The Fibonacci partition triangles

Philipp Fahr, Claus Michael Ringel

Abstract. In two previous papers we have presented partition formulae for the Fibonacci numbers motivated by the appearance of the Fibonacci numbers in the representation theory of the 3-Kronecker quiver and its universal cover, the 3-regular tree. Here we show that the basic information can be rearranged in two triangles. They are quite similar to the Pascal triangle of the binomial coefficients, but in contrast to the additivity rule for the Pascal triangle, we now deal with additivity along "hooks", or, equivalently, with additive functions for valued translation quivers. As for the Pascal triangle, we see that the numbers in these Fibonacci partition triangles are given by evaluating polynomials. We show that the two triangles can be obtained from each other by looking at differences of numbers, it is sufficient to take differences along arrows and knight's moves.

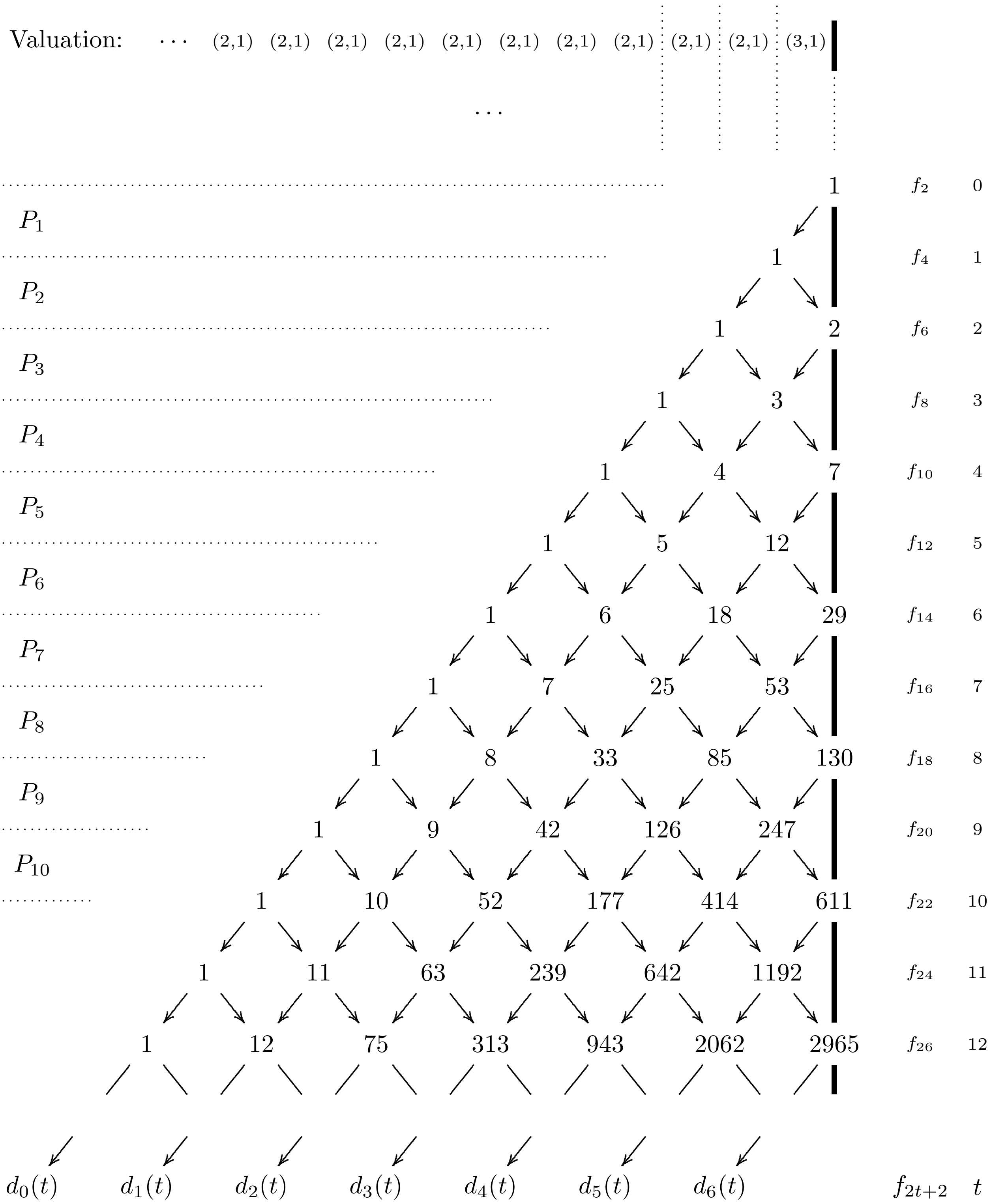
The aim of the paper is to rearrange the positive integers which are used in the partition formulae for the Fibonacci numbers as considered in [FR1,FR2]. For the even-index Fibonacci numbers we obtain a proper triangle which we call the even-index Fibonacci partition triangle. Second, what we call the odd-index Fibonacci partition triangle actually is a triangle only after removing one number (but it seems worthwhile to take this additional position into account). These arrangements of integers are quite similar to the Pascal triangle of the binomial coefficients. In particular, we will show that the numbers along the inclined lines are given by evaluating polynomials.

Let us recall that our work on the partition formulae was based on the appearance of the Fibonacci numbers in the representation theory of quivers and the present paper again relies on concepts which have been developed in this context. Namely, both triangles turn out to show additive functions on valued translation quivers. Translation quivers have been considered frequently in the representation theory of quivers and finite-dimensional algebras. The valued translation quivers can be used in order to describe module categories, but those with non-trivial valuation (as in the case of the Fibonacci partition triangles) have seldom be seen to be of importance when dealing with quivers.

