Morphisms determined by objects:

The case of modules over artin algebras.

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Abstract. Let Λ be an artin algebra. In his Philadelphia Notes, M. Auslander showed that any homomorphism between Λ-modules is right determined by a Λ-module C, but a formula for C which he wrote down has to be modified.

Let Λ be an artin algebra, the modules which we consider are usually finitely generated left Λ-modules. A morphism $\alpha\colon X \to Y$ of Λ-modules is said to be *right determined* by a Λ-module C provided the following condition is satisfied: given any morphism $\alpha'\colon X' \to Y$ such that $\alpha'\phi$ factors through α for any $\phi\colon C \to X'$, then α' itself factors through α. This definition is due to Auslander; the papers [A1] and [A2] are devoted to this concept. One of the main assertions of Auslander claims that any morphism $\alpha\colon X \to Y$ is right determined by $C = \operatorname{Tr} D(K) \oplus P(Q)$, see [A2], Theorem 2.6; here K is the kernel, Q the cokernel of α, and $\operatorname{Tr}(M)$ denotes the transpose, $D(M)$ the dual and $P(M)$ the projective cover of a module M.

The aim of this note is to show that this assertion is not correct as stated (in contrast to the weaker statement Theorem 3.17 (b) of [A1]). In section 1, we will present corresponding examples. The assertion has to be slightly modified: not the projective cover of Q is relevant, but the projective cover of the **socle** $\operatorname{soc} Q$ of Q.

Theorem. *Let $\alpha\colon X \to Y$ be a morphism. Let K be the kernel of α and Q the cokernel of α. Then α is right determined by* $\operatorname{Tr} D(K) \oplus P(\operatorname{soc} Q)$.

The modification of Auslander's treatment is formulated in Lemma 1 below (this should replace [A2] Lemma 2.1.b). Since Auslander's proof is somewhat hidden in two rather long papers, we feel that it may be appreciated if we provide a complete (and still short) direct proof of the Theorem. This will be done in section 2.

In the final section 3, we discuss the following problem: Given a homomorphism α, what is the structure of the modules C which right determine α and such that C belongs to $\operatorname{add} C'$ for any module C' which right determines α?

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1. Examples.

Consider the quiver of type $\mathbb{A}_3$ with linear orientation, say with simple modules indexed by $1, 2, 3$, such that $S(1)$ is projective, $S(3)$ is injective. Let $\alpha\colon S(1) \to P(3)$ be the inclusion map, thus the kernel is zero, and the projective cover of the cokernel is again $P(3)$. We claim that α is not right determined by $C = P(3)$. Consider the inclusion map $\alpha'\colon P(2) \to P(3)$. Obviously, α' cannot be factored through α. However, we have $\operatorname{Hom}(C, P(2)) = 0$,

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and the only map $\phi\colon C \to P(2)$ (the zero-map) has as composition with α' the zero-map $C \to P(3)$. But the zero-map $C \to P(3)$ factors through α, trivially.

Actually, an even easier example is given by the quiver $\mathbb{A}_2$, but here we deal with α being a zero map (some may consider this as a degenerate case, thus we presented first another example). Denote the two simple modules by $S(1)$ and $S(2)$, with $S(1)$ being projective, $S(2)$ being injective. We take as α the zero-map $0 \to P(2)$, its cokernel is $P(2)$ and already projective. But α is not right determined by $C = P(2)$, since the inclusion map $\alpha'\colon S(1) \to P(2)$ does not factor through α (after all, α is zero), whereas for any map $\phi\colon C \to S(1)$ (there is only the zero map) the composition $\alpha'\phi$ factors through α.

Remark. Let us stress that Auslander's claim is correct in case Λ is commutative, or, more generally, in case all the arrows of the quiver of Λ are loops. Namely, in this case (and only in this case) $\operatorname{add} P(M) = \operatorname{add} P(\operatorname{soc} M)$ for any Λ-module M.

2. The proof.

We start with the necessary amendment to Auslander's treatment.

