdots — \underset{n-1}{\circ} — \underset{n}{\circ}$$

with $a_{12} = -2$. Similarly, in the case $\mathbb{F}_4$, we label the vertices

$$\underset{1}{\circ} — \underset{2}{\circ} \Leftarrow \underset{3}{\circ} — \underset{4}{\circ}$$

with $a_{23} = -2$. We use an arbitrary ordering $\prec$ on the set of vertices $\{1, 2, \ldots, n\}$. Let $P(i)$ be the projective cover of S_i, for $1 \leqq i \leqq n$, and $Q(i)$ its injective envelope. Let $\mathbf{p}(i)$, $\mathbf{q}(i)$ be positive roots such that $M(\mathbf{p}(i)) = P(i)$, $M(\mathbf{q}(i)) = Q(i)$.

We deal with the Auslander-Reiten quiver Γ. We can write $\mathbf{a} = \tau^{-s}\mathbf{p}(i)$ for some $1 \leqq i \leqq n$ and some $s \geqq 0$. Since $\mathbf{a}$ is a long root, we have $i \geqq 2$ for $\mathbb{B}_n$, and $i \geqq 3$ for $\mathbb{F}_4$.

Consider first the case when $i < n$. Then there exists an arrow from $\mathbf{a}_2$ to $\mathbf{a}$ in Γ such that $\mathbf{a}_2$ lies in the τ-orbit of $\mathbf{p}(i+1)$; a corresponding non-zero map $M(\mathbf{a}_2) \to M(\mathbf{a})$ is injective and its cokernel is indecomposable, say of the form $M(\mathbf{a}_1)$, and $\mathbf{a}_1$ lies in the τ-orbit of $\mathbf{p}(n)$; here, we use that $\mathbf{a}$ is sincere, so that the "wing" with center $\mathbf{a}$ exists. In this way, we have obtained the two positive roots $\mathbf{a}_1$, $\mathbf{a}_2$ we were looking for.

This shows that we can assume that $i = n$. Case $\mathbb{B}_n$: Since $\mathbf{a}$ is sincere, there is a path $\mathbf{p}(1) \to \cdots \to \mathbf{a}$, and a path $\mathbf{a} \to \cdots \to \mathbf{q}(1)$, and an easy length consideration shows that both paths are sectional. In particular, the root $\mathbf{a}$ is uniquely determined. Up to duality, we can assume that $\mathbf{q}(1)$ is a simple root. It follows that we deal with the natural ordering $1 \prec 2 \prec \cdots \prec n$, since otherwise we would obtain some $i > 1$ with $\operatorname{Hom}(M, Q(i)) = 0$. As a consequence, $M(\mathbf{a}) = Q(n)$, and therefore $\mathbf{a} = 2\mathbf{e}_1 + \sum_{i=2}^{n} \mathbf{e}_i$. The case $n = 2$ has been excluded, thus $n \geqq 3$, and there is an exact sequence of the form

$$0 \to M\Big(\sum_{i=3}^{n} \mathbf{e}_i\Big) \to M \to M(2\mathbf{e}_1 + \mathbf{e}_2) \to 0 .$$

Case $\mathbb{F}_4$. Consider the sectional paths $\mathbf{p}(1) \to \cdots \to \mathbf{b}_1$ and $\mathbf{b}_2 \to \cdots \to \mathbf{q}(1)$, where both $\mathbf{b}_1, \mathbf{b}_2$ belong to the τ-orbit of $\mathbf{p}(4)$. We have $\mathbf{b}_1 = \tau^2 \mathbf{b}_2$. Note that $\operatorname{Hom}(P(1), \tau M(\mathbf{b}_2)) = 0$, thus the two modules $M(\mathbf{b}_1)$, $M(\mathbf{b}_2)$ are the only modules $N = M(\mathbf{c})$ with $\mathbf{c}$ in the τ-orbit of $\mathbf{p}(4)$ which satisfy $\operatorname{Hom}(P(1), N) \neq 0$. This shows that $M(\mathbf{a})$ is one of these two modules. Up to duality, we can assume that $\mathbf{b}_1$ or $\tau^{-1}\mathbf{b}_1$ is equal to $\mathbf{q}(4)$. In both cases, there exists an exact sequence

$$0 \to M(\tau^2 \mathbf{b}_1) \to M(\mathbf{b}_1) \to M(\mathbf{b}_2) \to 0 ,$$

thus, for $\mathbf{a} = \mathbf{b}_1$, we have found an exact sequence as required. It remains to consider the case $\mathbf{a} = \mathbf{b}_2$. Note that $\mathbf{b}_2 \neq \tau\mathbf{q}(4)$, since $\operatorname{Hom}(M(\tau\mathbf{q}(4)), M(\mathbf{q}(4))) = 0$, whereas we assume that $\mathbf{a}$ is sincere. It follows that $\mathbf{b}_2 = \mathbf{q}(4)$. But $\mathbf{q}(4)$ is sincere only in case we deal with the natural ordering $1 \prec 2 \prec 3 \prec 4$, and then we have the exact sequence

$$0 \to S_4 \to M(\mathbf{q}(4)) \to M(\mathbf{q}(3)) \to 0 ,$$

as required. This completes the proof.

Remark. In Proposition 5, we had to exclude the case $\mathbb{A}_1$, since otherwise $\mathbf{a}$ would be a simple root, thus $M(\mathbf{a})$ a simple representation. Also, we have excluded the case $\mathbb{G}_2$, since it has been discussed in detail before: the only Δ-orthogonal positive roots are the simple roots, and there are three sincere roots which are not of the form $\mathbf{e}_1 + \mathbf{e}_2$. Finally, consider the case $\mathbb{C}_n$:

$$\underset{1}{\circ} \Rightarrow \underset{2}{\circ} - \underset{3}{\circ} - \cdots - \underset{n-1}{\circ} - \underset{n}{\circ}$$

with $a_{21} = -2$. The only long positive roots are given by $\mathbf{e}_1 + 2\sum_{i=2}^{t} \mathbf{e}_i$, where $1 \neq t \leqq n$. We see that there is just one sincere long positive root, and it cannot be written in the form $\mathbf{a} + \mathbf{b}$, where $\mathbf{a}, \mathbf{b}$ are long positive roots. It follows that it cannot be written in the form $\mathbf{a} + \mathbf{b}$, where $\mathbf{a}, \mathbf{b}$ are Δ-orthogonal positive roots.

9. Skew commutators

Let **a**, **b** be roots. The linear combinations of **a**, **b** which are roots form a root system of type $\mathbb{A}_1 \times \mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{B}_2$, or $\mathbb{G}_2$.

Assume that the pair $(\mathbf{a}_1, \mathbf{a}_2)$ is $\vec{\Delta}$-orthogonal. Let $r = -\langle \mathbf{a}_1, \mathbf{a}_2 \rangle$. Then $M(\mathbf{a}_1)$, $M(\mathbf{a}_2)$ are orthogonal exceptional modules. The subcategory

$$\mathscr{C} = \mathscr{C}\big(M(\mathbf{a}_1) \oplus M(\mathbf{a}_2)\big)$$

is equivalent to the category of representations of a K-species $\mathscr{S}'$ of type $\vec{\Delta}'$, where Δ' is a Cartan matrix of rank 2. Here, we take $K = k$ in the simply-laced cases, and also in case at least one of the roots $\mathbf{a}_1$, $\mathbf{a}_2$ is a short root; in case both roots $\mathbf{a}_1$, $\mathbf{a}_2$ are long, let $K = \operatorname{End} M(\mathbf{a}_1)$. The last case can happen only for $r = 2$. Note that as a K-species, $\mathscr{S}'$ again is reduced. Let us fix an equivalence from the category of representations of $\mathscr{S}'$ onto $\mathscr{C}$, and denote it by v. Since the objects $M(\mathbf{a}_1)$, $M(\mathbf{a}_2)$ are the simple objects in $\mathscr{C}$, they are the images of the simple representations S_1' and S_2' of $\mathscr{S}'$ under v. We obtain an embedding of the Grothendieck group $K_0(\mathscr{S}') = \mathbb{Z}^2$ into $K_0(\mathscr{S})$, which again we denote by v (with $v(\mathbf{e}_i) = \mathbf{a}_i$, for $i = 1, 2$). Under this embedding, the bilinear form $\langle -, - \rangle$ of $K_0(\mathscr{S})$ restricts to a scalar multiple of the corresponding bilinear form of $K_0(\mathscr{S}')$, since $\mathscr{C}$ is an extension closed full subcategory of the category of representations of $\mathscr{S}$. In fact, in case $K = k$, we obtain the corresponding bilinear form itself, in case $K = \operatorname{End} M(\mathbf{a}_1)$, we obtain the r-multiple: given two representations M, N of $\mathscr{S}'$, we have

$$\begin{aligned}
\langle vM, vN \rangle &= \dim_k \operatorname{Hom}(vM, vN) - \dim_k \operatorname{Ext}^1(vM, vN) \\
&= [K:k]\big(\dim_K \operatorname{Hom}(M, N) - \dim_K \operatorname{Ext}^1(M, N)\big) \\
&= [K:k]\langle M, N \rangle,
\end{aligned}$$

and, in case $\mathbf{a}_1$, $\mathbf{a}_2$ both are long roots, then $[K:k] = 2 = r$.

