

**Morphisms determined by objects:
The case of modules over artin algebras.**

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Abstract. Let Λ be an artin algebra. In his Philadelphia Notes, M. Auslander showed that any homomorphism between Λ -modules is right determined by a Λ -module C , but a formula for C which he wrote down has to be modified.

Let Λ be an artin algebra, the modules which we consider are usually finitely generated left Λ -modules. A morphism $\alpha: X \rightarrow Y$ of Λ -modules is said to be *right determined* by a Λ -module C provided the following condition is satisfied: given any morphism $\alpha': X' \rightarrow Y$ such that $\alpha'\phi$ factors through α for any $\phi: C \rightarrow X'$, then α' itself factors through α . This definition is due to Auslander; the papers [A1] and [A2] are devoted to this concept. One of the main assertions of Auslander claims that any morphism $\alpha: X \rightarrow Y$ is right determined by $C = \text{Tr } D(K) \oplus P(Q)$, see [A2], Theorem 2.6; here K is the kernel, Q the cokernel of α , and $\text{Tr}(M)$ denotes the transpose, $D(M)$ the dual and $P(M)$ the projective cover of a module M .

The aim of this note is to show that this assertion is not correct as stated (in contrast to the weaker statement Theorem 3.17 (b) of [A1]). In section 1, we will present corresponding examples. The assertion has to be slightly modified: not the projective cover of Q is relevant, but the projective cover of the **socle** $\text{soc } Q$ of Q .

Theorem. *Let $\alpha: X \rightarrow Y$ be a morphism. Let K be the kernel of α and Q the cokernel of α . Then α is right determined by $\text{Tr } D(K) \oplus P(\text{soc } Q)$.*

The modification of Auslander's treatment is formulated in Lemma 1 below (this should replace [A2] Lemma 2.1.b). Since Auslander's proof is somewhat hidden in two rather long papers, we feel that it may be appreciated if we provide a complete (and still short) direct proof of the Theorem. This will be done in section 2.

In the final section 3, we discuss the following problem: Given a homomorphism α , what is the structure of the modules C which right determine α and such that C belongs to $\text{add } C'$ for any module C' which right determines α ?

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1. Examples.

Consider the quiver of type \mathbb{A}_3 with linear orientation, say with simple modules indexed by 1, 2, 3, such that $S(1)$ is projective, $S(3)$ is injective. Let $\alpha: S(1) \rightarrow P(3)$ be the inclusion map, thus the kernel is zero, and the projective cover of the cokernel is again $P(3)$. We claim that α is not right determined by $C = P(3)$. Consider the inclusion map $\alpha': P(2) \rightarrow P(3)$. Obviously, α' cannot be factored through α . However, we have $\text{Hom}(C, P(2)) = 0$,

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and the only map $\phi: C \rightarrow P(2)$ (the zero-map) has as composition with α' the zero-map $C \rightarrow P(3)$. But the zero-map $C \rightarrow P(3)$ factors through α , trivially.

Actually, an even easier example is given by the quiver \mathbb{A}_2 , but here we deal with α being a zero map (some may consider this as a degenerate case, thus we presented first another example). Denote the two simple modules by $S(1)$ and $S(2)$, with $S(1)$ being projective, $S(2)$ being injective. We take as α the zero-map $0 \rightarrow P(2)$, its cokernel is $P(2)$ and already projective. But α is not right determined by $C = P(2)$, since the inclusion map $\alpha': S(1) \rightarrow P(2)$ does not factor through α (after all, α is zero), whereas for any map $\phi: C \rightarrow S(1)$ (there is only the zero map) the composition $\alpha'\phi$ factors through α .

Remark. Let us stress that Auslander's claim is correct in case Λ is commutative, or, more generally, in case all the arrows of the quiver of Λ are loops. Namely, in this case (and only in this case) $\text{add } P(M) = \text{add } P(\text{soc } M)$ for any Λ -module M .

2. The proof.

We start with the necessary amendment to Auslander's treatment.