Lemma 1. *Let $\alpha\colon X \to Y$ and $\alpha'\colon X' \to Y$ be morphisms. Let Q' be the direct sum of all the simple modules which occur in the socle of the cokernel of α. Assume that for any map $\phi\colon P(Q') \to X'$, the composition $\alpha'\phi$ factors through α. Then the image of α' is contained in the image of α.*

Proof. We assume that the image of α' is not contained in the image of α and want to derive a contradiction. Let us denote by $\gamma\colon Y \to Q$ the cokernel map for α. By assumption, $\gamma\alpha' \neq 0$. Let U be the image of $\gamma\alpha'$, with epimorphism $\epsilon\colon X' \to U$ and monomorphism $\mu\colon U \to Q$, thus $\mu\epsilon = \gamma\alpha'$. Since U is a non-zero submodule of Q, we may consider a simple submodule S of U, this is a simple module which occurs in the socle of Q. Thus there is a non-zero map $\overline{\phi}\colon P(Q') \to U$ (with image S). Since $P(Q')$ is projective and ϵ is an epimorphism, we can lift $\overline{\phi}$ and obtain a map $\phi\colon P(Q') \to X'$ with $\epsilon\phi = \overline{\phi}$. Thus, we are in the situation mentioned in the statement of the Lemma: there is given a map $\phi\colon P(Q') \to X'$, and by the assumption of the Lemma, we know that the composition $\alpha'\phi$ factors through α, say $\alpha'\phi = \alpha\phi'$ for some $\phi'\colon P(Q') \to X$. Therefore

$$\mu\overline{\phi} = \mu\epsilon\phi = \gamma\alpha'\phi = \gamma\alpha\phi' = 0,$$

since γ is the cokernel of α. But μ is a monomorphism, therefore $\overline{\phi} = 0$, a contradiction.

Let us continue, as promised, with the complete proof of the Theorem. The only prerequisite which we will use is the existence of almost split sequences. To be precise: we will need for any indecomposable non-injective module M a non-split short exact sequence

$$0 \to M \xrightarrow{\sigma} N \xrightarrow{\rho} \operatorname{Tr} D(M) \to 0,$$

such that for any map $\zeta\colon M \to N'$ which is not a split monomorphism, there is $\zeta'\colon N \to N'$ with $\zeta = \zeta'\sigma$.

Lemma 2. *Let $\alpha\colon X \to Y$ be a morphism with kernel K and image I. Let $\alpha'\colon X' \to Y$ be a morphism with image contained in I. Assume that for* *any map $\phi\colon \operatorname{Tr} D(K) \to X'$, the composition $\alpha'\phi$ factors through α. Then α' factors through α.*

Proof. We may assume that $Y = I$, thus there is given the exact sequence η with epimorphism $\alpha\colon X \to Y$ and kernel $\mu\colon K \to X$. We form the induced exact sequence η' with respect to α', thus there is the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K & \xrightarrow{\mu} & X & \xrightarrow{\alpha} & Y & \longrightarrow & 0 & \qquad & \eta \\
 & & \| & & \uparrow{\scriptstyle \beta'} & & \uparrow{\scriptstyle \alpha'} & & & & \\
0 & \longrightarrow & K & \xrightarrow{\nu} & W & \xrightarrow{\beta} & X' & \longrightarrow & 0 & \qquad & \eta'
\end{array}$$

If η' is a split exact sequence, then α' factors through α.

Let us assume that α' does not factor through α, in order to derive a contradiction, again. Thus η' is not a split exact sequence. Write $K = \bigoplus K_i$ with indecomposable modules K_i and projection maps $\pi_i\colon K \to K_i$. Since η' does not split, there is some index i such that the exact sequences induced from η' by the map π_i does not split. We have the following commutative diagram with exact rows which do not split:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K & \xrightarrow{\nu} & W & \xrightarrow{\beta} & X' & \longrightarrow & 0 & \qquad & \eta' \\
 & & {\scriptstyle \pi_i}\downarrow & & {\scriptstyle \pi_i'}\downarrow & & \| & & & & \\
0 & \longrightarrow & K_i & \xrightarrow{\nu_i} & W_i & \xrightarrow{\beta_i} & X' & \longrightarrow & 0 & \qquad & \eta_i'
\end{array}$$