Since Δ' is of rank 2, it is of the form $\mathbb{A}_1 \times \mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{B}_2$, or $\mathbb{G}_2$. In case $r = 0$, the only indecomposable objects in $\mathscr{C}$ are the two modules $M(\mathbf{a}_1)$, $M(\mathbf{a}_2)$, thus let us assume that $r > 0$. For the connected rank 2 cases, we have seen above how to express the indecomposable as skew commutators, and we claim that we obtain corresponding formulae when we replace $\mathbf{e}_i$ by $\mathbf{a}_i$; in case of two long roots $\mathbf{a}_1$, $\mathbf{a}_2$, we also have to replace v by v^2. It is sufficient to consider $\mathscr{H}' = \mathscr{H}_*(\vec{\Delta}') \otimes \mathbb{Q}(v)$, and the ring homomorphism $\mathscr{H}' \to \mathscr{H}$ which sends E_i to $\langle \mathbf{a}_i \rangle$, and v to $v^{[K:k]}$. In this way, we see that the recipe outlined in the introduction is based on our calculations in the rank 2 cases.

Of course, we may use the information provided by Proposition 5 in order to improve the inductive construction of the elements $X(\mathbf{a})$. For any $\vec{\Delta}$-orthogonal pair $(\mathbf{a}, \mathbf{b})$, with $r = r_{\mathbf{a}}^{\mathbf{b}} \geqq 1$, we have defined

$$X(\mathbf{a} + \mathbf{b}) = X(\mathbf{b})\, X(\mathbf{a}) - v^{-r} X(\mathbf{a})\, X(\mathbf{b}),$$

and we wonder which additional elements we have to take care off. We may assume that Δ is connected and not of type $\mathbb{G}_2$. In the simply-laced cases, we will have obtained a PBW-basis in this way, thus, we only have to consider the cases $\mathbb{B}_n$, $\mathbb{C}_n$ and $\mathbb{F}_4$. We work with the graph and the labelling as presented in the last section.

For $\mathbb{B}_n$, the only positive root to be considered in addition is $2\mathbf{e}_1 + \mathbf{e}_2$; for $\mathbb{C}_n$; we have to deal with all the additional roots $\mathbf{e}_1 + \sum_{i=2}^{t} \mathbf{e}_i$ for $2 \leqq i \leqq t$; and for $\mathbb{F}_4$, we have to take into account the roots $2\mathbf{e}_2 + \mathbf{e}_3$ and $2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$. For these roots $\mathbf{a}$, we have to construct $X(\mathbf{a})$ as outlined in the introduction, as the $\frac{1}{[2]}$-multiple of a proper commutator.

10. The $\mathscr{A}$-form $U_{\mathscr{A}}^{+}$

Recall that we denote $\mathscr{A} = \mathbb{Z}(v, v^{-1}]$ and that $U_{\mathscr{A}}^{+}$ is the $\mathscr{A}$-subalgebra of U^{+} generated by the elements $E_i^{(t)}$.

Proposition 6. *The $\mathscr{A}$-algebra $\mathscr{H}_*(\vec{\Delta})$ is generated by the elements $\mathbf{e}_i^{(t)}$, with $1 \leqq i \leqq n$ and $t \geqq 1$.*

This is known, see [R6], [R7]. For the convenience of the reader, we outline below a direct argument.

Corollary. *The isomorphism $\eta\colon U^{+} \to \mathscr{H}$ of $\mathbb{Q}(v)$-algebras defined by $\eta(E_i) = \mathbf{e}_i$ maps $U_{\mathscr{A}}^{+}$ onto $\mathscr{H}_*(\vec{\Delta})$.*

In order to present a proof of Proposition 6, we need the following lemma:

Lemma 6. *For any $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$,*

$$\mathbf{e}_1^{(*d_1)} * \cdots * \mathbf{e}_n^{(*d_n)} = \sum_{\mathbf{dim}\,\alpha = \mathbf{d}} v^{-\zeta(\alpha,\alpha)} \langle \alpha \rangle .$$

Proof. We have $\mathbf{e}_i^{(*d_i)} = \langle d_i \mathbf{e}_i \rangle = v^{-d_i\varepsilon_i + d_i^2\varepsilon_i} d_i \mathbf{e}_i$, according to Lemma 2. Let us denote $\beta = \bigoplus_{i=1}^{n} d_i \mathbf{e}_i$. Thus $\dim \mathbf{d} = \dim \beta = \sum d_i \varepsilon_i$, and $\varepsilon(\beta) = \sum d_i^2 \varepsilon_i$. Also,

$$\zeta(\beta, \beta) = \sum_{i<j} \zeta(d_i \mathbf{e}_i, d_i \mathbf{e}_j),$$

since for $i \geqq j$, we have $\zeta(d_i \mathbf{e}_i, d_j \mathbf{e}_j) = 0$.

Any module M with dimension vector $\mathbf{d}$ has a unique filtration

$$M = M_0 \supseteqq M_1 \supseteqq \cdots \supseteqq M_n = 0$$

with factors M_{i-1}/M_i isomorphic to $d_i S_i$, since $\operatorname{Ext}^1(S_i, S_j) = 0$ for $i \geqq j$. This shows that the Hall polynomial $\phi^{\alpha}_{d_1\mathbf{e}_1,\ldots,d_n\mathbf{e}_n}$ is equal to 1, for any α with $\mathbf{dim}\,\alpha = \mathbf{d}$.

Also, for $i<j$, we have $\langle d_i\mathbf{e}_i, d_j\mathbf{e}_j\rangle = -\zeta(d_i\mathbf{e}_i, d_j\mathbf{e}_j)$, therefore

$$\begin{aligned}\mathbf{e}_1^{(*d_1)} * \cdots * \mathbf{e}_n^{(*d_n)} &= v^{-\sum d_i\varepsilon_i + \sum d_i^2\varepsilon_i} d_1\mathbf{e}_1 * \cdots * d_n\mathbf{e}_n \\ &= v^{-\sum d_i\varepsilon_i + \sum d_i^2\varepsilon_i} v^{-\sum_{i<j}\langle d_i\mathbf{e}_i, d_j\mathbf{e}_j\rangle} \sum_{\mathbf{dim}\,\alpha=\mathbf{d}} [\alpha] \\ &= v^{-\dim\beta+\varepsilon(\beta)-\zeta(\beta,\beta)} \sum_{\mathbf{dim}\,\alpha=\mathbf{d}} [\alpha] \\ &= v^{-\dim\mathbf{d}+\langle\mathbf{d},\mathbf{d}\rangle} \sum_{\mathbf{dim}\,\alpha=\mathbf{d}} [\alpha].\end{aligned}$$

We insert $[\alpha] = v^{\dim\alpha-\varepsilon(\alpha)}\langle\alpha\rangle$ and note that $\langle\mathbf{d},\mathbf{d}\rangle = \langle\mathbf{dim}\,\alpha, \mathbf{dim}\,\alpha\rangle = \varepsilon(\alpha) - \zeta(\alpha,\alpha)$. In this way, we obtain the formula as stated.

Proof of Proposition 6. According to Lemma 2, $\mathbf{e}_i^{(t)} = \langle\mathbf{e}_i\rangle^{(t)} = \langle t\mathbf{e}_i\rangle$, thus the elements $\mathbf{e}_i^{(t)}$ belong to $\mathscr{H}_*(\vec{\Delta})$.

Let $\mathscr{G}$ be the subring of $\mathscr{H}_*(\vec{\Delta})$ generated by the $\mathbf{e}_i^{(t)}$, with $1 \leqq i \leqq n$ and $t \geqq 1$. Let us show that any $\alpha \in \mathscr{B}$ belongs to $\mathscr{G}$. We use induction on $\dim\alpha$. If there are at least two different positive roots $\mathbf{a}_i$ with $\alpha(\mathbf{a}_i) \neq 0$, then we use Proposition 1′ in order to see that $\langle\alpha\rangle = \langle\alpha(\mathbf{a}_1)\mathbf{a}_1\rangle * \cdots * \langle\alpha(\mathbf{a}_m)\mathbf{a}_m\rangle$. By assumption, we know that $\dim\alpha(\mathbf{a}_i)\mathbf{a}_i < \dim\alpha$, for all i; thus by induction, all the elements $\langle\alpha(\mathbf{a}_i)\mathbf{a}_i\rangle$ belong to $\mathscr{G}$. This shows that $\langle\alpha\rangle$ belongs to $\mathscr{G}$. It remains to be seen that for any positive root $\mathbf{a}$, and any $t \in \mathbb{N}_0$, the element $\langle t\mathbf{a}\rangle$ belongs to $\mathscr{G}$.

We apply Lemma 6 for $\mathbf{d} = t\mathbf{a}$. Note that $\zeta(t\mathbf{a}, t\mathbf{a}) = 0$, thus

$$\langle t\mathbf{a}\rangle = \mathbf{e}_1^{(*d_1)} * \cdots * \mathbf{e}_n^{(*d_n)} - \sum_{\substack{\mathbf{dim}\,\beta=\mathbf{d}\\ \beta\neq t\mathbf{a}}} v^{-\zeta(\beta,\beta)}\langle\beta\rangle.$$

If $\beta \neq t\mathbf{a}$ is given with $\mathbf{dim}\,\beta = \mathbf{d}$, then β cannot be a multiple of a root, thus there are at least two different roots $\mathbf{a}_i$ with $\beta(\mathbf{a}_i) \neq 0$, and therefore we know already that $\langle\beta\rangle$ belongs to $\mathscr{G}$. Of course, also $\mathbf{e}_1^{(*d_1)} * \cdots * \mathbf{e}_n^{(*d_n)}$ belongs to $\mathscr{G}$, therefore $\langle t\mathbf{a}\rangle$ belongs to $\mathscr{G}$. This completes the proof.

We obtain the following consequence, where $X_1, \ldots, X_m$ generates a PBW-basis of U^+ as constructed above.

