The number of complete exceptional sequences for a Dynkin algebra

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Abstract. The Dynkin algebras are the hereditary artin algebras of finite representation type. The paper determines the number of complete exceptional sequences for any Dynkin algebra. The calculations may be considered as a categorification of dealing with maximal chains in the lattice of non-crossing partitions of Dynkin types.

1. Introduction.

We consider Dynkin algebras Λ, these are the hereditary artin algebras of finite representation type. Note that the indecomposable Λ-modules correspond bijectively to the positive roots of a Dynkin diagram $\Delta(\Lambda)$; such a Dynkin diagram is the disjoint union of connected diagrams and the connected Dynkin diagrams are of the form $\mathbb{A}_n, \mathbb{B}_n, \dots, \mathbb{G}_2$. Let us remark that the vertices i of $\Delta(\Lambda)$ correspond bijectively to the simple Λ-modules, there is an edge between two vertices if and only if there is a non-trivial extension between the corresponding simple modules (in one of the two possible directions), and the lacing (in the cases $\mathbb{B}_n, \mathbb{C}_n, \mathbb{F}_4, \mathbb{G}_2$) records the relative size of the endomorphism rings of the simple modules, see [DR1] or [DR2]. We call Λ a *Dynkin algebra* of type $\Delta(\Lambda)$.

Given a Dynkin algebra with n simple modules, an exceptional sequence of length t is a sequence $M_1, \dots, M_t$ of indecomposable modules such that $\operatorname{Hom}(M_i, M_j) = 0 = \operatorname{Ext}^1(M_i, M_j)$ for $i > j$. The length of an exceptional sequence is bounded by n and any exceptional sequence $(M_1, \dots, M_t)$ can be extended to an exceptional sequence $(M_1, \dots, M_n)$; in case $t = n-1$, the extension is unique (for all these assertions, see [CB] and [R2]). The exceptional sequences of length n are said to be *complete*.

Let $e(\operatorname{mod}\Lambda)$ be the number of complete exceptional sequences in the module category $\operatorname{mod}\Lambda$ (let us stress the following: when we refer to the number of modules of some kind or the number of sequences of modules, then we mean of course the number of isomorphism classes). In case Λ is the path algebra of a quiver, the number of complete exceptional sequences has been determined by Seidel [S] in 2001. The aim of this note is to finalize these investigation by dealing also with the Dynkin diagrams which are not simply laced.

There are direct connections between the representation theory of a Dynkin algebra Λ and the lattice L of non-crossing partitions of type $\Delta(\Lambda)$: As Ingalls and Thomas [IT] have shown, the lattice of the thick subcategories of $\operatorname{mod}\Lambda$ can be identified with L. Hubery and Krause [HK] have pointed out that this identification provides a bijection between the

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complete exceptional sequences for Λ and the maximal chains in L. Thus, the calculations may also be considered as a categorification of results concerning non-crossing partitions.

As we will see, the number $e(\operatorname{mod}\Lambda)$ only depends on $\Delta = \Delta(\Lambda)$, thus we may write $e(\Delta)$ instead of $e(\operatorname{mod}\Lambda)$. Also, the shuffle lemma presented in section 2 shows that it is sufficient to look at the connected Dynkin diagrams Δ.

The following table exhibits the numbers $e(\Delta)$ for any connected Dynkin diagram Δ:

Δ	$\mathbb{A}_n$	$\mathbb{B}_n, \mathbb{C}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
$e(\Delta)$	$(n+1)^{n-1}$	n^n	$2(n-1)^n$	$2^9\cdot 3^4$	$2\cdot 3^{12}$	$2\cdot 3^5\cdot 5^7$	$2^4\cdot 3^3$	$2\cdot 3$

It seems to be of interest that the numbers $e(\Delta)$ have only few different prime factors, all of them being rather small. Using the table, one easily verifies the following remarkable formula

$$\boxed{e(\Delta) = \frac{n!\, h(\Delta)^n}{|W(\Delta)|}}$$

where $W(\Delta)$ is the Weyl group of type Δ and $h(\Delta)$ the corresponding Coxeter number. Here are these numbers (see, for example, the appendix of [B]):

Δ	$\mathbb{A}_n$	$\mathbb{B}_n, \mathbb{C}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
$h(\Delta)$	$n+1$	$2n$	$2(n-1)$	$2^2\cdot 3$	$2\cdot 3^2$	$2\cdot 3\cdot 5$	$2^2\cdot 3$	$2\cdot 3$
$\|W(\Delta)\|$	$(n+1)!$	$2^n n!$	$2^{n-1}n!$	$2^7 3^4 5$	$2^{10}3^4 5\cdot 7$	$2^{14}3^5 5^2 7$	$2^7\cdot 3^2$	$2^2\cdot 3$

Unfortunately, our proof does not provide any illumination of the formula (and we should admit that the observation that the formula holds is stolen from Chapoton [C], see the end of the introduction).