Lemma 1. *Let $\alpha: X \rightarrow Y$ and $\alpha': X' \rightarrow Y$ be morphisms. Let Q' be the direct sum of all the simple modules which occur in the socle of the cokernel of α . Assume that for any map $\phi: P(Q') \rightarrow X'$, the composition $\alpha'\phi$ factors through α . Then the image of α' is contained in the image of α .*

Proof. We assume that the image of α' is not contained in the image of α and want to derive a contradiction. Let us denote by $\gamma: Y \rightarrow Q$ the cokernel map for α . By assumption, $\gamma\alpha' \neq 0$. Let U be the image of $\gamma\alpha'$, with epimorphism $\epsilon: X' \rightarrow U$ and monomorphism $\mu: U \rightarrow Q$, thus $\mu\epsilon = \gamma\alpha'$. Since U is a non-zero submodule of Q , we may consider a simple submodule S of U , this is a simple module which occurs in the socle of Q . Thus there is a non-zero map $\bar{\phi}: P(Q') \rightarrow U$ (with image S). Since $P(Q')$ is projective and ϵ is an epimorphism, we can lift $\bar{\phi}$ and obtain a map $\phi: P(Q') \rightarrow X'$ with $\epsilon\phi = \bar{\phi}$. Thus, we are in the situation mentioned in the statement of the Lemma: there is given a map $\phi: P(Q') \rightarrow X'$, and by the assumption of the Lemma, we know that the composition $\alpha'\phi$ factors through α , say $\alpha'\phi = \alpha\phi'$ for some $\phi': P(Q') \rightarrow X$. Therefore

$$\mu\bar{\phi} = \mu\epsilon\phi = \gamma\alpha'\phi = \gamma\alpha\phi' = 0,$$

since γ is the cokernel of α . But μ is a monomorphism, therefore $\bar{\phi} = 0$, a contradiction.

Let us continue, as promised, with the complete proof of the Theorem. The only prerequisite which we will use is the existence of almost split sequences. To be precise: we will need for any indecomposable non-injective module M a non-split short exact sequence

$$0 \rightarrow M \xrightarrow{\sigma} N \xrightarrow{\rho} \text{Tr } D(M) \rightarrow 0,$$

such that for any map $\zeta: M \rightarrow N'$ which is not a split monomorphism, there is $\zeta': N \rightarrow N'$ with $\zeta = \zeta'\sigma$.

Lemma 2. *Let $\alpha: X \rightarrow Y$ be a morphism with kernel K and image I . Let $\alpha': X' \rightarrow Y$ be a morphism with image contained in I . Assume that for*

any map $\phi: \text{Tr } D(K) \rightarrow X'$, the composition $\alpha'\phi$ factors through α . Then α' factors through α .

Proof. We may assume that $Y = I$, thus there is given the exact sequence η with epimorphism $\alpha: X \rightarrow Y$ and kernel $\mu: K \rightarrow X$. We form the induced exact sequence η' with respect to α' , thus there is the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\mu} & X & \xrightarrow{\alpha} & Y & \longrightarrow & 0 & \eta \\ & & \parallel & & \uparrow \beta' & & \uparrow \alpha' & & & \\ 0 & \longrightarrow & K & \xrightarrow{\nu} & W & \xrightarrow{\beta} & X' & \longrightarrow & 0 & \eta' \end{array}$$

If η' is a split exact sequence, then α' factors through α .

Let us assume that α' does not factor through α , in order to derive a contradiction, again. Thus η' is not a split exact sequence. Write $K = \bigoplus K_i$ with indecomposable modules K_i and projection maps $\pi_i: K \rightarrow K_i$. Since η' does not split, there is some index i such that the exact sequences induced from η' by the map π_i does not split. We have the following commutative diagram with exact rows which do not split:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\nu} & W & \xrightarrow{\beta} & X' & \longrightarrow & 0 & \eta' \\ & & \pi_i \downarrow & & \pi_i' \downarrow & & \parallel & & & \\ 0 & \longrightarrow & K_i & \xrightarrow{\nu_i} & W_i & \xrightarrow{\beta_i} & X' & \longrightarrow & 0 & \eta_i' \end{array}$$