Theorem 4. *The elements* $X_1^{(*\alpha(1))} * \cdots * X_m^{(*\alpha(m))}$ *with* $\alpha(1), \ldots, \alpha(m,) \in \mathbb{N}_0$ *form an* $\mathscr{A}$*-basis of* $U_{\mathscr{A}}^+$.

11. The subalgebra $\mathscr{H}|\mathscr{M}$

Let $\mathscr{S}$ be a reduced k-species of type $\vec{\Delta}$. We denote by rep-$\mathscr{S}$ the category of all representations of $\mathscr{S}$. Recall that a subcategory $\mathscr{M}$ of rep-$\mathscr{S}$ is said to be *closed under direct summands* provided for every module M in $\mathscr{M}$, all its direct summands belong to $\mathscr{M}$. A subcategory $\mathscr{M}$ of rep-$\mathscr{S}$ will be said to be *closed under potential extensions* provided for M_1, M_2 in $\mathscr{M}$, and $\phi_{M_1M_2}^M \neq 0$, also M belongs to $\mathscr{M}$. (Let us stress that it may happen that

$\phi^M_{M_1 M_2} \neq 0$, whereas there is no exact sequence of the form $0 \to M_2 \to M \to M_1 \to 0$; the evaluation of the Hall-polynomial $\phi^M_{M_1 M_2} \neq 0$ at $|k|$ may be zero: as an example, for $\mathbb{D}_4$, there are indecomposable modules M, M_1, M_2 with $\phi^M_{M_1 M_2} = q - 2$.

The following is obvious from the definition of the Hall multiplication: *Let $\mathcal{M}$ be a subcategory of* rep-$\mathcal{S}$ *which is closed under potential extension. Let $\mathcal{H}|\mathcal{M}$ be the $\mathbb{Q}(v)$-subspace of $\mathcal{H}$ generated by the elements $[M]$ with $M \in \mathcal{M}$. Then $\mathcal{H}|\mathcal{M}$ is a subring. Similarly, let $\mathcal{H}_*(\vec{\Delta})|\mathcal{M}$ be the $\mathcal{A}$-submodule of $\mathcal{H}_*(\vec{\Delta})$ generated by the elements $[M]$ with $M \in \mathcal{M}$. Then $\mathcal{H}_*(\vec{\Delta})|\mathcal{M}$ is a subring.*

Proposition 7. *Let $\mathcal{M}$ be a subcategory of* rep-$\mathcal{S}$ *which is closed under direct summands and potential extensions. Let $\mathcal{N} \subseteq \mathcal{M}$ be a subcategory, and assume that any representation in $\mathcal{M}$ has a filtration with factors in $\mathcal{N}$. Then $\mathcal{H}|\mathcal{M}$ is generated as a $\mathbb{Q}(v)$-subalgebra by the elements $[N]$ with N in $\mathcal{N}$.*

The proof is similar to that of Proposition 4: Let $\mathcal{G}$ be the $\mathbb{Q}(v)$-subalgebra generated by the elements $[N]$ with N in $\mathcal{N}$. In order to show that any $[M]$, with M in $\mathcal{M}$ belongs to $\mathcal{G}$, we use induction. Let $M = M(\alpha)$ for some $\alpha \in \mathcal{B}$. Note that M is the direct sum of the modules $M(\alpha(\mathbf{a})\mathbf{a})$, with $\mathbf{a} \in \Phi^+$. If $\alpha(\mathbf{a}_i) \neq 0$ for at least two different positive roots $\mathbf{a}_i$, then all the modules $M(\alpha(\mathbf{a})\mathbf{a})$ have smaller length, and by induction $[M(\alpha(\mathbf{a})\mathbf{a})]$ belongs to $\mathcal{G}$. Proposition 1′ shows that also $[M]$ itself belongs to $\mathcal{G}$. It remains to consider the case $\alpha = t\mathbf{a}$ for some positive root $\mathbf{a}$ and some $t \geqq 1$. By assumption, $M = M(t\mathbf{a})$ has a filtration with factors in $\mathcal{N}$, thus, there are modules $N_1, \ldots, N_s$ with $\phi^M_{N_1,\ldots,N_s} \neq 0$. Write

$$[N_1] * \cdots * [N_s] = \sum_{\mathbf{dim}\,\beta = t\mathbf{a}} c_\beta \cdot [M(\beta)],$$

with coefficients $c_\beta \in \mathbb{Q}(v)$. Note that $[M(t\mathbf{a})]$ occurs with a non-zero factor $c_{t\mathbf{a}}$. Thus

$$[M] = c_{t\mathbf{a}}^{-1}\Big([N_1] * \cdots * [N_s] - \sum_{\substack{\mathbf{dim}\,\beta = t\mathbf{a} \\ \beta \neq t\mathbf{a}}} c_\beta \cdot [M(\beta)]\Big).$$

If $\beta \neq t\mathbf{a}$ is given with $\mathbf{dim}\,\beta = \mathbf{d}$, then there are at least two different roots $\mathbf{a}_i$ with $\beta(\mathbf{a}_i) \neq 0$, and therefore we know already that $[M(\beta)]$ belongs to $\mathcal{G}$. Altogether, this shows that $[M]$ belongs to $\mathcal{G}$.

For example, if i is a vertex for $\vec{\Delta}$, let rep-$\mathcal{S}\langle i\rangle$ be the subcategory of all representations which do not have S_i as a direct summand. By construction, this subcategory is closed under direct summands. If i is a sink or a source for $\vec{\Delta}$, then rep-$\mathcal{S}\langle i\rangle$ is also closed under potential extensions (note that for i a sink, S_i is projective; similarly, for i a source, S_i is injective). In these two cases, we will consider

$$\mathcal{H}\langle i\rangle = \mathcal{H}|\text{rep-}\mathcal{S}\langle i\rangle,$$

$$\mathcal{H}_*(\vec{\Delta})\langle i\rangle = \mathcal{H}_*(\vec{\Delta})|\text{rep-}\mathcal{S}\langle i\rangle.$$

The set of isomorphism classes of representations in rep-$\mathcal{S}\langle i\rangle$ will be denoted by $\mathcal{B}\langle i\rangle$, thus $\mathcal{H}\langle i\rangle$ is the free $\mathbb{Q}(v)$-module with basis $\mathcal{B}\langle i\rangle$.

Lemma 7. *Let i be a sink or a source. Let $\mathcal{N}$ be the set of indecomposable representations N with dimension vector $\mathbf{e}_j + t\mathbf{e}_i$ for some $j \neq i$, and some $t \geqq 0$. Then any representation in* rep-$\mathcal{S}\langle i\rangle$ *has a filtration with factors in $\mathcal{N}$.*

Proof. We consider the case where i is a sink. The other case follows by duality. Since i is a sink, S_i is projective. Thus, a representation M belongs to rep-$\mathcal{S}\langle i\rangle$ if and only if $\mathrm{Hom}(M, S_i) = 0$. In particular, we see that in this case the subcategory rep-$\mathcal{S}\langle i\rangle$ is closed under factor modules.

Let M be a representation in rep-$\mathcal{S}\langle i\rangle$. Let M' be the maximal submodule of M without composition factor of the form S_i. The composition factors of M' belong to $\mathcal{N}$, thus it remains to exhibit a filtration of M/M' with factors in $\mathcal{N}$. By definition of M', the socle of M/M' is a direct sum of copies of S_i. It follows that the socle of M/M' is contained in the radical of M/M', since $\mathrm{Hom}(M, S_i) = 0$. We can assume that $M/M' \neq 0$. Let M'' be a submodule of M containing M' such that $N = M''/M'$ is local and of Loewy length 2. Then clearly N belongs to $\mathcal{N}$. Also, M/M'' again belongs to rep-$\mathcal{S}\langle i\rangle$, thus by induction M/M'' has a filtration with factors in $\mathcal{N}$. This completes the proof.

Of course, if N is an indecomposable representations with dimension vector $\mathbf{e}_j + t\mathbf{e}_i$ for some $j \neq i$, and some $t \geqq 0$, then $t \leqq -a_{ij}$, and, conversely such indecomposable representations do exist.

Consider again the case where i is a sink. The bimodule ${}_jM_i$ has, as a right F_i-vector space, dimension $-a_{ij}$ (since its k-dimension is $-\varepsilon_i a_{ij}$). Let M' be an F_i-subspace of ${}_jM_i$ of codimension t. We construct a representation of $\mathcal{S}$ by attaching F_j to j, and ${}_jM_i/M'$ to i, and we use the projection map $F_j \otimes {}_jM_i \to {}_jM_i/M'$for the arrow $j \to i$.

The case when i is a source follows by duality; of course, we also can write down an explicit recipe: attach again F_j to j, an F_i-subspace M'' of $\mathrm{Hom}_{F_j}({}_iM_j, F_j)$ to i, and use as map $M'' \otimes {}_iM_j \to F_j$ the evaluation map.

Corollary. *Let i be a sink or a source. The $\mathbb{Q}(v)$-algebra $\mathcal{H}\langle i\rangle$ is generated by the elements $\mathbf{e}_j + t\mathbf{e}_i$, where $j \neq i$, and $0 \leqq t \leqq -a_{ij}$.*

12. Reflection functors

Let i be a vertex of $\vec{\Delta}$. Let $\sigma_i\vec{\Delta}$ be obtained from $\vec{\Delta}$ by changing the orientation of all arrows which have i as starting point or end point.

Let $\mathcal{S}$ be a reduced k-species of type $\vec{\Delta}$. Let $\sigma_i\mathcal{S}$ be the k-species obtained from $\mathcal{S}$ by replacing ${}_rM_s$ by its k-dual, if $r = i$ or $s = i$; note that $\sigma_i\mathcal{S}$ is a reduced k-species of type $\sigma_i\vec{\Delta}$.