As we have mentioned, for Λ the path algebra of a quiver (thus for the typical Dynkin algebras of type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_m$), the numbers $e(\Lambda)$ have been determined already by Seidel [S] in 2001. The essential cases which were missing are the Dynkin algebras of type $\mathbb{B}_n$. The inductive strategy of proof works for all types. However, we also will show a direct relationship between the cases $\mathbb{B}_n$ and $\mathbb{A}_{n-1}$, and this could be used directly in order to complete Seidel's considerations.

Here is an outline of the proof: we will use induction on the rank n of Λ. If M is an indecomposable Λ-module, let $M^\perp$ be the full subcategory of $\operatorname{mod}\Lambda$ consisting of all modules N such that $\operatorname{Hom}(M,N) = 0 = \operatorname{Ext}^1(M,N)$. Since M is exceptional, one knows that $M^\perp$ is (equivalent to) the module category of a representation-finite hereditary artin algebra of rank $n-1$ (see [GL] or [Sc]), thus by induction we may assume to know $e(M^\perp)$. Obviously, the complete exceptional sequences $(M_1,\dots,M_n)$ with $M_n = M$ correspond bijectively to the complete exceptional sequences in $M^\perp$, thus $e(M^\perp)$ is the number of complete exceptional sequences in $\operatorname{mod}\Lambda$ whose last entry is M. In section 3 we will see that there is a vertex i_M of Δ such that $e(M^\perp) = e(\Delta(i_M))$, where $\Delta(i)$ is obtained from $\Delta = \Delta(\Lambda)$ by deleting the vertex i and all the edges involving i. Thus

$$e(\Delta) = \sum\nolimits_M e(\Delta(i_M)),$$

and therefore there is the following reduction formula

$$e(\Delta) = \frac{h}{2} \sum\nolimits_{i \in \Delta_0} e(\Delta(i))$$

where h is the Coxeter number for Δ (see section 4). In section 5 we will use the reduction formula in order to obtain the entries of the table, here we have to proceed case by case. The proof of cases $\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n$ relies on some well-known recursion formulas which go back to Abel [Ab], see the Appendix. Conversely, one may observe that the interpretation using complete exceptional sequences provides a categorification of these formulas. Since we deal with artin algebras (and not more generally with artinian rings), the diagrams which arise are the Dynkin diagrams $\mathbb{A}_n, \dots \mathbb{G}_2$. If one is interested in all the finite Coxeter diagrams (thus also in $\mathbb{I}_2(m), \mathbb{H}_3, \mathbb{H}_4$), one may consider in the same way corresponding artinian rings: they are known to exist for $\mathbb{I}_2(5), \mathbb{H}_3, \mathbb{H}_4$ (see [DRS] and [O]), this will be done in [FR].

The general frame. The calculations presented here can be seen in a broader frame, since the representation theory of hereditary artinian rings has turned out to be an intriguing tool for dealing with various questions in different parts of mathematics. In particular, there is a strong relationship to the theory of (generalized) non-crossing partitions (see [Ag]) as observed first by Fomin and Zelevinsky. As Ingalls and Thomas [IT] have shown, given the path algebra Λ of a finite directed quiver of type Δ, there is a bijection between the thick subcategories of $\operatorname{mod}\Lambda$ with a generator and the non-crossing partitions of type Δ (and this result can easily be extended to arbitrary hereditary artin algebras Λ). Hubery and Krause [HK] have pointed out that in this way one obtains a bijection between the complete exceptional sequences in $\operatorname{mod}\Lambda$ and the maximal chains in the poset $\mathrm{NC}(\Delta)$ of non-crossing partitions of type Δ, or, equivalently, the factorizations of a fixed Coxeter element c in $W(\Delta)$ as a product of n reflections. Thus, the numbers $e(\Delta)$ calculated here for the Dynkin diagrams Δ via representation theory are nothing else than the numbers of maximal chains in $\mathrm{NC}(\Delta)$ (in the Dynkin case considered here, this poset is even a lattice) or the numbers of factorizations of c as a product of n reflections. The latter numbers for $\Delta = \mathbb{A}_n, \mathbb{B}_n, \mathbb{D}_n$ have been determined in a famous letter [D] of Deligne to Looijenga. The numbers of maximal chains in $\mathrm{NC}(\Delta)$ have been calculated for the cases $\mathbb{A}_n$, $\mathbb{B}_n$ and $\mathbb{D}_n$ by Kreweras [K], Reiner [Rei] and Athanasiadis-Reiner [AR], respectively, and in general by Chapoton [C] and Reading [Rea], see also Chapuy-Stump [CS]. It seems that the term $n!h^n/|W|$ is mentioned first by Chapoton [C].