Since $\nu_i: K_i \rightarrow W_i$ is a monomorphism which does not split, we see that K_i cannot be injective, thus there is an almost split sequence

$$0 \rightarrow K_i \xrightarrow{\sigma_i} V_i \xrightarrow{\rho_i} \text{Tr } D(K_i) \rightarrow 0,$$

and ν_i can be factored as $\nu_i = \nu_i' \sigma_i$ for some $\nu_i': V_i \rightarrow W_i$. Thus we obtain the following commutative square on the left, and therefore also the map $\phi: \text{Tr } D(K_i) \rightarrow X'$ with a commutative square on the right:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i & \xrightarrow{\nu_i} & W_i & \xrightarrow{\beta_i} & X' & \longrightarrow & 0 & \eta_i' \\ & & \parallel & & \uparrow \nu_i' & & \uparrow \phi & & & \\ 0 & \longrightarrow & K_i & \xrightarrow{\sigma_i} & V_i & \xrightarrow{\rho_i} & \text{Tr } D(K_i) & \longrightarrow & 0 & \omega_i \end{array}$$

By assumption, the map $\phi\alpha': \text{Tr } D(K_i) \rightarrow Y$ factors through α , that means there is $\phi': \text{Tr } D(K_i) \rightarrow X$ with $\alpha\phi' = \alpha'\phi$. Now, W is the pullback of α, α' , thus there is a map $\phi'': \text{Tr } D(K_i) \rightarrow W$ such that $\beta\phi'' = \phi$ and $\beta'\phi'' = \phi'$. It follows that

$$\phi = \beta\phi'' = \beta_i\pi_i'\phi''.$$

But if ϕ factors through β_i , then the exact sequence ω_i induced from η_i' by ϕ has to split. This is a contradiction, since ω_i is an Auslander-Reiten sequence, thus non-split.

Proof of the Theorem. Let $\alpha: X \rightarrow Y$ be a morphism with kernel K and cokernel Q and let $C = \text{Tr } D(K) \oplus P(\text{soc } Q)$. Let $\alpha': X' \rightarrow Y$ be a morphism such that $\alpha'\phi$ factors through α for any map $\phi: C \rightarrow X'$. If Q' is the direct sum of all the simple modules which occur in $\text{soc } Q$, then $P(Q')$ is a direct summand of C . Thus, for any map $\phi: P(Q') \rightarrow X'$, the composition

$\alpha'\phi$ factors through α . Lemma 1 asserts that the image of α' is contained in the image of α . Now we use that $\text{Tr } D(K)$ is a direct summand of C , thus for any map $\phi: \text{Tr } D(K) \rightarrow X'$, the composition $\alpha'\phi$ factors through α . Thus we can apply Lemma 2 in order to see that α' factors through α . This shows that α is right determined by C .

Example. Let us add an example which may be illuminating, albeit it is extremely special. Let Λ be the path algebra of a finite directed quiver. Let b be a vertex of the quiver and assume that there are s arrows starting in b , say $b \rightarrow a_i$ with $1 \leq i \leq s$, and that there are t arrows ending in b , say $c_j \rightarrow b$ with $1 \leq j \leq t$. For any vertex x , we denote by $S(x)$ the simple module with support x , by $P(x)$ the projective cover of $S(x)$, by $I(x)$ the injective envelope of $S(x)$.

Let α be a non-zero map $X = P(b) \rightarrow I(b) = Y$, this is the homomorphism which we want to look at. Note that the image of α is $S(x)$. The kernel of α is the radical of $P(b)$, thus the direct sum of the modules $P(a_i)$ with $1 \leq i \leq s$. The cokernel of α is the factor module of $I(b)$ modulo its socle, thus it is the direct sum of the modules $I(c_j)$ with $1 \leq j \leq t$. The projective cover of the socle of $I(c_j)$ is $P(c_j)$. Altogether we see: the theorem asserts that α is right determined by the module

$$C = \bigoplus_{i=1}^s \text{Tr } D(P(a_i)) \oplus \bigoplus_{j=1}^t P(c_j).$$

But this module C is precisely the middle term of the almost split sequence starting in $P(b)$.