Let us assume that i is a sink for $\vec{\Delta}$. We denote by σ_i^+, the Bernstein-Gelfand-Ponomarev reflection functor, see [BGP], [DR2]: it is an equivalence

$$\sigma_i^+ : \text{rep-}\mathcal{S}\langle i\rangle \to \text{rep-}\sigma_i\mathcal{S}\langle i\rangle .$$

Theorem 5. *Let i be a sink. The functor σ_i^+ yields an $\mathscr{A}$-algebra isomorphism*

$$\sigma_i \colon \mathscr{H}_*(\vec{\Delta})\langle i\rangle \to \mathscr{H}_*(\sigma_i\vec{\Delta})\langle i\rangle$$

where

$$\sigma_i\langle M\rangle = \langle \sigma_i^+ M\rangle .$$

Proof. The subcategories rep-$\mathscr{S}\langle i\rangle$, rep-$\sigma_i\mathscr{S}\langle i\rangle$, are closed under extensions. Let N_1, N_2 be in rep-$\mathscr{S}\langle i\rangle$. If M is a module with a submodule M_1 isomorphic to N_2, such that M/M_1 is isomorphic to N_1, then M belongs to rep-$\mathscr{S}\langle i\rangle$, and

$$\dim_k M = \dim_k N_1 + \dim_k N_2 ,$$

and

$$\dim_k \sigma_i^+ M = \dim_k \sigma_i^+ N_1 + \dim_k \sigma_i^+ N_2 .$$

Also, we may calculate the Hall-polynomials, as well as the bilinear form between modules in rep-$\mathscr{S}\langle i\rangle$ inside this subcategory, and therefore

$$\phi^M_{N_1N_2} = \phi^{\sigma_i^+M}_{\sigma_i^+N_1\sigma_i^+N_2} \quad\text{and}\quad \langle \mathbf{dim}\, N_1, \mathbf{dim}\, N_2\rangle = \langle \mathbf{dim}\,\sigma_i^+ N_1, \mathbf{dim}\,\sigma_i^+ N_2\rangle .$$

Recall that

$$\langle N_1\rangle * \langle N_2\rangle = \sum v^{c(M)}\phi^M_{N_1N_2}\langle M\rangle ,$$

with $c(M) = \varepsilon(N_1) + \varepsilon(N_2) + \langle \mathbf{dim}\, N_1, \mathbf{dim}\, N_2\rangle - \varepsilon(M)$. Similarly, we have

$$\langle \sigma_i^+ N_1\rangle * \langle \sigma_i^+ N_2\rangle = \sum v^{c(M)}\phi^M_{\sigma_i^+N_1\sigma_i^+N_2}\langle M\rangle ,$$

with the same function c. It follows that

$$\begin{aligned}\sigma_i(\langle N_1\rangle * \langle N_2\rangle) &= \sigma_i\big(\sum v^{c(M)}\phi^M_{N_1N_2}\langle M\rangle\big)\\ &= \sum v^{c(M)}\phi^M_{N_1N_2}\langle \sigma_i^+ M\rangle\\ &= \sum v^{c(M)}\phi^{\sigma_i^+M}_{\sigma_i^+N_1\sigma_i^+N_2}\langle \sigma_i^+ M\rangle\\ &= \langle \sigma_i^+ N_1\rangle * \langle \sigma_i^+ N_2\rangle\\ &= \sigma_i\langle N_1\rangle * \sigma_i\langle N_2\rangle .\end{aligned}$$

This shows that σ_i is a ring homomorphism.

Example. Let i be a sink. Let $j \neq i$, and $0 \leqq t \leqq -a_{ij}$. Then

$$\sigma_i\langle \mathbf{e}_j + t\mathbf{e}_i\rangle = \langle \mathbf{e}_j + (-a_{ij} - t)\,\mathbf{e}_i\rangle .$$

Proof. Recall that we denote by $\bar{\sigma}_i$ the reflection in $\mathbb{Z}^n$ at $\mathbf{e}_i$ with respect to the symmetric bilinear form $(-,-)$, thus $\bar{\sigma}_i(\mathbf{e}_j) = \mathbf{e}_j - \dfrac{2(\mathbf{e}_i, \mathbf{e}_j)}{(\mathbf{e}_i, \mathbf{e}_j)}\,\mathbf{e}_i$ and $(\mathbf{e}_i, \mathbf{e}_j) = \varepsilon_i a_{ij}$, whereas $(\mathbf{e}_i, \mathbf{e}_i) = 2\varepsilon_i$. This shows that $\bar{\sigma}_i(\mathbf{e}_j) = \mathbf{e}_j - a_{ij}\mathbf{e}_i$, and that $\bar{\sigma}_i(\mathbf{e}_i) = -\mathbf{e}_i$, thus

$\bar{\sigma}_i(\mathbf{e}_j + t\mathbf{e}_i) = \mathbf{e}_j + (-a_{ij} - t)\mathbf{e}_i$. On the other hand, for M an indecomposable representation of $\mathcal{S}$ different from S_i, we have $\mathbf{dim}\,\sigma_i^+(M) = \bar{\sigma}_i\,\mathbf{dim}\,M$, see [DR2].

Lusztig has proposed several braid group operations on U. In particular, consider for any i the following operator $T''_{i,1}$ defined for the canonical generators E_i, F_i, K_μ of U by

$$\begin{aligned}
T''_{i,1}(E_i) &= -F_i K_i^{\varepsilon_i},\\
T''_{i,1}(F_i) &= -K_i^{-\varepsilon_i} E_i,\\
T''_{i,1}(E_j) &= \sum_{r+s=-a_{ij}} (-1)^r v^{-\varepsilon_i r} E_i^{(s)} E_j E_i^{(r)} \quad \text{for} \quad j \neq i,\\
T''_{i,1}(F_j) &= \sum_{r+s=-a_{ij}} (-1)^r v^{\varepsilon_i r} F_i^{(r)} F_j E_i^{(s)} \quad \text{for} \quad j \neq i,\\
T''_{i,1}(K_\mu) &= K_{\bar{\sigma}_i(\mu)},
\end{aligned}$$

this defines an automorphism $T''_{i,1}$ of U.

Theorem 6. *Let i be a sink. The homomorphism σ_i is the restriction of $T''_{i,1}$ to $\mathcal{H}_*(\vec{\Delta})\langle i\rangle$.*

We show that $T''_{i,1}$ and σ_i have the same effect on a generating set of $\mathcal{H}\langle i\rangle$. We have seen above that the elements $\langle \mathbf{e}_j + t\mathbf{e}_i\rangle$ with $j \neq i$ and $0 \leqq t \leqq a_{ij}$ form such a generating set, and that $\sigma_i\langle \mathbf{e}_j + t\mathbf{e}_i\rangle = \langle \mathbf{e}_j + (-a_{ij} - t)\mathbf{e}_i\rangle$.

On the other hand, Lusztig has shown in [L4], 37.2.5, that

$$T''_{i,1}(x_{i,j;1,t;-1}) = x'_{i,j;1,-a_{ij}-t;-1}$$

where

$$\begin{aligned}
x_{i,j;1,t;-1} &= \sum_{r+s=t} (-1)^r v^{-\varepsilon_i r(-a_{ij}-t+1)} E_i^{(r)} E_j E_i^{(s)},\\
x'_{i,j;1,t;-1} &= \sum_{r+s=t} (-1)^r v^{-\varepsilon_i r(-a_{ij}-t+1)} E_i^{(s)} E_j E_i^{(r)}.
\end{aligned}$$

By Proposition 3, we know that $x_{i,j;1,t;-1} = \langle \mathbf{e}_j + t\mathbf{e}_i\rangle_{\vec{\Delta}}$, since i is a sink for $\vec{\Delta}$, and that $x'_{i,j;1,t;-1} = \langle \mathbf{e}_j + t\mathbf{e}_i\rangle_{\sigma_i\vec{\Delta}}$, since i a source for $\sigma_i\vec{\Delta}$. Here, we have added to $\langle \mathbf{e}_j + t\mathbf{e}_i\rangle$ the indices $\vec{\Delta}$ and $\sigma_i\vec{\Delta}$, respectively, in order to point out the relevant orientation. This completes the proof.

13. Construction of the PBW-basis, using a braid group operation

We recall from Lusztig [L4] that the operators $T''_{i,1}$, where i runs through the vertices of the graph of Δ, define a braid group operation on U. The considerations above allow to see that our generating sequences for PBW-bases can be obtained from the generators $E_1, \ldots, E_n$ using this braid group operation (in the simply-laced cases, a similar result was pointed out by Lusztig in [L3]).

Recall that a sequence $i_m, \ldots, i_1$ is called a *sink sequence* for $\vec{\Delta}$, provided i_m is a sink for $\vec{\Delta}$, and for any $1 \leqq t < m$, the vertex i_t is a sink for the orientation $\sigma_{i_{t+1}} \cdots \sigma_{i_m}\vec{\Delta}$.

Consider a $\vec{\Delta}$-admissible ordering $\mathbf{a}_1, \ldots, \mathbf{a}_m$ of the positive roots. There exists a sequence $i_1, \ldots, i_m$ of vertices of $\vec{\Delta}$, with the following properties:

(1) The sequence $i_m, \ldots, i_1$ is a sink sequence for $\vec{\Delta}$.

(2) We have $\mathbf{a}_j = \bar{\sigma}_{i_1}\bar{\sigma}_{i_2}\cdots\bar{\sigma}_{i_{j-1}}(\mathbf{e}_{i_j})$.