The present paper only relies on well-known properties of the module category of an artin algebra. On the other hand, the result presented here, and indeed also the main steps of our proof, may be considered as a categorification of the considerations of Deligne and Reading.

The authors of the present paper are strongly indebted to H. Krause, C. Stump and H. Thomas for pointing out pertinent references concerning non-crossing partitions, the relevance of the numbers $e(\Delta)$, and to M. Baake for those concerning the binomial convolution. The references [AR], [Rei] were provided by Thomas, the references [D], [CS] and [Rea] by Krause. Also, we learned from Krause that in the context of simple singularities, the numbers $e(\Delta)$ for simply laced Dynkin diagrams Δ have been presented in 1974 by Looijenga [L].

2. The shuffle lemma.

Lemma 1 (Shuffle Lemma). *Let Λ, Λ' be representation-finite hereditary artin algebras of ranks n, n' respectively. Then*

$$e(\operatorname{mod}(\Lambda \times \Lambda')) = \binom{n+n'}{n} e(\operatorname{mod}\Lambda) e(\operatorname{mod}\Lambda').$$

Proof. Let $(E_1, \dots, E_n)$ be a (complete) exceptional sequence in $\operatorname{mod}\Lambda$ and let $(E'_1, \dots, E'_{n'})$ be a (complete) exceptional sequence in $\operatorname{mod}\Lambda'$. Let I be a subset of $\{1, 2, \dots, n+n'\}$ of cardinality n, say let $I = \{i_1 < i_2 < \cdots < i_n\}$ and let $\{j_1 < j_2 < \cdots < j_{n'}\}$ be its complement. Let $(M_1, \dots, M_{n+n'})$ be defined by $M_{i_t} = E_t$ for $1 \le t \le n$ and $M_{j_t} = E'_t$ for $1 \le t \le n'$. Then clearly $(M_1, \dots, M_{n+n'})$ is a complete exceptional sequence in $\operatorname{mod}(\Lambda \times \Lambda')$ and every complete exceptional sequence in $\operatorname{mod}(\Lambda \times \Lambda')$ is obtained in this way. Thus, fixing a subset I of cardinality n, the number of complete exceptional sequences $(M_1, \dots, M_{n+n'})$ in $\operatorname{mod}(\Lambda \times \Lambda')$ with M_i in $\operatorname{mod}\Lambda$ for all $i \in I$ is equal to $e(\operatorname{mod}\Lambda)e(\operatorname{mod}\Lambda')$, and the number of such subsets I is just $\binom{n+n'}{n}$. This completes the proof.

3. The category $M^\perp$.

Let Λ be a representation-finite hereditary artin algebra of rank n. Let $\Delta = \Delta(\Lambda)$. Given a vertex i of Δ, let $\Delta(i)$ be obtained from Δ by deleting the vertex i and the edges involving i (it is of course a disjoint union of Dynkin diagrams).

Let τ be the Auslander-Reiten translation for Λ. For every indecomposable Λ-module M, there is a natural number t such that $\tau^t M$ is indecomposable projective, thus $\tau^t M = P(i_M)$ for a (uniquely determined) vertex i_M of Δ.

Let M be an indecomposable module. It is known that the category $M^\perp$ is equivalent to a module category $\operatorname{mod}\Lambda'$ where Λ' is a representation-finite hereditary artin algebra of rank $n-1$.