In sections 1 and 2 we will exhibit the even-index, and the odd-index triangle, respectively. Our main task will be to show in which way the two triangles can be obtained from each other, see sections 3 and 4. The relationship which we will encounter shows that the two triangles are intimately connected. The proof provided here relies on the categorification of the Fibonacci pairs given in [FR2], it will be given in section 5. The final section 6 provides some further remarks and open questions. For known properties of the Fibonacci numbers one may consult the book of Koshy [K], see also [W].

The investigation is based on the Fibonacci partition formulae established in [FR1] for the Fibonacci numbers with even index, and in the PhD thesis [F] of Fahr for those with odd index, see also [FR2], but also on further discussions of the authors during the time when Fahr was a PhD student at Bielefeld. The final version was written by Ringel.

1. The even-index Fibonacci partition triangle



First, let us explain the numbers displayed in the triangle as well as the two columns on the right. The rows of the triangle are indexed by $t = 0, 1, 2, \ldots$ as shown on the right. In any row, say the row t, the entries will be labeled, from left to right, by $d_0(t), d_1(t), \ldots, d_i(t), \ldots$, with $i \le \frac{t}{2}$; for example, for $t = 5$ we have

$$d_0(5) = 1, \quad d_1(5) = 5, \quad d_2(5) = 12.$$

For some calculations it seems convenient to define $d_i(t)$ for all $i \in \mathbb{Z}$ as follows: we let $d_i(t) = 0$ for $i < 0$ and we define $d_i(t) = d_{t-i}(t)$ for $i > \frac{t}{2}$ (thus $d_i(t) = d_{t-i}(t)$ for all $i \in \mathbb{Z}$).

The entries $d_i(t)$ displayed in the triangle are calculated inductively as follows: We start with $d_0(t) = 1$ for all $t \ge 0$. Now let $i \ge 1$. Then, for $1 \le i < \frac{t}{2}$ (and $t \ge 3$) let

$$d_i(t) = 2d_{i-1}(t-1) + d_i(t-1) - d_{i-1}(t-2),$$

whereas for $i = \frac{t}{2}$ (thus $t = 2i$)

$$d_i(2i) = 3d_{i-1}(2i-1) - d_{i-1}(2i-2).$$

So using the convention that $d_i(t) = d_{t-i}(t)$ for $i > \frac{t}{2}$, we see that the rule for $i = \frac{t}{2}$ is the same as that for $i < \frac{t}{2}$.

The entries $d_i(t)$ displayed in the triangle have been considered already in the paper [FR1] (but labeled differently): they are derived from the numbers $a_s[j]$ of [FR1] according to the rule

$$d_i(t) = a_{\lceil t/2 \rceil}[t - 2i]$$

for $0 \le i \le \frac{t}{2}$ (given any real number α, we denote by $\lceil \alpha \rceil$ the smallest integer z with $\alpha \le z$). For the convenience of the reader, we will provide at the end of the paper a visual concordance which allows to compare the notion used here with that of the previous paper [FR1].

Using the notation $d_i(t)$, the partition formulae of [FR1] can be written in a unified way, as follows:

Partition formula for the Fibonacci numbers f_n with even index n:

$$\boxed{3\left(\sum_{0 \le i < t/2} 2^{t-2i-1} d_i(t)\right) + d_{t/2}(t) \; = f_{2t+2}}$$

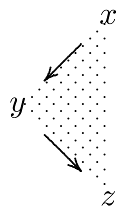
where we set $d_\alpha(t) = 0$ if $\alpha \notin \mathbb{Z}$. This shows that *the row t of the triangle yields the Fibonacci number f_{2t+2}*. Thus, we have added the even-index Fibonacci numbers in a column on the right. As examples, let us look at the rows $t = 5$ and $t = 6$, they yield

$(t = 5)$	$3(2^4 \cdot 1 + 2^2 \cdot 5 + 2^0 \cdot 12)$	$= 144 = f_{12}$
$(t = 6)$	$3(2^5 \cdot 1 + 2^3 \cdot 6 + 2^1 \cdot 18) + 29$	$= 377 = f_{14}$

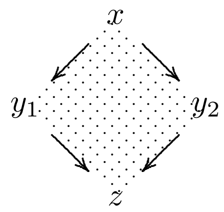
The numbers of the triangle are connected by arrows: **we consider the triangle** as a quiver, or better **as a valued translation quiver.** Let us recall the definition (see for example [HPR]).

A *quiver* $\Gamma = (\Gamma_0, \Gamma_1)$ *without multiple arrows* is given by a set Γ_0, called the set of *vertices,* and a subset Γ_1 of $\Gamma_0 \times \Gamma_0$. An element $\alpha = (x, y)$ of Γ_1 with $x, y \in \Gamma_0$ is called an *arrow,* and usually one writes $\alpha\colon x \to y$ and calls x the *starting* vertex, y the *terminal* vertex of α. Given vertices x, z of Γ, we denote by x^+ the set of vertices y with an arrow $x \to y$, and by z^- the set of vertices y with an arrow $y \to z$.