(We may construct this sequence $i_1, \ldots, i_m$ as follows: Let C be the Coxeter transformation for $\vec{\Delta}$. For any j, there exist some power C^s such that $C^s(\mathbf{a}_j)$ is a positive root, but $C^{s+1}(\mathbf{a}_j)$ is not positive. Then there exists a unique vertex i_j such that $\langle C^s(\mathbf{a}_j), \mathbf{e}_{i_j}\rangle > 0$, this is the vertex we are interested in. In terms of representation theory: the representation $M(C^s(\mathbf{a}_j))$ is indecomposable projective, thus has a unique simple factor module, namely $M(\mathbf{e}_{i_j})$.)

We fix a $\vec{\Delta}$-admissible ordering $\mathbf{a}_1, \ldots, \mathbf{a}_m$ of the positive roots and we set $X_j = \langle \mathbf{a}_j \rangle$.

Theorem 7.

$$X_j = T''_{i_1,1} T''_{i_2,1} \cdots T''_{i_{j-1},1}(E_{i_j}).$$

Proof. Since we deal with a sink sequence, the operations $T''_{i,1}$ are given by σ_i, see Theorem 6, thus we can use the reflection functors σ_i^+, see Theorem 5. This shows that

$$T''_{i_1,1} T''_{i_2,1} \cdots T''_{i_{j-1},1}(E_{i_i}) = \langle \bar{\sigma}_{i_1}\bar{\sigma}_{i_2}\cdots\bar{\sigma}_{i_{j-1}}(S_{i_j})\rangle,$$

where S_{i_j} is the simple representation of $\bar{\sigma}_{i_{j-1}}\cdots\bar{\sigma}_{i_2}\bar{\sigma}_{i_1}\mathscr{S}$ corresponding to the vertex i_j. However,

$$\langle \bar{\sigma}_{i_1}\bar{\sigma}_{i_2}\cdots\bar{\sigma}_{i_{j-1}}(S_{i_j})\rangle = \langle \bar{\sigma}_{i_1}\bar{\sigma}_{i_2}\cdots\bar{\sigma}_{i_{j-1}}(\mathbf{e}_{i_j})\rangle = X_j.$$

This completes the proof.

Appendix 1. The rank 2 cases

For all the rank 2 cases, we are going to present the multiplication table for one of the generating sequences for a PBW-basis, explicitly.

Case $\mathbb{A}_2$. Let

$$S_1 = \langle \mathbf{e}_2 \rangle, \quad X_2 = \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle, \quad X_3 = \langle \mathbf{e}_1 \rangle.$$

Then

$$X_2 * X_1 = vX_1 * X_2,$$

$$X_3 * X_2 = vX_2 * X_3,$$

$$X_3 * X_1 = v^{-1}X_1 * X_3 + X_2.$$

Actually, for arbitrary $\mathbb{A}_n$, an explicit presentation of $\mathscr{H}$ by generators and relations, using as generating set the generating sequence for a PBW-basis, will be given in Appendix 2.

Case $\mathbb{B}_2$, with $\varepsilon_1 = 2$. The elements

$$X_1 = \langle \mathbf{e}_2 \rangle, \quad X_2 = \langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle, \quad X_3 = \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle, \quad X_4 = \langle \mathbf{e}_1 \rangle$$

satisfy the following relations:

$$\begin{aligned}
X_2 * X_1 &= v^2 X_1 * X_2, \\
X_3 * X_2 &= v^2 X_2 * X_3, \\
X_4 * X_3 &= v^2 X_3 * X_4, \\
X_3 * X_1 &= X_1 * X_3 + [2] X_2, \\
X_4 * X_2 &= X_2 * X_4 + (v^2 - 1) X_3^{(*2)}, \\
X_4 * X_1 &= v^{-2} X_1 * X_4 + X_3 .
\end{aligned}$$

Proof. The vanishing of a skew commutator for X_i, X_{i+1} follows from the general considerations concerning Auslander-Reiten quivers. We have seen above how to write X_2 and X_3 as skew commutators. This yields the fourth and the sixth equality. It remains to show the fifth equality.

We have $X_4 = \langle \mathbf{e}_1 \rangle = \mathbf{e}_1$, and $X_2 = \langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle = v^{-4+2}(\mathbf{e}_1 + 2\mathbf{e}_2)$, thus

$$X_4 * X_2 = v^{-2} \mathbf{e}_1 * (\mathbf{e}_1 + 2\mathbf{e}_2).$$

Note that $\langle \mathbf{e}_1, \mathbf{e}_1 + 2\mathbf{e}_2 \rangle = -2$. The Hall polynomial $\phi^{\mathbf{e}_1 \oplus (\mathbf{e}_1 + 2\mathbf{e}_2)}_{\mathbf{e}_1, \mathbf{e}_1 + 2\mathbf{e}_2}$ is given by q^2. On the other hand, there are $q^2 + 1$ images of non-zero maps $M(\mathbf{e}_1 + 2\mathbf{e}_2) \to M(2(\mathbf{e}_1 + \mathbf{e}_2))$. The number of images of the form $M(\mathbf{e}_1 + \mathbf{e}_2)$ is $q + 1$, the remaining ones are of the form $M(\mathbf{e}_1 + 2\mathbf{e}_2)$. This shows that the Hall polynomial $\phi^{2(\mathbf{e}_1 + \mathbf{e}_2)}_{\mathbf{e}_1, \mathbf{e}_1 + 2\mathbf{e}_2}$ is given by $q^2 - q = v^4 - v^2$. Thus, we see that

$$\begin{aligned}
X_4 * X_2 &= v^{-2} \mathbf{e}_1 * (\mathbf{e}_1 + 2\mathbf{e}_2) \\
&= v^{-4} v^4 \mathbf{e}_1 \oplus (\mathbf{e}_1 + 2\mathbf{e}_2) + v^{-4}(v^4 - v^2)(2(\mathbf{e}_1 + \mathbf{e}_2)) \\
&= X_2 * X_4 + v^{-4+2}(v^4 - v^2) X_3^{(*2)},
\end{aligned}$$

here, we use that $X_2 * X_4 = v^{-2}(\mathbf{e}_1 + 2\mathbf{e}_2) * \mathbf{e}_1 = \mathbf{e}_1 \oplus (\mathbf{e}_1 + 2\mathbf{e}_2)$ and that

$$X_3^{(*2)} = \langle 2(\mathbf{e}_1 + \mathbf{e}_2) \rangle = v^{-6+4}(2(\mathbf{e}_1 + \mathbf{e}_2)).$$

Case $\mathbb{G}_2$, with $\varepsilon_1 = 3$. We denote the elements as follows:

$$\begin{aligned}
&X_1 = \langle \mathbf{e}_2 \rangle, && X_2 = \langle \mathbf{e}_1 + 3\mathbf{e}_2 \rangle, && X_3 = \langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle, \\
&X_4 = \langle 2\mathbf{e}_1 + 3\mathbf{e}_2 \rangle, && X_5 = \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle, && X_6 = \langle \mathbf{e}_1 \rangle.
\end{aligned}$$

We have the following relations: Of course, as usual, we have

$$X_{i+1} * X_i = v^3 X_i * X_{i+1} \quad \text{for all} \quad 1 \leqq i \leqq 5.$$

In addition

$$
\begin{aligned}
X_3 * X_1 &= \quad vX_1 * X_3 + [3]X_2,\\
X_4 * X_2 &= v^3 X_2 * X_4 + (v^6 - v^4 - v^2 + 1)X_3^{(*3)},\\
X_5 * X_3 &= \quad vX_3 * X_5 + [3]X_4,\\
X_6 * X_4 &= v^3 X_4 * X_6 + (v^6 - v^4 - v^2 + 1)X_5^{(*3)},\\
X_4 * X_1 &= \qquad X_1 * X_4 + (v^3 - v^{-1})X_3^{(*2)},\\
X_5 * X_2 &= \qquad X_2 * X_5 + (v^3 - v^{-1})X_3^{(*2)},\\
X_6 * X_3 &= \qquad X_3 * X_6 + (v^3 - v^{-1})X_5^{(*2)},\\
X_5 * X_1 &= v^{-1}X_1 * X_5 + [2]X_3,\\
X_6 * X_2 &= v^{-3}X_2 * X_6 + (v^2 - 1)X_3 * X_5 + (v^3 - v - v^{-1})X_4,\\
X_6 * X_1 &= v^{-3}X_1 * X_6 + X_5.
\end{aligned}
$$

Consider $X_6 * X_3$. Note: $X_6 = \langle \mathbf{e}_1 \rangle = \mathbf{e}_1$. $X_3 = \langle \mathbf{e}_1 + 2\mathbf{e}_2 \rangle = v^{-5+1}(\mathbf{e}_1 + 2\mathbf{e}_2)$, and $\langle \mathbf{e}_1, \mathbf{e}_1 + 2\mathbf{e}_2 \rangle = -3$. The Hall polynomial $\phi^{\mathbf{e}_1 \oplus (\mathbf{e}_1 + 2\mathbf{e}_2)}_{\mathbf{e}_1, \mathbf{e}_1 + 2\mathbf{e}_2}$ is given by q^3. On the other hand, there are $q^3 + 1$ images of non-zero maps $M(\mathbf{e}_1 + 2\mathbf{e}_2) \to M(2(\mathbf{e}_1 + \mathbf{e}_2))$. The number of images of the form $M(\mathbf{e}_1 + \mathbf{e}_2)$ is $q + 1$, the remaining ones are of the form $M(\mathbf{e}_1 + 2\mathbf{e}_2)$. This shows that the Hall polynomial $\phi^{2(\mathbf{e}_1 + \mathbf{e}_2)}_{\mathbf{e}_1, \mathbf{e}_1 + 2\mathbf{e}_2}$ is given by $q^3 - q = v^6 - v^2$. Thus, we see that

$$
\begin{aligned}
X_6 * X_3 &= v^{-4}\mathbf{e}_1 * (\mathbf{e}_1 + 2\mathbf{e}_2)\\
&= v^{-7}v^6 \mathbf{e}_1 \oplus (\mathbf{e}_1 + 2\mathbf{e}_2) + v^{-7}(v^6 - v^2)\big(2(\mathbf{e}_1 + \mathbf{e}_2)\big)\\
&= X_3 * X_6 + (v^{-3} - v^{-1})X_5^{(*2)},
\end{aligned}
$$

here, we use that $X_3 * X_6 = v^{-4}(\mathbf{e}_1 + 2\mathbf{e}_2) * \mathbf{e}_1 = v^{-1}\mathbf{e}_1 \oplus (\mathbf{e}_1 + 2\mathbf{e}_2)$, and that $X_5^{(*2)} = \langle 2(\mathbf{e}_1 + \mathbf{e}_2) \rangle = v^{-8+4}(2(\mathbf{e}_1 + \mathbf{e}_2))$. This proves the assertion concerning $X_6 * X_3$. The shift by τ yields a similar formula for $X_4 * X_1$, by duality, we obtain the corresponding result for $X_5 * X_2$.