Lemma 2. *Let M be an indecomposable module and assume that $M^\perp$ is equivalent to the module category $\operatorname{mod}\Lambda'$. Then Λ' has type $\Delta(i_M)$.*

Proof. First, assume that $M = P(i)$ is indecomposable projective, thus $i = i_M$. Let ϵ_i be an idempotent of Λ such that $P(i) = \Lambda\epsilon_i$. Then $M^\perp$ is the set of Λ-modules N with $\operatorname{Hom}(P(i), N) = 0$, thus the set of $\Lambda/\Lambda\epsilon_i\Lambda$-modules. On the other hand, we have $\Delta(\Lambda/\Lambda\epsilon_i\Lambda) = \Delta(i)$.

Now assume that M is indecomposable and not projective. There is a slice $\mathcal{S}$ (in the sense of [R2]) in the Auslander-Reiten quiver of Λ such that M is a sink for $\mathcal{S}$. Let $M_1, \dots M_n$ be the indecomposable modules in $\mathcal{S}$, one from each isomorphism class, and we assume that $M_n = M$. Since M is a sink of $\mathcal{S}$, we know that $\operatorname{Hom}(M, M_i) = 0$ for $1 \le i \le n-1$, thus the modules $M_1, \dots, M_{n-1}$ belong to $M^\perp$. Let $T = \bigoplus_{i=1}^{n-1} M_i$, then T is a tilting module for $M^\perp = \operatorname{mod} \Lambda'$ (it has no self-extensions and enough indecomposable direct summands). Since $\mathcal{S}$ is a slice, we know that the endomorphism ring of $\bigoplus_{i=1}^{n} M_i$ is hereditary, thus also $\operatorname{End}(T)^{\mathrm{op}}$ is hereditary and the Dynkin diagram $\Delta(\operatorname{End}(T)^{\mathrm{op}})$ is just $\Delta(i_M)$. A tilting module with hereditary endomorphism ring is a slice module (see for example [R3], section 1.2). Thus T is a slice module for $\operatorname{mod} \Lambda'$ and therefore Λ' and $\operatorname{End}(T)^{\mathrm{op}}$ have the same Dynkin type. This shows that the Dynkin type of Λ' is $\Delta(i_M)$.

4. The reduction formula.

We assume by induction that $e(\operatorname{mod} \Lambda')$ only depends on $\Delta(\Lambda')$ for any representation-finite hereditary artin algebra Λ' of rank $n' < n$.

Proposition. *Let Λ be a representation-finite hereditary artin algebra of rank n and type Δ. Then*

$$e(\operatorname{mod} \Lambda) = \frac{h}{2} \sum\nolimits_{i \in \Delta_0} e(\Delta(i)).$$

This reduction formula shows that $e(\operatorname{mod} \Lambda)$ only depends on $\Delta = \Delta(\Lambda)$.

Proof. If M is an indecomposable Λ-module, then we have seen in section 3 that $M^\perp$ is equivalent to the module category $\operatorname{mod} \Lambda'$, where Λ' is of type $\Delta(i_M)$. Thus

$$e(M^\perp) = e(\Delta(i_M)).$$

For any vertex i of Δ, let $m(i)$ be the length of the τ-orbit of $P(i)$, thus there are precisely $m(i)$ indecomposable modules M such that $i_M = i$. Therefore

$$e(\operatorname{mod} \Lambda) = \sum\nolimits_M e(M^\perp) = \sum\nolimits_M e(\Delta(i_M)) = \sum\nolimits_i m(i) e(\Delta(i)).$$

We have to distinguish two cases. First, assume that Δ is not of the form $\mathbb{A}_n$ or $\mathbb{D}_{2m+1}$ or $\mathbb{E}_6$. In this case, we have $m(i) = \frac{h}{2}$ for any vertex i of Δ. Therefore

$$\sum\nolimits_i m(i) e(\Delta(i)) = \sum\nolimits_i \frac{h}{2} e(\Delta(i)).$$

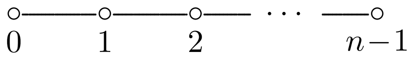
Second, assume that Δ is equal to $\mathbb{A}_n$, or $\mathbb{D}_{2m+1}$ or $\mathbb{E}_6$. Thus, there is a (unique) automorphism ρ of Δ of order 2. One knows that $m(i) + m(\rho(i)) = h$ for all vertices i of Δ. The automorphism ρ shows that $e(\Delta(\rho(i))) = e(\Delta(i))$, thus

$$\begin{aligned} 2 \sum\nolimits_i m(i) e(\Delta(i)) &= \sum\nolimits_i m(i) e(\Delta(i)) + \sum\nolimits_i m(\rho(i)) e(\Delta(\rho(i))) \\ &= \sum\nolimits_i (m(i) + m(\rho(i)) e(\Delta(i)) \\ &= \sum\nolimits_i h \cdot e(\Delta(i)). \end{aligned}$$

Dividing by 2 we obtain the required formula.