A *translation quiver* $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ is given by a quiver (Γ_0, Γ_1) without multiple arrows, a subset $\Gamma_0^p \subseteq \Gamma_0$ and an injective function $\tau\colon (\Gamma_0 \setminus \Gamma_0^p) \to \Gamma_0$ such that for any vertex z in $\Gamma_0 \setminus \Gamma_0^p$, one has $(\tau z)^+ = z^-$. The function τ is called the *translation,* the vertices in Γ_0^p are said to be the *projective* vertices, those not in the image of τ the *injective* vertices. Given a non-projective vertex z, one says that $\tau z, z^-, z$ is the mesh ending in z; in the examples which we consider, these sets z^- consist of either one or two elements, thus we deal with meshes of the following form, where $x = \tau z$:



and

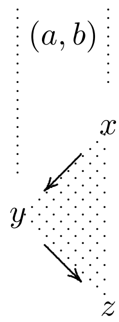


A *valued translation quiver* $\Gamma = (\Gamma_0, \Gamma_1, \tau, v)$ is a translation quiver with two functions $v', v''\colon \Gamma_1 \to \mathbb{N}_1$ such that

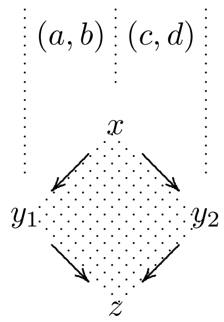
$$v'(\tau z, y) = v''(y, z) \quad \text{and} \quad v''(\tau z, y) = v'(y, z),$$

for any arrow $y \to z$ in Γ, where z is a non-projective vertex; we write $v = (v', v'')$ and call v the *valuation* of Γ.

Dealing with meshes as exhibited above, it is sufficient to write down the valuation for the arrows pointing south-east, say as follows:



and



this means on the right that $v'(y_1, z) = a, v''(y_1, z) = b$ (thus $v'(x, y_1) = b, v''(x, y_1) = a$), and that $v'(x, y_2) = c, v''(x, y_2) = d$ (thus $v'(y_2, z) = d, v''(y_2, z) = c$), and similarly, on the left, that $v'(y, z) = a, v''(y, z) = b$ and $v'(x, y) = b, v''(x, y) = a$.

A function $g\colon \Gamma_0 \to \mathbb{Z}$ is called *additive* provided

$$g(z) + g(\tau z) = \sum\nolimits_{y \in z^-} v'(y, z) g(y)$$

for all non-projective vertices z of Γ (this is said to be the corresponding *mesh relation*). Thus, looking again at the meshes displayed above, we must have, on the right

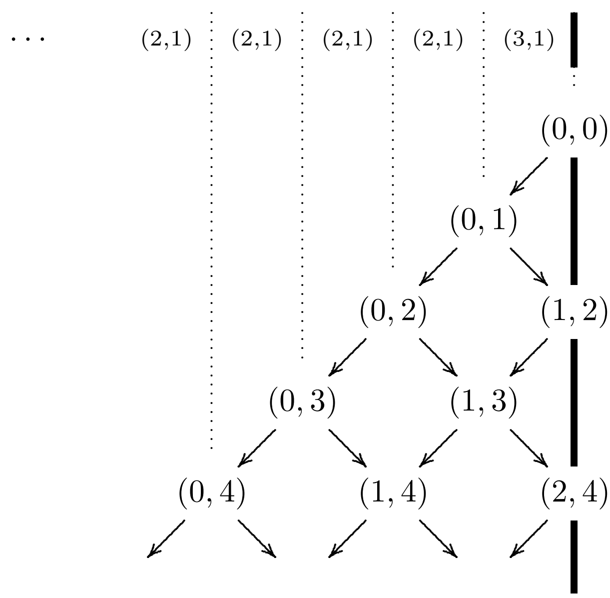
$$g(z) = -g(x) + ag(y_1) + dg(y_2),$$

and, on the left

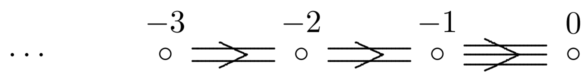
$$g(z) = -g(x) + ag(y).$$

The even-index triangle which we have presented above is the following translation quiver Γ^{ev}: its vertices are the pairs (i,t) with $i,t \in \mathbb{N}_0$ such that $2i \le t$, with arrows $(i,t) \to (i,t+1)$ (we draw them as pointing south-west) and, for $2i < t$, with arrows $(i,t) \to (i+1,t+1)$ (drawn as pointing south-east), such that $\tau(i,t) = (i-1,t-2)$ provided $i \ge 1$, whereas the vertices of the form $(0,t)$ are projective. The vertices of the form $(i,2i)$ are said to lie on the *pylon*.

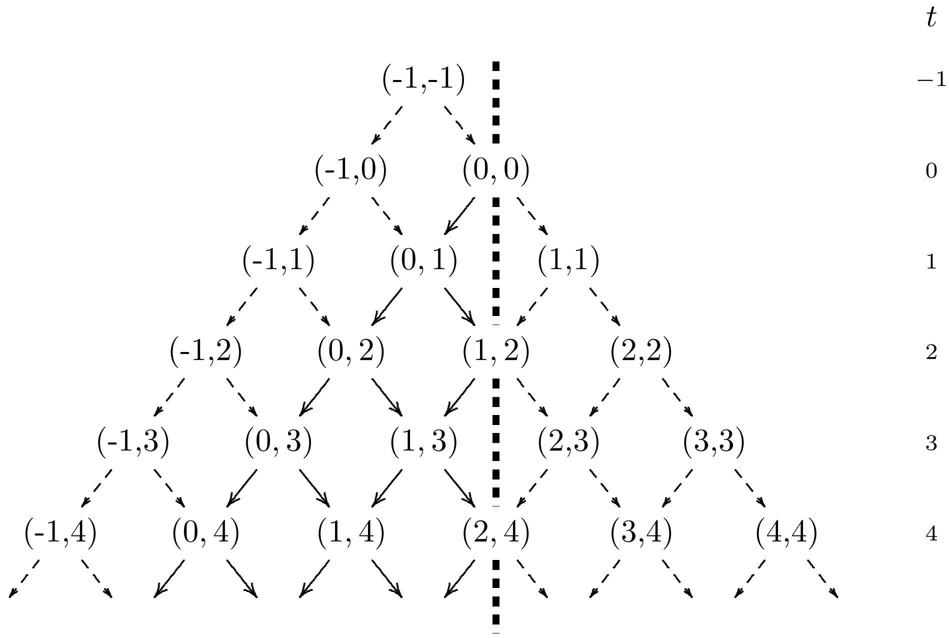
We use the following valuation: the valuation of the arrows ending on the pylon will be $(3,1)$, all other south-east arrows have valuation $(2,1)$. Here is part of the triangle, the pylon being marked as a black line; the upper row records the valuation:



Note that Γ^{ev} is a subquiver of the valued translation quiver $\mathbb{Z}\Delta^{\mathrm{ev}}$, where Δ^{ev} is the valued graph



(for the construction of $\mathbb{Z}\Delta$, where Δ is a valued quiver, see [HPR]). If necessary, then we consider the underlying translation quiver of Γ^{ev} and of Γ^{odd} (a valued translation quiver which will be defined in section 2) as a subquiver of the translation quiver $\mathbb{Z}\mathbb{A}_\infty^\infty$; the latter quiver has as vertex set the set $\mathbb{Z} \times \mathbb{Z}$, there are arrows $(a,b) \to (a,b+1)$ and $(a,b) \to (a+1,b+1)$, and the translation is given by $(a,b) \mapsto (a-1,b-2)$, for all $a,b \in \mathbb{Z}$. Actually, for our considerations it always will be sufficient to deal with the following subquiver of $\mathbb{Z}\mathbb{A}_\infty^\infty$:



For better orientation, we have inserted the pylon for $\Gamma^{\rm ev}$ as a dotted vertical line (in the case of $\Gamma^{\rm odd}$ this will be the position of the left pylon).

If g is an additive function on the translation quiver $\Gamma^{\rm ev}$, we write $g_i(t)$ instead of $g(i,t)$. Note that any additive function g on $\Gamma^{\rm ev}$ is uniquely determined by the values $g(p)$ with p projective, thus by the values of g_0.

By definition, the function d presented in the even-index triangle is an *additive function on* $\Gamma^{\rm ev}$. And *d is the unique additive function on $\Gamma^{\rm ev}$ such that $d_0(t) = 1$ for all $t \geq 0$* (thus with value 1 on the projective vertices).

Proposition 1. *The function $g\colon \Gamma_0^{\rm ev} \to \mathbb{Z}$ is additive on $\Gamma^{\rm ev}$ if and only if it satisfies the following hook condition for all $t \geq 1$:*

$$g_i(t) = g_i(t-1) + \sum_{0 \leq j < i} g_j(t-i+j)$$

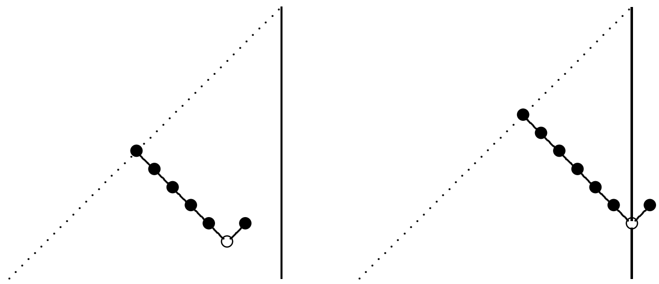
provided $2i < t$, and, for $t = 2i$:

$$g_i(2i) = g_{i-1}(2i-1) + \sum_{0 \leq j < i} g_j(i+j).$$

If we define $g_i(2i-1) = g_{i-1}(2i-1)$ (thus adding a column on the right of the triangle, with vertex $(i, 2i-1)$ in the row with index $2i-1$), then the second condition in Proposition 1 has the same form as the first condition, with $t = 2i$, namely

$$g_i(2i) = g_i(2i-1) + \sum_{0 \leq j < i} g_j(2i-i+j),$$

thus all conditions concern "hooks" in the even-index triangle as follows (the value at the circle being obtained by adding the values at the bullets):



Here we should insert a remark concerning the difference between the additivity property we encounter for the Fibonacci partition triangles and the additivity property of the Pascal triangle. The additivity property for the Pascal triangle means that any coefficient is the sum of the two upper neighbors (one of them may be zero, if we are on the boundary). The hook condition means that we have to add not only the values of the two upper neighbors, but that we have to deal with the values on a hook (but all the summands are still taken with multiplicity one). In contrast, the additivity property for a valued translation quiver (with arrows pointing downwards) means that the value at the vertex z is obtained by first taking a certain linear combination of the values at the two upper neighbors (using positive coefficients which may be different from 1) and then subtracting the value at τz. Note that, in general, such a mesh relation always involves subtraction, thus it may lead to negative numbers. Of course, in our case, the equivalent hook condition shows that we stay inside the set of positive integers.