Consider $X_6 * X_2$. We have $X_6 = \langle \mathbf{e}_1 \rangle = \mathbf{e}_1$, $X_2 = \langle \mathbf{e}_1 + 3\mathbf{e}_2 \rangle = v^{-6+3}(\mathbf{e}_1 + 3\mathbf{e}_2)$, and $\langle \mathbf{e}_1, \mathbf{e}_1 + 3\mathbf{e}_2 \rangle = -6$. Therefore

$$
\begin{aligned}
X_6 * X_2 &= v^{-3}\mathbf{e}_1 * (\mathbf{e}_1 + 3\mathbf{e}_2)\\
&= v^{-9}\big(c_1 \mathbf{e}_1 \oplus (\mathbf{e}_1 + 3\mathbf{e}_2) + c_2(\mathbf{e}_1 + \mathbf{e}_2) \oplus (\mathbf{e}_1 + 2\mathbf{e}_2) + c_3(2\mathbf{e}_1 + 3\mathbf{e}_2)\big),
\end{aligned}
$$

where

$$
\begin{aligned}
c_1 &= \phi^{\mathbf{e}_1 \oplus (\mathbf{e}_1 + 3\mathbf{e}_2)}_{\mathbf{e}_1, \mathbf{e}_1 + 3\mathbf{e}_2},\\
c_2 &= \phi^{(\mathbf{e}_1 + \mathbf{e}_2) \oplus (\mathbf{e}_1 + 2\mathbf{e}_2)}_{\mathbf{e}_1, \mathbf{e}_1 + 3\mathbf{e}_2},\\
c_3 &= \phi^{2\mathbf{e}_1 + 3\mathbf{e}_2}_{\mathbf{e}_1, \mathbf{e}_1 + 3\mathbf{e}_2}
\end{aligned}
$$

are the various Hall polynomials. Clearly, $c_1 = \phi^{\mathbf{e}_1 \oplus (\mathbf{e}_1 + 3\mathbf{e}_2)}_{\mathbf{e}_1, \mathbf{e}_1 + 3\mathbf{e}_2} = q^3$. The last polynomial has been determined in [R4], it is $c_3 = \phi^{2\mathbf{e}_1 + 3\mathbf{e}_2}_{\mathbf{e}_1, \mathbf{e}_1 + 3\mathbf{e}_2} = q^3 - q^2 - q$.

Thus, it remains to calculate c_2. Consider maps $f\colon M(\mathbf{e}_1 + 3\mathbf{e}_2) \to M(\mathbf{e}_1 + 2\mathbf{e}_2)$ and $f'\colon M(\mathbf{e}_1 + 3\mathbf{e}_2) \to M(\mathbf{e}_1 + \mathbf{e}_2)$. The corresponding map

$$\begin{bmatrix} f \\ f' \end{bmatrix} : M(\mathbf{e}_1 + 3\mathbf{e}_2) \to M(\mathbf{e}_1 + 2\mathbf{e}_2) \oplus M(\mathbf{e}_1 + \mathbf{e}_2)$$

is injective if and only if $f \neq 0$ and f' cannot be factored through f. Since $\operatorname{End}(M(\mathbf{e}_1 + 3\mathbf{e}_2))$ operates transitively on $\operatorname{Hom}(M(\mathbf{e}_1 + 3\mathbf{e}_2), M(\mathbf{e}_1 + 2\mathbf{e}_2))$, we can fix some projection $\pi\colon M(\mathbf{e}_1 + 3\mathbf{e}_2) \to M(\mathbf{e}_1 + 2\mathbf{e}_2)$ and we obtain all images of injective maps

$$M(\mathbf{e}_1 + 3\mathbf{a}_2) \to M(\mathbf{e}_1 + 2\mathbf{e}_2) \oplus M(\mathbf{e}_1 + \mathbf{e}_2)$$

by using only the maps $\begin{bmatrix} \pi \\ f' \end{bmatrix}$, where f' does not factor through π. Now assume there is given another map $f''\colon M(\mathbf{e}_1 + 3\mathbf{e}_2) \to M(\mathbf{e}_1 + \mathbf{e}_2)$ such that $\begin{bmatrix} \pi \\ f' \end{bmatrix}$ and $\begin{bmatrix} \pi \\ f'' \end{bmatrix}$ have the same image. The projectivity of $M(\mathbf{e}_1 + 3\mathbf{e}_2)$ shows that there exists an automorphism ϱ of $M(\mathbf{e}_1 + 3\mathbf{e}_2)$ such that $\pi\varrho = \pi$ and $f'\varrho = f''$. The first equality implies that $\varrho = 1$, thus $f' = f''$. Also, since π is surjective, the multiplication by π yields a bijection between

$$\operatorname{Hom}(M(\mathbf{e}_1 + 2\mathbf{e}_2), M(\mathbf{e}_1 + \mathbf{e}_2))$$

and the set of those elements of

$$\operatorname{Hom}(M(\mathbf{e}_1 + 3\mathbf{e}_2), M(\mathbf{e}_1 + \mathbf{e}_2))$$

which factor through π. This shows that

$$c_2 = \phi^{(\mathbf{e}_1 + \mathbf{e}_2) \oplus (\mathbf{e}_1 + 2\mathbf{e}_2)}_{\mathbf{e}_1, \mathbf{e}_1 + 3\mathbf{e}_2} = q^{\varepsilon(\mathbf{e}_1 + 3\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2)} - q^{\varepsilon(\mathbf{e}_1 + 2\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2)} = q^3 - q^2.$$

Also, we note that

$$\begin{aligned} X_2 * X_6 &= \langle \mathbf{e}_1 \oplus (\mathbf{e}_1 + 3\mathbf{e}_2) \rangle = v^{-9+9} \mathbf{e}_1 \oplus (\mathbf{e}_1 + 3\mathbf{e}_2), \\ X_3 * X_5 &= \langle (\mathbf{e}_1 + \mathbf{e}_2) \oplus (\mathbf{e}_1 + 2\mathbf{e}_2) \rangle = v^{-9+4} (\mathbf{e}_1 + \mathbf{e}_2) \oplus (\mathbf{e}_1 + 2\mathbf{e}_2), \\ X_4 &= \langle 2\mathbf{e}_1 + 3\mathbf{e}_2 \rangle = v^{-9+3} (2\mathbf{e}_1 + 3\mathbf{e}_2). \end{aligned}$$

Therefore

$$\begin{aligned} X_6 * X_2 &= v^{-9} \big(v^6 \mathbf{e}_1 \oplus (\mathbf{e}_1 + 3\mathbf{e}_2) + (v^6 - v^4)(\mathbf{e}_1 + \mathbf{e}_2) \oplus (\mathbf{e}_1 + 2\mathbf{e}_2) \\ &\qquad + (v^6 - v^4 - v^2)(2\mathbf{e}_1 + 3\mathbf{e}_2) \big) \\ &= v^{-9} v^6 v^0 X_2 * X_6 + v^{-9} (v^6 - v^4) v^5 X_3 * X_5 + v^{-9} (v^6 - v^4 - v^2) v^6 X_4 . \end{aligned}$$

Consider $X_4 * X_2$. We have $X_4 = v^{-9+3}(2\mathbf{e}_1 + 3\mathbf{e}_2)$, $X_2 = v^{-6+3}(\mathbf{e}_1 + 3\mathbf{e}_2)$, and $\langle 2\mathbf{e}_1 + 3\mathbf{e}_2, \mathbf{e}_1 + 3\mathbf{e}_2 \rangle = -3$. Thus

$$X_4 * X_2 = v^{-6-3-3}(c_1((\mathbf{e}_1 + 3\mathbf{e}_2) \oplus (2\mathbf{e}_1 + 3\mathbf{e}_2)) + c_2(3(\mathbf{e}_1 + 2\mathbf{e}_2)))$$

with

$$c_1 = \phi^{(\mathbf{e}_1+3\mathbf{e}_2)\oplus(2\mathbf{e}_1+3\mathbf{e}_2)}_{2\mathbf{e}_1+3\mathbf{e}_2,\mathbf{e}_1+3\mathbf{e}_2},$$

$$c_2 = \phi^{3(\mathbf{e}_1+2\mathbf{e}_2)}_{2\mathbf{e}_1+3\mathbf{e}_2,\mathbf{e}_1+3\mathbf{e}_2}.$$

We calculate the Hall polynomials. We have $c_1 = \phi^{(\mathbf{e}_1+3\mathbf{e}_2)\oplus(2\mathbf{e}_1+3\mathbf{e}_2)}_{2\mathbf{e}_1+3\mathbf{e}_2,\mathbf{e}_1+3\mathbf{e}_2} = q^6 = v^{12}$.