Since $\nu_i\colon K_i \to W_i$ is a monomorphism which does not split, we see that K_i cannot be injective, thus there is an almost split sequence

$$0 \to K_i \xrightarrow{\sigma_i} V_i \xrightarrow{\rho_i} \operatorname{Tr} D(K_i) \to 0,$$

and ν_i can be factored as $\nu_i = \nu_i'\sigma_i$ for some $\nu_i'\colon V_i \to W_i$. Thus we obtain the following commutative square on the left, and therefore also the map $\phi : \operatorname{Tr} D(K_i) \to X'$ with a commutative square on the right:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K_i & \xrightarrow{\nu_i} & W_i & \xrightarrow{\beta_i} & X' & \longrightarrow & 0 & \qquad & \eta_i' \\
 & & \| & & \uparrow{\scriptstyle \nu_i'} & & \uparrow{\scriptstyle \phi} & & & & \\
0 & \longrightarrow & K_i & \xrightarrow{\sigma_i} & V_i & \xrightarrow{\rho_i} & \operatorname{Tr} D(K_i) & \longrightarrow & 0 & \qquad & \omega_i
\end{array}$$

By assumption, the map $\phi\alpha'\colon \operatorname{Tr} D(K_i) \to Y$ factors through α, that means there is $\phi'\colon \operatorname{Tr} D(K_i) \to X$ with $\alpha\phi' = \alpha'\phi$. Now, W is the pullback of α, α', thus there is a map $\phi''\colon \operatorname{Tr} D(K_i) \to W$ such that $\beta\phi'' = \phi$ and $\beta'\phi'' = \phi'$. It follows that

$$\phi = \beta\phi'' = \beta_i\pi_i'\phi''.$$

But if ϕ factors through β_i, then the exact sequence ω_i induced from η_i' by ϕ has to split. This is a contradiction, since ω_i is an Auslander-Reiten sequence, thus non-split.

Proof of the Theorem. Let $\alpha\colon X \to Y$ be a morphism with kernel K and cokernel Q and let $C = \operatorname{Tr} D(K) \oplus P(\operatorname{soc} Q)$. Let $\alpha'\colon X' \to Y$ be a morphism such that $\alpha'\phi$ factors through α for any map $\phi\colon C \to X'$. If Q' is the direct sum of all the simple modules which occur in $\operatorname{soc} Q$, then $P(Q')$ is a direct summand of C. Thus, for any map $\phi\colon P(Q') \to X'$, the composition

$\alpha'\phi$ factors through α. Lemma 1 asserts that the image of α' is contained in the image of α. Now we use that $\operatorname{Tr} D(K)$ is a direct summand of C, thus for any map $\phi\colon \operatorname{Tr} D(K) \to X'$, the composition $\alpha'\phi$ factors through α. Thus we can apply Lemma 2 in order to see that α' factors through α. This shows that α is right determined by C.

Example. Let us add an example which may be illuminating, albeit it is extremely special. Let Λ be the path algebra of a finite directed quiver. Let b be a vertex of the quiver and assume that there are s arrows starting in b, say $b \to a_i$ with $1 \le i \le s$, and that there are t arrows ending in b, say $c_j \to b$ with $1 \le j \le t$. For any vertex x, we denote by $S(x)$ the simple module with support x, by $P(x)$ the projective cover of $S(x)$, by $I(x)$ the injective envelope of $S(x)$.

Let α be a non-zero map $X = P(b) \to I(b) = Y$, this is the homomorphism which we want to look at. Note that the image of α is $S(x)$. The kernel pf α is the radical of $P(b)$, thus the direct sum of the modules $P(a_i)$ with $1 \le i \le s$. The cokernel of α is the factor module of $I(b)$ modulo its socle, thus it is the direct sum of the modules $I(c_j)$ with $1 \le j \le t$. The projective cover of the socle of $I(c_j)$ is $P(c_j)$. Atogether we see: the theorem asserts that α is right determined by the module

$$C = \bigoplus_{i=1}^{s} \operatorname{Tr} D(P(a_i)) \oplus \bigoplus_{j=1}^{t} P(c_j).$$

But this module C is precisely the middle term of the almost split sequence starting in $P(b)$.