Proof of Proposition 1, by induction on t. First, assume that $2i < t$. Then the additivity formula for g yields:

$$\begin{aligned} g_i(t) &= 2g_{i-1}(t-1) + g_i(t-1) - g_{i-1}(t-2) \\ &= g_i(t-1) + g_{i-1}(t-1) + (g_{i-1}(t-1) - g_{i-1}(t-2)) \\ &= g_i(t-1) + g_{i-1}(t-1) + \sum_{0 \le j < i-1} g_j(t-i+j) \\ &= g_i(t-1) + \sum_{0 \le j < i} g_j(t-i+j). \end{aligned}$$

Similarly, for $t = 2i > 0$ we get:

$$\begin{aligned} g_i(2i) &= 3g_{i-1}(2i-1) - g_{i-1}(2i-2) \\ &= g_{i-1}(2i-1) + g_{i-1}(2i-1) + (g_{i-1}(2i-1) - g_{i-1}(2i-2)) \\ &= g_{i-1}(2i-1) + g_{i-1}(2i-1) + \sum_{0 \le j < i-1} g_j(i+j) \\ &= g_{i-1}(2i-1) + \sum_{0 \le j < i} g_j(i+j). \end{aligned}$$

The converse is shown in the same way.

Corollary 1. *The function $d_i(t)$ is for $t \geq 2i-1$ a polynomial of degree i, for any* $i \geq 0$, it is a monic linear combination of the binomial coefficients $\binom{t}{n}$.

Proof: We use induction on i. For $i=0$, we deal with the constant polynomial $d_0(t) = 1 = \binom{t}{0}$. Now let $i > 0$. Then we have

$$d_i(t) - d_i(t-1) = \sum_{0 \leq j < i} d_j(t-i+j),$$

for $t \geq 2i$, and the right side is by induction a monic linear combination of the binomial coefficients $\binom{t}{0}, \binom{t}{1}, \dots \binom{t}{i-1}$. Thus g_i has to be a monic linear combination of the binomial coefficients $\binom{t}{0}, \binom{t}{1}, \dots \binom{t}{i}$. In particular, $d_i(t)$ is a polynomial of degree i.

Let us stress that our convention to define $d_i(t)$ for $t \geq 0$ and all i using the rule $g_{t-i}(t) = g_i(t)$ (as well as $g_i(t) = 0$ for $i < 0$) is inconvenient with respect to the polynomiality assertion. For example, this means that $d_2(2) = 1$, whereas the function $d_2(t)$ for $t \geq 3$ is given by the polynomial $p_2(t) = \frac{1}{2}(t^2+t-6)$, and $p_2(2) = 0$. Using our convention, the functions $d_i(t)$ with $t \geq 0$ is only eventually polynomial (one says that a function $f\colon \mathbb{N}_0 \to \mathbb{Z}$ is *eventually polynomial* of degree t provided there is some natural number n_0 such that the restriction of f to the set $\{n \mid n \geq n_0\}$ is a polynomial function of degree t).

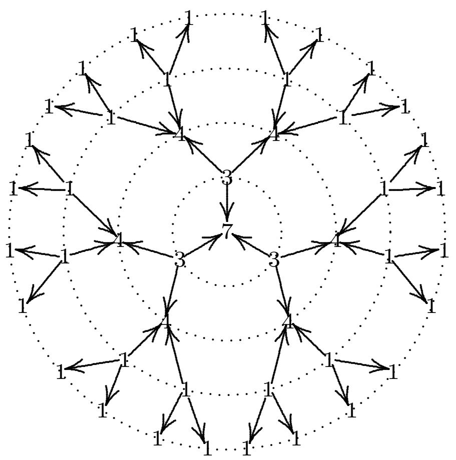
It follows that the function $d_i(t)$ with $t \geq 2i-1$ is determined by the functions $d_0(t), \dots, d_{i-1}(t)$, as well as one special value, for example the pylon value $d_i(2i)$. In this way, we see that the pylon values determine the whole triangle.

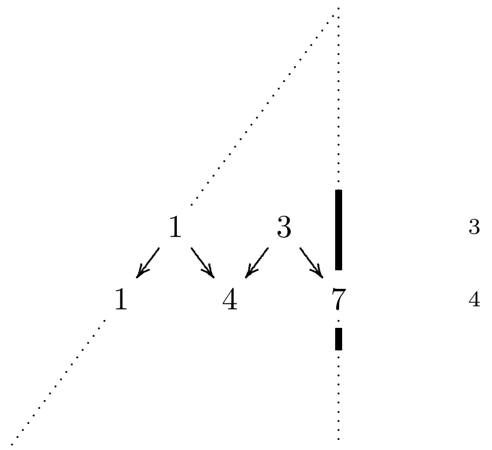
Here are the first polynomials $d_i(t)$:

$$\begin{aligned}
d_0(t) &= 1 = \binom{t}{0}, \\
d_1(t) &= t = \binom{t}{1}, \\
d_2(t) &= \tfrac{1}{2}(t^2+t-6) = \binom{t}{2} + \binom{t}{1} - 3\binom{t}{0}, \\
d_3(t) &= \tfrac{1}{6}(t^3+3t^2-22t-18) = \binom{t}{3} + 2\binom{t}{2} - 3\binom{t}{1} - 3\binom{t}{0}.
\end{aligned}$$

It would be interesting to know a general formula for the polynomials $d_i(t)$.

Looking back at the page showing the even-index triangle, we still have to explain the entries P_i in the left column. This concerns the Fibonacci modules $P_i = P_i(x)$ considered in [FR1] and [FR2], these are indecomposable representations of the quiver obtained from the 3-regular tree by choosing a bipartite orientation (if i is even, x has to be a sink, otherwise a source; in the following picture on the left, x is the vertex in the center). For example, the top of the Fibonacci module P_4 has length $f_8 = 21$ and its socle has length $f_{10} = 55$. The dimension vector of P_4 is of the form as shown below on the left; the corresponding two rows $t=3$ and $t=4$ of the triangle (copied here on the right) display one of the many walks in the support of P_4 which goes from the boundary to the center x.