On the other hand, given three maps f, f', f'': $M(\mathbf{e}_1 + 3\mathbf{e}_2) \to M(\mathbf{e}_1 + \mathbf{e}_2)$, the corresponding map

$$\begin{bmatrix} f \\ f' \\ f'' \end{bmatrix} : M(\mathbf{e}_1 + 3\mathbf{e}_2) \to 3M(\mathbf{e}_1 + \mathbf{e}_2)$$

is injective if and only if f, f', f'' are linearly independent over $k = \operatorname{End}(M(\mathbf{e}_1 + \mathbf{e}_2))$. This shows that the number of injective maps $M(\mathbf{e}_1 + 3\mathbf{e}_2) \to 3M(\mathbf{e}_1 + \mathbf{e}_2)$ is given by the polynomial $(q^3 - 1)(q^3 - q)(q^3 - q^2)$. Of course, different triples will yield the same image if and only if they are obtained from each other by the multiplication using an automorphism of $M(\mathbf{e}_1 + 3\mathbf{e}_2)$, and the number of such automorphisms is given by the polynomial $q^3 - 1$. Note that the cokernel of any injective map $M(\mathbf{e}_1 + 3\mathbf{e}_2) \to 3M(\mathbf{e}_1 + \mathbf{e}_2)$ is of the form $M(3(\mathbf{e}_1 + 2\mathbf{e}_2))$. Altogether we see that

$$c_2 = \phi^{3(\mathbf{e}_1+2\mathbf{e}_2)}_{2\mathbf{e}_1+3\mathbf{e}_2,\mathbf{e}_1+3\mathbf{e}_2} = (q^3 - q)(q^3 - q^2) = v^{12} - v^{10} - v^8 + v^6.$$

Thus:

$$\begin{aligned} X_4 * X_2 &= v^{-12}(v^{12}(\mathbf{e}_1 + 3\mathbf{e}_2) \oplus (2\mathbf{e}_1 + 3\mathbf{e}_2) + (v^{12} - v^{10} - v^8 + v^6)(3(\mathbf{e}_1 + 2\mathbf{e}_2))) \\ &= v^3 X_2 * X_4 + (v^6 - v^4 - v^2 + 1) X_3^{(*3)}, \end{aligned}$$

since

$$X_2 * X_4 = \langle (\mathbf{e}_1 + 3\mathbf{e}_2) \oplus (2\mathbf{e}_1 + 3\mathbf{e}_2) \rangle = v^{-15+12}(\mathbf{e}_1 + 3\mathbf{e}_2) \oplus (2\mathbf{e}_1 + 3\mathbf{e}_2)$$

and $X_3^{(*3)} = \langle 3(\mathbf{e}_1 + 2\mathbf{e}_2) \rangle = v^{-15+9}(3(\mathbf{e}_1 + 2\mathbf{e}_2))$. This completes the consideration of $X_4 * X_2$. The shift by τ^{-1} yields the corresponding result for $X_6 * X_2$.

Appendix 2. The case $\mathbb{A}_n$

A PBW-basis for this case has been exhibited by Yamane [Y1], [Y2]. We will show that his basis can be derived easily from the considerations above. For the convenience of the reader, we choose an analogous indexing, and label the cases in the same way as he did.

We consider the following orientation:

$$\underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \cdots \longrightarrow \underset{n-1}{\circ} \longrightarrow \underset{n}{\circ}.$$

Let $P(i)$ be the projective cover of S_i, for $1 \leqq i \leqq n$, and set $P(n+1) = 0$. Note that for $i < j$, there is an embedding $P(j) \subset P(i)$, and we denote by $M_{ij} = P(i)/P(j)$ the corresponding factor module. In this way, $S_i = M_{i,i+1}$, and M_{ij} is a serial module of length $j - i$. The Auslander-Reiten quiver has the following shape:

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & M_{1,n+1} & & & & & & & & \\
 & & & & & & & \nearrow & & \searrow & & & & & & & \\
 & & & & & & M_{2,n+1} & & & & M_{1,n} & & & & & & \\
 & & & & & \nearrow & & \searrow & & \nearrow & & \searrow & & & & & \\
 & & & & M_{3,n+1} & & & & M_{2,n} & & & & M_{1,n-1} & & & & \\
 & & & \cdot^{\cdot^{\cdot}} & & \ddots & & \cdot^{\cdot^{\cdot}} & & \ddots & & \cdot^{\cdot^{\cdot}} & & \ddots & & & \\
 & & & & & & & & & & & & & & M_{14} & & \\
 & & \cdot^{\cdot^{\cdot}} & & & & & & & & & \ddots & & \nearrow & & \searrow & \\
 & M_{n-1,n+1} & & & & & & & & & & & M_{24} & & & & M_{13} \\
\nearrow & & \searrow & & \cdot^{\cdot^{\cdot}} & & & & & & \ddots & & \nearrow & \searrow & & \nearrow & \searrow \\
M_{n,n+1} & & & M_{n-1,n} & & & & & & & & M_{34} & & & M_{23} & & M_{12}
\end{array}
$$

Some general features should be mentioned: If M_{rs}, M_{ij} are indecomposable modules, then $\dim \operatorname{Hom}(M_{rs}, M_{ij}) \leqq 1$, and $\dim \operatorname{Ext}^1(M_{rs}, M_{ij}) \leqq 1$. We write $\langle rs, ij \rangle = \langle \mathbf{dim}\, M_{rs}, \mathbf{dim}\, M_{ij} \rangle$, it follows that $-1 \leqq \langle rs, ij \rangle \leqq 1$, and for $(r, s) \neq (i, j)$, we have $-1 \leqq \langle rs, ij \rangle + \langle ij, rs \rangle \leqq 1$.

Given a pair of modules M_{ij}, M_{rs} with $i < r$ or with $i = r$, and $j < s$, there are six different cases to be considered.

Case (I): $i = r < j < s$. There exists an epimorphism $M_{rs} \to M_{ij}$. Thus

$$\langle rs, ij \rangle = 1, \quad \langle ij, rs \rangle = 0 .$$

Case (II): $i < r < s < j$. There are no homomorphisms and no extensions between M_{rs} and M_{ij}, thus

$$\langle rs, ij \rangle = 0, \quad \langle ij, rs \rangle = 0 .$$

Case (III): $i < r < j = s$. There exists an inclusion $M_{rs} \to M_{ij}$. Thus

$$\langle rs, ij \rangle = 1, \quad \langle ij, rs \rangle = 0 .$$

Case (IV): $i < r < j < s$. There exists an exact sequence

$$0 \to M_{rs} \to M_{is} \oplus M_{rj} \to M_{ij} \to 0,$$

in particular, we have a non-zero map $M_{rs} \to M_{rj} \to M_{ij}$. Thus

$$\langle rs, ij \rangle = 1, \quad \langle ij, rs \rangle = -1 .$$

Case (V): $i < j = r < s$. There exists an exact sequence

$$0 \to M_{rs} \to M_{is} \to M_{ij} \to 0,$$

and $\operatorname{Hom}(M_{rs}, M_{ij}) = 0$. Thus

$$\langle rs, ij\rangle = 0, \quad \langle ij, rs\rangle = -1.$$

Case (VI): $i<j<r<s$. There are no homomorphisms and no extensions between M_{rs} and M_{ij}, thus again

$$\langle rs, ij\rangle = 0, \quad \langle ij, rs\rangle = 0.$$

These considerations are sufficient for writing down the automorphism $\iota_{ij} = \iota_{\mathbf{dim}\, M_{ij}}$. The corresponding skew derivation $\delta_{ij} = \delta_{\langle M_{ij}\rangle}$ will be zero in the cases (I), (II), (III) and (VI), thus it remains to consider the cases (IV) and (V). Since in these cases $\dim \operatorname{Ext}^1(M_{ij}, M_{rs}) = 1$, we see that $\delta_{ij}(\langle M_{rs}\rangle)$ will be a multiple of an element of our basis.

Let $m(ij) = \dim M_{ij} = j - i$. Thus $\langle M_{ij}\rangle = v^{-m(ij)+1}[M_{ij}]$.