5. The different cases.

Type $\mathbb{A}_n$. This concerns the following diagram



We have $\Delta(i) = \mathbb{A}_i \sqcup \mathbb{A}_{n-i-1}$, therefore, by the shuffle lemma and induction,

$$e(\Delta(i)) = \binom{n-1}{i} e(A_i) e(\mathbb{A}_{n-i-1}) = \binom{n-1}{i}(i+1)^{i-1}(n-i)^{n-i-2}.$$

Thus we have to calculate

$$\sum_{i=0}^{n-1} e(\Delta(i)) = \sum_{i=0}^{n-1} \binom{n-1}{i}(i+1)^{i-1}(n-i)^{n-i-2},$$

but this is the coefficient $F(n-1)$ of the power series $F = A * A$, see the appendix, and the formula (1) asserts that $F(n-1) = 2(n+1)^{n-2}$.

Now $h = n+1$, thus

$$\frac{h}{2}\sum_{i=1}^{n} e(\Delta(i)) = \frac{n+1}{2} 2(n+1)^{n-2} = (n+1)^{n-1}.$$

Type $\mathbb{B}_n$: The relationship between $\mathbb{B}_n$ and $\mathbb{A}_{n-1}$.

Let us show directly the following relationship:

$$e(\mathbb{B}_n) = n^2 \cdot e(\mathbb{A}_{n-1}).$$

Proof. Let Λ be a hereditary artin algebra of type $\mathbb{B}_n$. Let P be the indecomposable projective Λ-module such that $\mathbf{dim}\, P$ is a short root. If $(M_1, \dots, M_n)$ is an exceptional sequence in $\operatorname{mod}\Lambda$, then there is precisely one index i such that $\mathbf{dim}\, M_i$ is a short root (see [R2]). Thus, let $\mathcal{E}_i(\operatorname{mod}\Lambda)$ be the set of exceptional sequences in $\operatorname{mod}\Lambda$ such that $\mathbf{dim}\, M_i$ is a short root, and let $e_i(\operatorname{mod}\Lambda)$ the cardinality of $\mathcal{E}_i(\operatorname{mod}\Lambda)$. If $i < n$, and $(M_1, \dots, M_n)$ belongs to $\mathcal{E}_i(\operatorname{mod}\Lambda)$, then there is a uniquely determined element $(M_1, \dots, M_{i-1}, M_{i+1}, M_i^*, M_{i+2}, \dots, M_n)$ in $\mathcal{E}_{i+1}(\operatorname{mod}\Lambda)$ and every element of $\mathcal{E}_{i+1}(\operatorname{mod}\Lambda)$ is obtained in this way (again, see [R2]). This shows that $e_i(\operatorname{mod}\Lambda) = e_{i+1}(\operatorname{mod}\Lambda)$ and therefore

$$e(\operatorname{mod}\Lambda) = \sum_{i=1}^{n} e_i(\operatorname{mod}\Lambda) = n \cdot e_n(\operatorname{mod}\Lambda).$$

There are precisely n indecomposable modules M such that $\mathbf{dim}\, M$ is a short root, namely the modules in the τ-orbit $\mathcal{O}(P)$ of P. For any module M in $\mathcal{O}(P)$, the exceptional sequences $(M_1, \dots, M_n)$ with $M_n = M$ correspond bijectively to the exceptional sequences in $M^\perp$, and $M^\perp$ is equivalent to a module category $\operatorname{mod}\Lambda_M$ with Λ_M a hereditary artin algebra of type $\mathbb{A}_{n-1}$. This shows that

$$e_n(\operatorname{mod}\Lambda) = \sum\nolimits_{M\in\mathcal{O}(P)} e(M^\perp) = n\cdot e(\mathbb{A}_{n-1}).$$

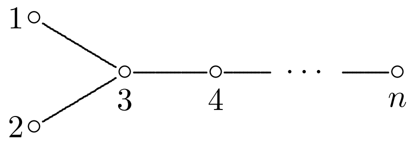
This completes the proof.

Type $\mathbb{C}_n$. There is the corresponding formula

$$e(\mathbb{C}_n) = n^2\cdot e(\mathbb{A}_{n-1})$$

(with a similar proof).