To be precise, the numbers $d_i(t)$ are categorified by the modules $P_n = P_n(x)$ as follows: they provide the Jordan-Hölder multiplicities of these modules. First, let us consider the socle of P_t; the composition factors in the socle of P_t are of the form $S(z)$ with $0 \leq D(x,z) \leq t$ and $D(x,z) \equiv t \mod 2$, where, $D(x,z)$ denotes the distance between x, z in the 3-regular tree (and $S(z)$ is the one-dimensional representation with $\dim S(z)_z \neq 0$). There is the following multiplicity formula:

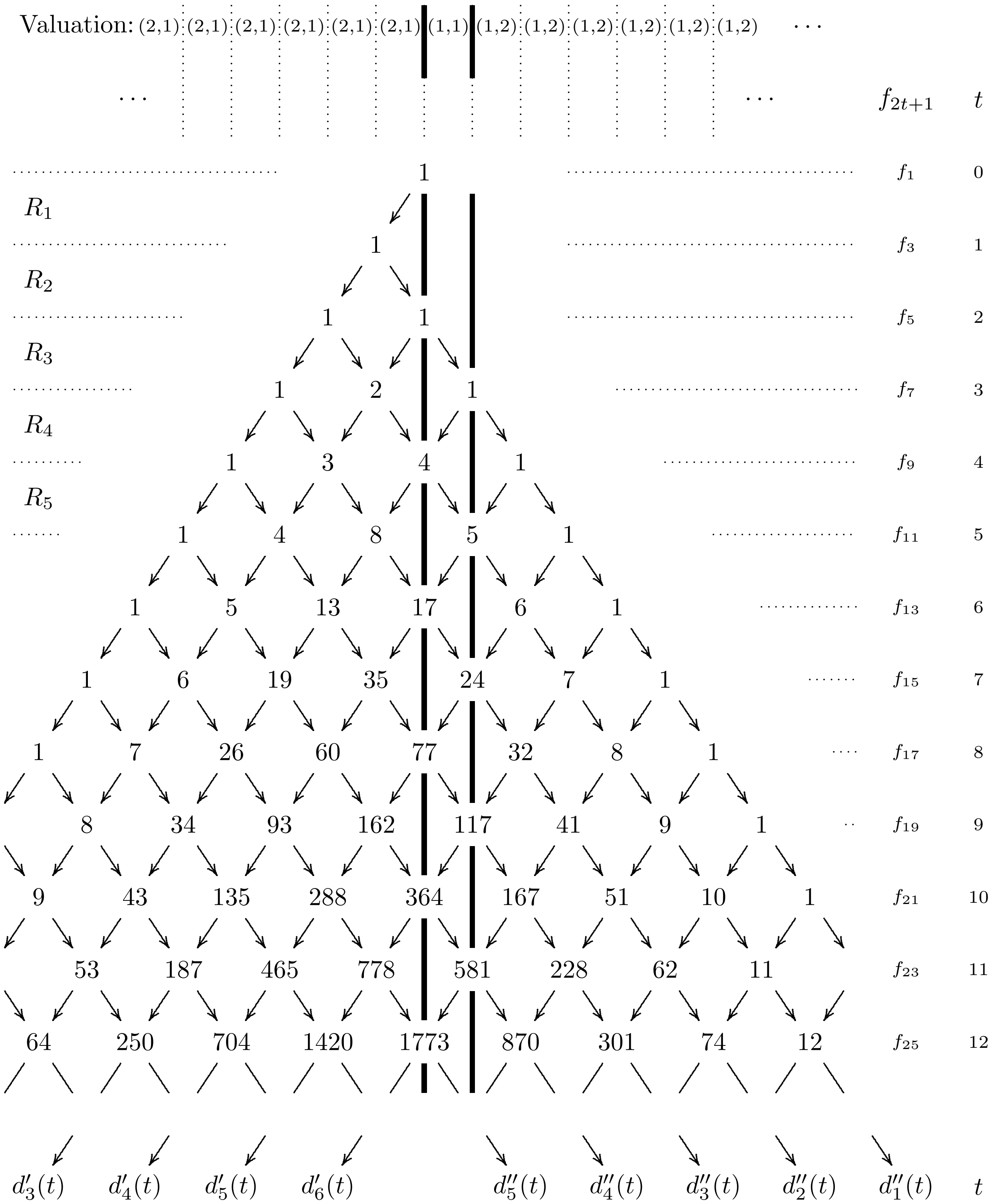
$$\boxed{d_i(t) = \dim P_t(x)_z \qquad \text{for} \quad D(x,z) = |t - 2i|.}$$

(Remark: In case we assume, as we usually do, that $i \leq \frac{t}{2}$, then we may just write $D(x,z) = t - 2i$. However, it is sometimes convenient to consider also the values $d_i(t)$ for $i > \frac{t}{2}$, where, by definition, $d_i(t) = d_{t-i}(t)$. There is the second convention that $d_i(t) = 0$ for $i < 0$ and all t; again, the rule above is valid also for $i < 0$, since $t - 2i > t$ for $i < 0$, and $\dim P_t(x)_z = 0$ in case $D(x,z) > t$.)

Note that the socle of $P_t(x)$ is isomorphic to the top of $P_{t+1}(x)$, thus $d_i(t)$ is also equal to the dimension of $\dim P_{t+1}(x)_z$ where $D(x,z) = |t - 2i|$.