Case (IV). The exact sequence

$$0 \to M_{rs} \to M_{is} \oplus M_{rj} \to M_{ij} \to 0$$

shows that $\delta_{ij}(\langle M_{rs}\rangle)$ is a multiple of $\langle M_{is} \oplus M_{rj}\rangle$. We determine the corresponding coefficient. Obviously, the Hall polynomial is

$$\phi^{M_{is}\oplus M_{rj}}_{M_{ij}M_{rs}} = q - 1,$$

thus the coefficient of $[M_{is} \oplus M_{rj}]$ in $[M_{ij}] * [M_{rs}]$ is $v^{\langle ij,rs\rangle}(q-1) = v^{-1}(q-1)$. Let us note that there are no homomorphisms between M_{is} and M_{rj}, thus $\dim \operatorname{End}(M_{is} \oplus M_{rj}) = 2$; it follows that $\langle M_{is} \oplus M_{rj}\rangle = v^{-m(ij)-m(rs)+2}[M_{is} \oplus M_{rj}]$, and that $\langle M_{is} \oplus M_{rj}\rangle = \langle M_{is}\rangle * \langle M_{rj}\rangle$. On the other hand, we note that

$$\langle M_{ij}\rangle = v^{-m(ij)+1}[M_{ij}], \quad \langle M_{rs}\rangle = v^{-m(rs)+1}[M_{rs}].$$

Altogether, we see that

$$\delta_{ij}(\langle M_{rs}\rangle) = (v - v^{-1})\langle M_{is}\rangle * \langle M_{rj}\rangle.$$

Case (V). The exact sequence

$$0 \to M_{rs} \to M_{is} \to M_{ij} \to 0$$

shows that $\delta_{ij}(\langle M_{rs}\rangle)$ is a multiple of $\langle M_{is}\rangle$. Here, the Hall polynomial $\phi^{M_{is}}_{M_{ij}M_{rs}}$ is equal to 1, thus the coefficient of $[M_{is}]$ in $[M_{ij}] * [M_{rs}]$ is $v^{\langle ij,rs\rangle} = v^{-1}$. We have $\langle M_{ij}\rangle = v^{-m(ij)+1}$, $\langle M_{rs}\rangle = v^{-m(rs)+1}$ and $\langle M_{is}\rangle = v^{-m(ij)-m(rs)+1}$. Altogether, we see that

$$\delta_{ij}(\langle M_{rs}\rangle) = \langle M_{is}\rangle.$$

We use the notation $X_{ij} = \langle M_{ij}\rangle$. We have shown that

$$X_{ij} * X_{rs} = v \cdot X_{rs} * X_{ij}, \qquad \text{in case} \quad \text{(I), (III)},$$
$$X_{ij} * X_{rs} = X_{rs} * X_{ij}, \qquad \text{(II), (VI)},$$
$$X_{ij} * X_{rs} = X_{rs} * X_{ij} + (v - v^{-1}) X_{is} * X_{rj}, \qquad \text{(IV)},$$
$$X_{ij} * X_{rs} = v^{-1} X_{rs} * X_{ij} + X_{is}, \qquad \text{(V)}.$$

Condition (V) asserts that we may define X_{ij} inductively by

$$X_{is} = X_{ij} * X_{js} - v^{-1} X_{js} * X_{ij}, \quad \text{for} \quad i < j < s,$$

starting with $X_{i,i+1} = \mathbf{e}_i$. These elements in U^+ have been presented already by Jimbo in [J2] (with v replaced by v^{-1}). In order to rewrite these generators and relations in the form presented by Yamane [Y1], [Y2], we have to adjoin a square root t of v (this element t is denoted by q in Yamane; but of course, in our presentation, we have $q = v^2 = t^4$; there is a good reason to stick to the notation $q = v^2$, since in our approach, this q is usually evaluated at prime powers, namely at the cardinality of some finite field).

Let $\mathbb{Q}(t)$ be the rational function field in one variable t, let $v = t^2$, and consider $\mathscr{H}_t = \mathscr{H} \otimes_{\mathbb{Q}(v)} \mathbb{Q}(t)$. We define

$$E_{ij} = t^{j-i-1} X_{ij} (= t^{-j+i+1} [M_{ij}]).$$

Then we obtain the following relations:

$$E_{ij} * E_{rs} = t^2 \cdot E_{rs} * E_{ij}, \qquad \text{in case} \quad \text{(I), (III)},$$
$$E_{ij} * E_{rs} = E_{rs} * E_{ij}, \qquad \text{(II), (VI)},$$
$$E_{ij} * E_{rs} = E_{rs} * E_{ij} + (t^2 - t^{-2}) E_{is} * E_{rj}, \qquad \text{(IV)},$$
$$t^2 E_{ij} * E_{rs} = E_{rs} * E_{ij} + t \cdot E_{is}, \qquad \text{(V)}.$$

Proof. The extra factors t^{j-i-1}, t^{s-r-1} cancel in all but the last equality. The last equality can be written as follows: $E_{ij} * E_{rs} = v^{-1} \cdot E_{ji} * E_{rs} + t^{-1} E_{is}$, and this relation is directly derived from the relations before.

Our identification of U^+ and $\mathscr{H}$ yields an identification of $U_t^+ = U^+ \otimes_{\mathbb{Q}(v)} \mathbb{Q}(t)$ and $\mathscr{H}_t$. We have $E_{i,i+1} = E_i$, and the last relation above shows that for $j - i > 1$, we have $E_{ij} = t E_{i,j-1} * E_{j-1,j} - t^{-1} E_{j-1,j} E_{i,j-1}$. This shows that the elements E_{ij} of U_t^+ coincide with the elements e_{ij} as introduced by Yamane.

References

[BGP] *I.N. Bernstein, I.M. Gelfand, V.A. Ponomarev*, Coxeter functors and Gabriel's theorem, Uspechi Mat. Nauk **28** (1973); Russian Math. Surv. **28** (1973), 17–32.

[CB] *W.W. Crawley-Boevey*, Exceptional sequences of representations of quivers, Proc. ICRA VI, Carleton-Ottawa Math. LNS **14** (1992).

[DR1] *V. Dlab, C. M. Ringel*, On algebras of finite representation type, J. Algebra **33** (1975), 306–394.

[DR2] *V. Dlab, C. M. Ringel*, Indecomposable representations of graphs and algebras, Mem. Amer. Math. Soc. **173** (1976).

[D] *V. G. Drinfeld*, Hopf algebras and the Yang-Baxter equation, Soviet. Math. Dokl. **32** (1985), 254–258.

[Ga] *P. Gabriel*, Unzerlegbare Darstellungen I, Manuscr. Math. **6** (1972), 71–103.

[Go] *K. R. Goodearl*, Uniform ranks of prime factors of skew polynomial rings, Ring Theory, Proc. Ohio State – Denison Conference 1992, to appear.

[GL] *K. R. Goodearl*, and *E. S. Letzter*, Prime factor algebras of the coordinate ring of quantum matrices, Proc. Amer. Math. Soc., to appear.

[Gr] *J. A. Green*, Hall algebras, hereditary algebras and quantum groups, Invent. math., to appear.

[H] *J. E. Humphreys*, Introduction to Lie Algebras and Representation Theory, Springer Grad. Texts Math. **9** (1972).

[J1] *M. Jimbo*, A q-analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation, Letters Math. Phys. **10** (1985), 63–69.

[J2] *M. Jimbo*, A q-analogue of $U(gl(N+1))$, Hecke algebra, and the Yang-Baxter equation, Letters Math. Phys. **11** (1986), 247–252.

[KT1] *S. M. Khoroshkin, V. N. Tolstoy*, Universal R-matrix for quantized (super)algebras, Comm. Math. Phys. **141** (1991), 599–617.

[KT2] *S. M. Khoroshkin, V. N. Tolstoy*, The Cartan-Weyl basis and the universal R-matrix for quantum Kac-Moody algebras and superalgebras, Proc. Second Wigner Symp. Goslar 1991, to appear.

[KT3] *S. M. Khoroshkin, V. N. Tolstoy*, Twisting of quantum (super)algebras, connection of Drinfeld's and Cartan-Weyl realizations for quantum affine algebras, Max-Planck-Institut Bonn, Preprint MPI/94-23.

[LS] *S. Levendorskii, Y. Soibelman*, Algebras of functions on compact quantum groups, Schubert cells and quantum tori, Comm. Math. Phys. **139** (1991), 141–170.

[L1] *G. Lusztig*, Finite dimensional Hopf algebras arising from quantum groups, J. Amer. Math. Soc. **3** (1990), 257–296.

[L2] *G. Lusztig*, Quantum groups at roots of 1, Geom. Dedicata (1990).

[L3] *G. Lusztig*, Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. **3** (1990), 447–498.

[L4] *G. Lusztig*, Introduction to quantum groups, Progr. Math. **110**, Birkhäuser, 1993.

[R1] *C. M. Ringel*, Representations of K-species and bimodules, J. Algebra **41** (1976), 269–302.

[R2] *C. M. Ringel*, Tame algebras and integral quadratic forms, Springer Lect. Notes Math. **1099** (1984).

[R3] *C. M. Ringel*, Hall algebras, In: Topics in Algebra, Banach Center Publ. **26** (1990), 433–447.

[R4] *C. M. Ringel*, Hall polynomials for the representation-finite hereditary algebras, Adv. Math. **84** (1990), 137–178.

[R5] *C. M. Ringel*, Hall algebras and quantum groups, Invent. math. **101** (1990), 583–592.

[R6] *C. M. Ringel*, From representations of quivers via Hall and Loewy algebras to quantum groups, In: Proc. Int. Conf. Algebra, Novosibirsk 1989, Contemp. Math. **131.2** (1992), 381–401.

[R7] *C. M. Ringel*, Hall algebras revisted, Israel Math. Conf. Proc. **7** (1993), 171–176.

[R8] *C. M. Ringel*, The braid group action on the set of exceptional sequences, Proc. Oberwolfach Conf. Abelian groups and modules 1993, Contemp. Math., to appear.

[Ro] *M. Rosso*, Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra, Comm. Math. Phys. **117** (1988), 581–593.

[S] *A. Schofield*, Semi-invariants of quivers, J. London Math. Soc. **43** (1991), 383–395.

[T] *M. Takeuchi*, The q-bracket product and quantum enveloping algebras of classical type, J. Math. Soc. Japan **42** (1990), 605–628.

[X] *N. Xi*, Root vectors in quantum groups, Preprint.

[Y1] *H. Yamane*, A Poincaré-Birkhoff-Witt theorem for the quantum group of type A_N, Proc. Japan Acad. **64** (A) (1988), 385–386.

[Y2] *H. Yamane*, A Poincaré-Birkhoff-Witt theorem for quantized universal enveloping algebras of type A_N, Publ. RIMS, Kyoto Univ. **25** (1989), 503–520.

Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld
e-mail: ringel@mathematik.uni-bielefeld.de

Eingegangen 1. Oktober 1993, in revidierter Fassung 24. Februar 1995