This should not come as a surprise. Namely, let X' be an indecomposable module and assume that there is a non-zero map $\alpha'\colon X' \to Y = I(b)$. Then there is a map $\beta' : P(b) \to X'$ with composition $\alpha'\beta' = \alpha$. Now either β' is invertible so that α' factors through α, or else β' is not invertible and α' does not factor through α. In the latter case, β' factors through the minimal left almost split map $\gamma\colon P(b) \to C$ starting in $P(b)$, this means that there is some $\phi\colon C \to X'$ with $\beta' = \phi\gamma$. But if we look at the composition of ϕ and α, then one should be aware that no non-zero map $C \to I(b)$ factors through α.

3. Minimal right determinators.

Given a morphism $\alpha\colon X \to Y$, we may try to split off non-zero direct summands of X which lie in the kernel of Y. If this is not possible, then α is said to be *right minimal.* In general, we may write $X = X_0 \oplus X_1$ with X_0 contained in the kernel of α and such that $\alpha|X_1$ is right minimal; then we call the kernel of $\alpha|X_1$ the *essential kernel* of α (note that it is unique up to isomorphism). Looking at the proof of Lemma 2, one immediately realizes that it is sufficient to take $\operatorname{Tr} D(K')$ instead of $\operatorname{Tr} D(K)$, with K' the essential kernel of α. Thus the Theorem we discuss can be strengthened as follows: *Any morphism $X \to Y$ is right determined by* $\operatorname{Tr} D(K') \oplus P(\operatorname{soc} Q)$*, where* K' is the essential kernel and Q the cokernel of α.

In case $\alpha\colon X \to Y$ is a monomorphism, and C right determines α, then it is easy to see that $P(\operatorname{soc} Y)$ has to belong to $\operatorname{add} C$. But in general, given a morphism α with cokernel Q, and a module C which right determines α, then the module $P(\operatorname{soc} Q)$ does not have to belong to $\operatorname{add} C$, as the following example shows:

Example. Consider the linearly ordered quiver of type $\mathbb{A}_5$ with simple modules $S(1), \dots, S(5)$ such that $\operatorname{Ext}^1(S(i{+}1), S(i)) \neq 0$ for $1 \le i \le 4$. Endow it with one zero relation so that the indecomposable projective module $P(4)$ has length 3. Let $X = P(3)$, let $Y = P(5)/S(2)$ and let $\alpha\colon X \to Y$ be a non-zero map (thus the image of α is $S(3)$). Then $K = P(2)$ is the kernel and the essential kernel of α, and $\operatorname{Tr} D(K) = P(3)/P(1)$. Claim: *$\alpha$ is right determined by $\operatorname{Tr} D(K)$,* whereas $P(4) = P(\operatorname{soc} Q)$ does not belong to $\operatorname{add} \operatorname{Tr} D(K)$ (here, Q is the cokernel of α).

Namely, for every indecomposable module X' which has a map $\alpha'\colon X' \to Y$ not factoring through α, we have to find some map $\phi\colon \operatorname{Tr} D(K) \to X'$ such that also $\alpha'\phi$ does not factor through α. There are six such modules X', namely

$$P(3)/P(1),\ P(4),\ P(5),\ S(3),\ P(4)/S(2), P(5)/P(2).$$

For the first three modules X', take the embedding $\operatorname{Tr} D(K) \to X'$, for the remaining three modules X', there is a non-zero map $\operatorname{Tr} D(K) \to X'$ with image $S(3)$. In all cases, the composition $\alpha'\phi\colon \operatorname{Tr} D(K) \to Y$ has image $S(3)$, in particular $\alpha'\phi \neq 0$. Since $\operatorname{Hom}(\operatorname{Tr} D(K), X) = 0$, we see that $\alpha'\phi$ cannot be factored through α.

3. References.

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