We should add the following warning: the module $P_n(x)$ with n even is only defined in case x is a sink, for n odd, if x is a source; in particular, the modules $P_t(x)$ and $P_{t+1}(x)$ are **not** defined for the same quiver.

2. The odd-index Fibonacci partition triangle



Again, let us first explain the numbers displayed here. As before, the rows are indexed by $t = 0, 1, 2, \dots$ as shown on the right. Note that we deal here with a triangle only after removing the row $t = 0$; alternatively, we may interprete the display as being formed by two triangles, separated by the left pylon.

The row $t = 0$ consists of a single entry, namely $d'_0(0) = 1$ (and if one would like to deal with a proper triangle, one just could remove this entry).

The remaining rows have index $t \geq 1$, and in every such row there are precisely t entries, namely, from left to right, the numbers

$$d'_0(t), d'_1(t), \dots, d'_i(t), \dots, d'_{t-1}(t).$$

They are defined inductively as follows: We start with $d'_i(t) = 1$ for $i = 0$ and for $i = t-1$. The entries $d'_i(t)$ with $1 \leq i \leq \frac{t}{2}$ are given by the rule

$$d'_i(t) = 2d'_{i-1}(t-1) + d'_i(t-1) - d'_{i-1}(t-2),$$

those with $\frac{t}{2} < i < t-1$ by

$$d'_i(t) = d'_{i-1}(t-1) + 2d'_i(t-1) - d'_{i-1}(t-2)$$

(the two pylons mark the regions for the different recursion rules: the first one applies to the numbers on the left up to those on the left pylon, the second one applies to the numbers on the right starting with those on the right pylon).

It will be convenient to label the entries also from the right, thus let $d''_i(t) = d'_{t-i-1}(t)$ for $0 \leq i \leq t-1$. Usually, we will use the notation d' for the numbers up to the second pylon from the left, the notation d'' for those starting with the first pylon and going to the right. For example, for $t = 5$, the standard convention will be to use labels as follows:

$$d'_0(5) = 1, \quad d'_1(5) = 4, \quad d'_2(5) = 8, \quad d'_3(5) = d''_1(5) = 5, \quad d''_0(5) = 1.$$

If necessary, we will write $d'_i(t) = 0$ for $i < 0$. Also, let $d''_i(t) = 0$ for $i < 0$, unless $i = -1$ and $t = 0$. Using the rule $d''_i(t) = d'_{t-i-1}(t)$ we have defined in this way $d'_i(t)$ and $d''_i(t)$ for all $i \in \mathbb{Z}$ (and $t \geq 0$).

Note that the entries $d'_i(t)$ and $d''_i(t)$ displayed can also be calculated from the numbers $u_i[j]$ of the paper [FR2] according to the rule

$$\begin{aligned} d'_i(t) &= u_{\lceil t/2 \rceil}[t-2i], \\ d''_i(t) &= u_{\lceil t/2 \rceil}[-t+2+2i]. \end{aligned}$$

and we provide a visual concordance for comparing the notion used here with that of the previous paper [FR2].

Using this notation, the partition formulae of [FR2] again can be written in a unified form, as follows:

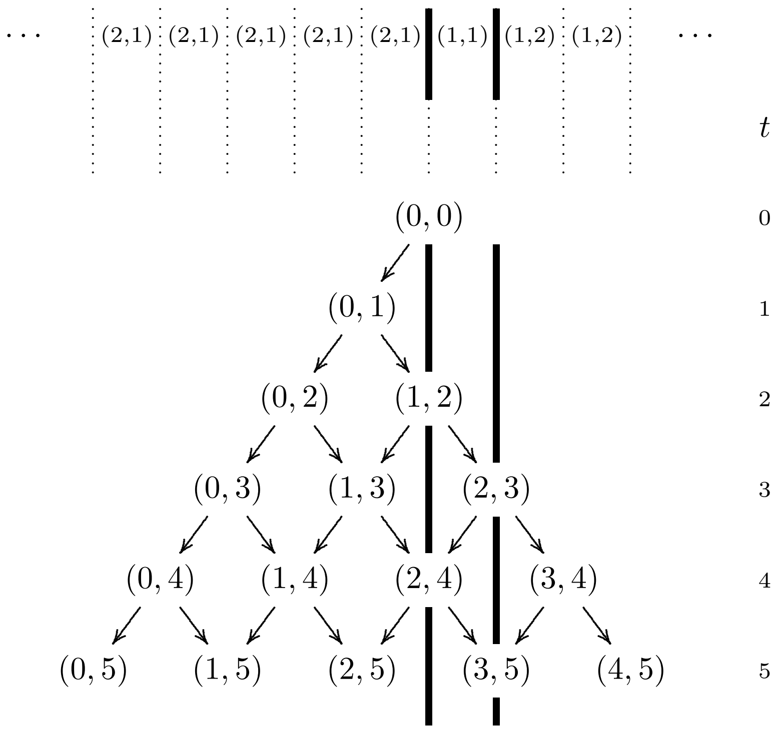
Partition formula for the Fibonacci numbers f_n with odd index n:

$$\boxed{\sum_{0\le i<t/2} 2^{t-2i} d'_i(t) + \sum_{0\le i<(t-3)/2} 2^{t-2i} d''_i(t) \;= f_{2t+1}}$$

Thus, *the row t of the triangle yields the Fibonacci number f_{2t+1}.* As examples, let us look at the rows $t=5$ and $t=6$, they yield

$$(t=5)\qquad (2^5\cdot 1+2^3\cdot 4+2^1\cdot 8)+(2^0\cdot 5+2^2\cdot 1)=\ \ 89=f_{11}$$
$$(t=6)\qquad (2^6\cdot 1+2^4\cdot 5+2^2\cdot 13+2^0\cdot 17)+(2^1\cdot 6+2^3\cdot 1)=233=f_{13}$$