Type $\mathbb{D}_n$. This concerns the following diagram



with $n \geq 4$. Actually, also the cases $n = 3$ and $n = 2$ are of interest: for $n = 3$, we have $\mathbb{D}_3 = \mathbb{A}_3$, for $n = 2$ we deal with $\mathbb{D}_2 = \mathbb{A}_1 \sqcup \mathbb{A}_1$.

Before we proceed, let us mention the following notation (see the appendix): For any $n \geq 0$, let $A(n) = (n+1)^{n-1}$ and $D(n) = (n-1)^n$.

For $k \geq 4$, we have $\Delta(k) = \mathbb{D}_{k-1} \sqcup \mathbb{A}_{n-k}$, thus the shuffling lemma yields

$$\begin{aligned} e(\Delta(k)) &= \tbinom{n-1}{k-1} e(\mathbb{D}_{k-1})\cdot e(\mathbb{A}_{n-k}) \\ &= \tbinom{n-1}{k-1} 2(k-1)^k\cdot(n-k+1)^{n-k-1} \\ &= \tbinom{n-1}{k-1} 2D(k-1)A(n-k). \end{aligned}$$

For $k = 3$, we have $\Delta(3) = \mathbb{A}_1 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_{n-3}$, and $D(2) = 1$, thus

$$\begin{aligned} e(\Delta(3)) &= \tfrac{(n-1)!}{1!1!(n-3)!} e(\mathbb{A}_{n-3}) \\ &= \tbinom{n-1}{2}\cdot 2\cdot(n-2)^{n-4} \\ &= \tbinom{n-1}{2}\cdot 2D(2)A(n-3) \end{aligned}$$

For $k = 1$ and $k = 2$, we have $\Delta(k) = \mathbb{A}_{n-1}$, therefore

$$e(\Delta(k)) = e(\mathbb{A}_{n-1}) = n^{n-2} = A(n-1),$$

thus the sum $e(\Delta(1)) + e(\Delta(2))$ is of the form

$$e(\Delta(1)) + e(\Delta(2)) = \tbinom{n-1}{0} 2D(0)A(n-1)$$

(since $D(0) = 1$).

Taking into account that $D(1) = 0$, we see that

$$\sum\nolimits_{k=1}^{n} e(\Delta(k)) = e(\Delta(1)) + e(\Delta(2)) + \sum\nolimits_{k=3}^{n} e(\Delta(k))$$
$$= \sum\nolimits_{k=1}^{n} \tbinom{n-1}{k-1} 2D(k-1)A(n-k)$$

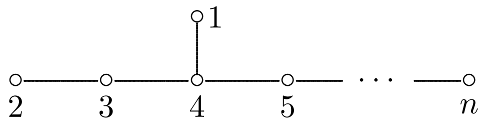
but this is the coefficient $G(n-1)$ of the power series $G = D * A$, see the appendix. The formula (3) in the appendix asserts that $G(n-1) = (n-1)^{n-1}$.

Since the Coxeter number for $\mathbb{D}_n$ is $h = 2(n-1)$, we have

$$\frac{h}{2} \sum\nolimits_{k=1}^{n} e(\Delta(k)) = (n-1) \cdot 2 \cdot (n-1)^{n-1} = 2(n-1)^n,$$

as we wanted to show.

Type $\mathbb{E}_n$. This concerns the following diagrams



and we will deal with the cases $n = 6, 7, 8$.

Type $\mathbb{E}_6$

i	$\Delta(i)$	$e(\Delta(i))$
1	$\mathbb{A}_5$	1296
2	$\mathbb{D}_5$	2048
3	$\mathbb{A}_1 \sqcup \mathbb{A}_4$	$\frac{5!}{1!4!}\ 1 \cdot 125$
4	$\mathbb{A}_2 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_2$	$\frac{5!}{2!1!2!}\ 3 \cdot 1 \cdot 3$

We see:

$$e(\mathbb{E}_6) = \frac{h}{2}\Big(e(\mathbb{A}_5) + 2e(\mathbb{D}_5) + 2e(\mathbb{A}_1 \sqcup \mathbb{A}_4) + e(\mathbb{A}_2 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_2)\Big) = 41\,472 = 2^9 3^4$$