This should not come as a surprise. Namely, let X' be an indecomposable module and assume that there is a non-zero map $\alpha': X' \rightarrow Y = I(b)$. Then there is a map $\beta': P(b) \rightarrow X'$ with composition $\alpha'\beta' = \alpha$. Now either β' is invertible so that α' factors through α , or else β' is not invertible and α' does not factor through α . In the latter case, β' factors through the minimal left almost split map $\gamma: P(b) \rightarrow C$ starting in $P(b)$, this means that there is some $\phi: C \rightarrow X'$ with $\beta' = \phi\gamma$. But if we look at the composition of ϕ and α , then one should be aware that no non-zero map $C \rightarrow I(b)$ factors through α .

3. Minimal right determinators.

Given a morphism $\alpha: X \rightarrow Y$, we may try to split off non-zero direct summands of X which lie in the kernel of α . If this is not possible, then α is said to be *right minimal*. In general, we may write $X = X_0 \oplus X_1$ with X_0 contained in the kernel of α and such that $\alpha|X_1$ is right minimal; then we call the kernel of $\alpha|X_1$ the *essential kernel* of α (note that it is unique up to isomorphism). Looking at the proof of Lemma 2, one immediately realizes that it is sufficient to take $\text{Tr } D(K')$ instead of $\text{Tr } D(K)$, with K' the essential kernel of α . Thus the Theorem we discuss can be strengthened as follows: *Any morphism $X \rightarrow Y$ is right determined by $\text{Tr } D(K') \oplus P(\text{soc } Q)$, where K' is the essential kernel and Q the cokernel of α .*

In case $\alpha: X \rightarrow Y$ is a monomorphism, and C right determines α , then it is easy to see that $P(\text{soc } Y)$ has to belong to $\text{add } C$. But in general, given a morphism α with cokernel Q , and a module C which right determines α , then the module $P(\text{soc } Q)$ does not have to belong to $\text{add } C$, as the following example shows:

Example. Consider the linearly ordered quiver of type A_5 with simple modules $S(1), \dots, S(5)$ such that $\text{Ext}^1(S(i+1), S(i)) \neq 0$ for $1 \leq i \leq 4$. Endow

it with one zero relation so that the indecomposable projective module $P(4)$ has length 3. Let $X = P(3)$, let $Y = P(5)/S(2)$ and let $\alpha: X \rightarrow Y$ be a non-zero map (thus the image of α is $S(3)$). Then $K = P(2)$ is the kernel and the essential kernel of α , and $\text{Tr } D(K) = P(3)/P(1)$. Claim: α is right determined by $\text{Tr } D(K)$, whereas $P(4) = P(\text{soc } Q)$ does not belong to $\text{add } \text{Tr } D(K)$ (here, Q is the cokernel of α).

Namely, for every indecomposable module X' which has a map $\alpha': X' \rightarrow Y$ not factoring through α , we have to find some map $\phi: \text{Tr } D(K) \rightarrow X'$ such that also $\alpha'\phi$ does not factor through α . There are six such modules X' , namely

$$P(3)/P(1), P(4), P(5), S(3), P(4)/S(2), P(5)/P(2).$$

For the first three modules X' , take the embedding $\text{Tr } D(K) \rightarrow X'$, for the remaining three modules X' , there is a non-zero map $\text{Tr } D(K) \rightarrow X'$ with image $S(3)$. In all cases, the composition $\alpha'\phi: \text{Tr } D(K) \rightarrow Y$ has image $S(3)$, in particular $\alpha'\phi \neq 0$. Since $\text{Hom}(\text{Tr } D(K), X) = 0$, we see that $\alpha'\phi$ cannot be factored through α .

3. References.

- [A1] Auslander, M.: Functors and morphisms determined by objects. In: Representation Theory of Algebras. Lecture Notes in Pure Appl. Math. 37. Marcel Dekker, New York (1978), 1-244. Also in: Selected Works of Maurice Auslander, Amer. Math. Soc. (1999).
- [A2] Auslander, M.: Applications of morphisms determined by objects. In: Representation Theory of Algebras. Lecture Notes in Pure Appl. Math. 37. Marcel Dekker, New York (1978), 245-327. Also in: Selected Works of Maurice Auslander, Amer. Math. Soc. (1999).
- [K] Krause, H.: Morphisms determined by objects in triangulated categories. arXiv:1110.5625.

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