Type $\mathbb{E}_7$

i	$\Delta(i)$	$e(\Delta(i))$
1	$\mathbb{A}_6$	16807
2	$\mathbb{D}_6$	46656
3	$\mathbb{A}_1 \sqcup \mathbb{A}_5$	$\frac{6!}{1!5!}\ 1 \cdot 1296$
4	$\mathbb{A}_2 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_3$	$\frac{6!}{2!1!3!}\ 3 \cdot 1 \cdot 16$
5	$\mathbb{A}_4 \sqcup \mathbb{A}_2$	$\frac{6!}{4!2!}\ 125 \cdot 3$
6	$\mathbb{D}_5 \sqcup \mathbb{A}_1$	$\frac{6!}{5!1!}\ 2048 \cdot 1$
7	$\mathbb{E}_6$	41472

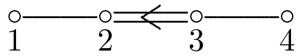
$$e(\mathbb{E}_7) = 1\,062\,882 = 2 \cdot 3^{12}$$

Type $\mathbb{E}_8$

i	$\Delta(i)$	$e(\Delta(i))$
1	$\mathbb{A}_7$	262144
2	$\mathbb{D}_7$	559872
3	$\mathbb{A}_1 \sqcup \mathbb{A}_6$	$\frac{7!}{1!6!}\, 1 \cdot 16807$
4	$\mathbb{A}_2 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_4$	$\frac{7!}{2!1!4!}\, 3 \cdot 1 \cdot 125$
5	$\mathbb{A}_4 \sqcup \mathbb{A}_3$	$\frac{7!}{4!3!}\, 125 \cdot 16$
6	$\mathbb{D}_5 \sqcup \mathbb{A}_2$	$\frac{7!}{5!2!}\, 2048 \cdot 3$
7	$\mathbb{E}_6 \sqcup \mathbb{A}_1$	$\frac{7!}{6!1!}\, 41472 \cdot 1$
8	$\mathbb{E}_7$	1062882

$$e(\mathbb{E}_8) = 37\,968\,750 = 2 \cdot 3^5 \cdot 5^7.$$

Type $\mathbb{F}_4$**.** This concerns the following diagram



i	$\Delta(i)$	$e(\Delta(i))$
1	$\mathbb{B}_3$	27
2	$\mathbb{A}_1 \sqcup \mathbb{A}_2$	$\frac{3!}{1!2!}\, 1 \cdot 3$
3	$\mathbb{A}_2 \sqcup \mathbb{A}_1$	$\frac{3!}{2!1!}\, 3 \cdot 1$
4	$\mathbb{C}_3$	27

$$e(\mathbb{F}_4) = 432 = 2^4 \cdot 3^3.$$

6. Appendix: The binomial convolution of some power series.

We are interested in the power series A, B, D with coefficients $A(n) = (n+1)^{n-1}$, $B(n) = n^n$, and $D(n) = (n-1)^n$, thus

$$A = \sum_n (n+1)^{n-1} T^n = 1 + T + 3T^2 + 16T^3 + 125T^4 + \dots$$

$$B = \sum_n n^n T^n = 1 + T + 4T^2 + 27T^3 + 256T^4 + \dots$$

$$D = \sum_n (n-1)^n T^n = 1 + T^2 + 8T^3 + 81T^4 + \dots.$$

The main result of the paper asserts that $e(\mathbb{A}_n) = A(n)$ and $e(\mathbb{B}_n) = e(\mathbb{C}_n) = B(n)$ for $n \geq 1$ and that $e(\mathbb{D}_n) = 2D(n)$ for $n \geq 2$.

Let $\mathbb{C}[[T]]$ be the set of formal power series in one variable T with coefficients in $\mathbb{C}$. Given power series $F = \sum_n F(n)T^n$ and $G = \sum_n G(n)T^n$, the *binomial convolution* $F * G$ is by definition the power series $\sum_n H(n)T^n$ with $H(n) = \sum_k \binom{n}{k} F(k)G(n–k)$ (see [GKP]).

Our proofs use two of the following identities, namely (1) and (3) (and we could use (2) in order to deal with the case $\mathbb{B}_n$):

Proposition.

$$A * A = \sum_{n \geq 0} 2(n+2)^{n-1} T^n \tag{1}$$

$$A * B = \sum_{n \geq 0} (n+1)^n T^n \tag{2}$$

$$A * D = \sum_{n \geq 0} n^n T^n = B \tag{3}$$

Proof. Let us recall Abel's identity [Ab]

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x-kz)^{k-1}(y+kz)^{n-k}$$

which is valid in any commutative ring with x being invertible. Several proofs can be found in Comptet [C]. Let us start with the proof of (2). We use Abel's identity for $x = 1, y = n, z = -1$.

$$(1+n)^n = \sum_{k=0}^{n} \binom{n}{k} (1+k)^{k-1}(n-k)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} A(k)B(n-k) = (A * B)(n).$$

For the proof of (3), we use Abel's identity for $x = 1, y = n-1$ and $z = -1$.

$$n^n = (1 + (n-1))^n = \sum_{k=0}^{n} \binom{n}{k} (1+k)^{k-1}(n-1-k)^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} A(k)D(n-k) = (A * D)(n).$$

For the proof of (1), we expand $(n+2)^{n-1}$ with $x = 1, y = n+1$ and $z = -1$,

$$(1 + (n+1))^{n-1} = \sum_{k=0}^{n} \binom{n-1}{k} (1+k)^{k-1}(n+1-k)^{n-1-k}, \tag{$*$}$$

note that we have added the summand $k = n$, but this can be done since by definition $\binom{n}{n-1} = 0$. Replacing the summation index k by $n-k$, and using the equality $\binom{n-1}{n-k} = \binom{n-1}{k-1}$, we see that we also have

$$(**) \qquad (1+(n+1))^{n-1} = \sum_{k=0}^{n} \binom{n-1}{k-1} (1+n-k)^{n-k-1}(k+1)^{k-1}.$$

Since $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$, the summation of $(*)$ and $(**)$ yields

$$\begin{aligned} 2(n+2)^{n-1} &= \sum_{k=0}^{n} \binom{n}{k} (k+1)^{k-1}(n-k+1)^{n-k-1} \\ &= \sum_{k=0}^{n} \binom{n}{k} A(k)A(n-k) = (A*A)(n). \end{aligned}$$

This completes the proof.

It seems to us that these binomial convolution formulas are very pretty; as an example, let us exhibit the coefficients of T^4 in $A*A,\ A*B,\ A*D$:

$$\begin{array}{llllllll} (A*A)(4) & 1\cdot1\cdot125+ & 4\cdot1\cdot16\ + & 6\cdot3\cdot3\ + & 4\cdot16\cdot1 & +1\cdot125\cdot1 & = & 2\cdot6^3 \\ (A*B)(4) & 1\cdot1\cdot256+ & 4\cdot1\cdot27\ + & 6\cdot3\cdot4\ + & 4\cdot16\cdot1 & +1\cdot125\cdot1 & = & 5^4 \\ (A*D)(4) & 1\cdot1\cdot\ 81+ & 4\cdot1\cdot\ 8\ + & 6\cdot3\cdot1\ + & 4\cdot16\cdot0 & +1\cdot125\cdot1 & = & 4^4 \end{array}$$

Finally, let us add some general information concerning the sequences A, B, D as provided by Sloane's On-Line Encyclopedia of Integer Sequences [Sl]. The sequence $A(n) = (n+1)^{n-1}$ is the Sloane sequence A000272, but shifted by 1, thus $A(n)$ is the *number of trees on* $n+1$ *labeled nodes.* The sequence $B(n) = n^n$ is the Sloane sequence A000312, the number $B(n)$ is the *number of functions from the set* $\{1, 2, ..., n\}$ *to itself.* The sequence $D(n) = (n{-}1)^n$ with $e(\mathbb{D}_n) = 2D(n)$ for $n \geq 2$ is the Sloane sequence A065440; the number $D(n)$ is the *number of functions from the set* $\{1, 2, ..., n\}$ *to itself without fixed points.*

Here are the first terms of the sequences $A, B, 2D$, namely $A(n), B(n), 2D(n)$, with $n \leq 10$; note that $A(n) = e(\mathbb{A}_n), B(n) = e(\mathbb{B}_n)$, for $n \geq 1$ and $2D(n) = e(\mathbb{D}_n)$, for $n \geq 2$.

n	$A(n)$	$B(n)$	$2D(n)$
0	1	1	2
1	1	1	0
2	3	4	2
3	16	27	16
4	125	256	162
5	1 296	3 125	2 048
6	16 807	46 656	31 250
7	262 144	823 543	559 872
8	4 782 969	12 777 216	11 529 602
9	100 000 000	387 420 489	268 435 456
10	2 357 947 691	10 000 000 000	6 973 568 802

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