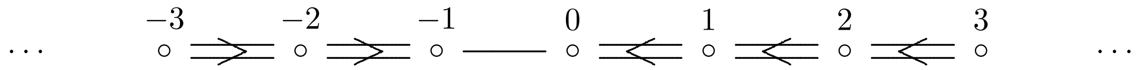
Let us describe in detail the translation quiver $\Gamma^{\rm odd}$ used here. Its vertices are the pair $(0,0)$ as well as all the pairs (i,t) with integers $0\le i<t$; there are the south-west arrows (i,t) to $(i,t+1)$ as well as the south-east arrows $(i,t)\to(i+1,t+1)$, where $(i,t)\ne(0,0)$. All the vertices of the form $(0,t)$ and $(i,i+1)$ with $i\ge 2$ are projective, and $\tau(i,t)=(i-1,t-2)$ for the remaining vertices. Now we mark two pylons: the first one consists of the vertices of the form $(i,2i)$, the second one of the vertices $(i,2i-1)$. All the arrows with both starting vertex and terminal vertex on a pylon have valuation $(1,1)$; the (south-east) arrows $(i,t)\to(i+1,t+1)$ with $2i<t$ have valuation $(2,1)$, the remaining south-east arrows have valuation $(1,2)$.



(Let us stress that this pair of pylons indicates completely different mesh rules than the single pylon used in the even-index triangle.)

The function d' is an additive function on $\Gamma^{\rm odd}$, this is the unique additive function on $\Gamma^{\rm odd}$ such that $d'_0(t)=1$ for all $t\ge 0$ and $d''_0(t)=1$ for all $t\ge 3$ (thus with values 1 on the projective vertices).

Remark. Note that Γ^{odd} can be considered as a subquiver of the valued translation quiver $\mathbb{Z}\Delta^{\mathrm{odd}}$, where Δ^{odd} is the valued graph



as mentioned at the end of section 3 of [FR2].

For additive functions g on Γ^{odd}, we write $g'_i(t) = g(i,t)$, usually for $2i \leq t+1$, and $g''_i(t) = g(t-1-i,t)$, usually for $2i \geq t$. There is again a hook characterization of additivity:

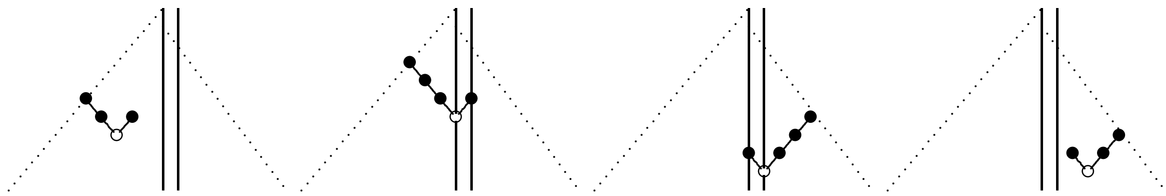
Proposition 2. *The function* $g\colon \Gamma^{\mathrm{odd}}_0 \to \mathbb{Z}$ *is additive on* Γ^{odd} *if and only if it satisfies the following hook conditions:*
(a) *If* $2i \leq t$, *then*

$$g'_i(t) = g'_i(t-1) + \sum_{0 \leq j < i} g'_j(t-i+j).$$

(b) *If* $2i > t$, *then*

$$g''_i(t) = g''_i(t-1) + \sum_{0 \leq j < i} g''_j(t-i+j).$$

The following pictures indicate the position of the hooks in the odd-index triangle (again, the value at the circle is obtained by adding the values at the bullets):



(a) (a) (b) (b)

Corollary 2. *The functions* d'_i, d''_i *are eventually polynomial of degree* i, *for any* $i \geq 0$. The corresponding polynomials are monic linear combinations of the binomial coefficients $\binom{t}{n}$.

To be precise: *The functions* $d'_i(t)$ *with* $t \geq 2i-1$ *and* $d''_i(t)$ *with* $t \geq 2i+2$ *are polynomials of degree* i.

The proofs of Proposition 2 and Corollary 2 are similar to those of Proposition 1 and Corollary 1.

As in the case of the even-index triangle, we see also for the odd-index triangle, that the pylon numbers determine completely all the other values of the triangle (we need only those for one of the two pylons).

Here are the first polynomials which occur for $d'_i(t)$ and $d''_i(t)$:

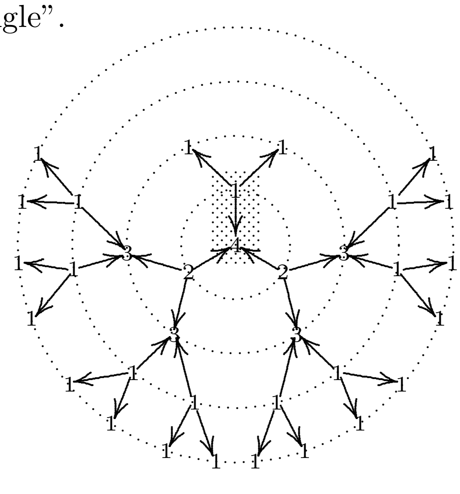
$$
\begin{aligned}
d'_0(t) &= 1 = \binom{t}{0},\\
d'_1(t) &= t-1 = \binom{t}{1} - \binom{t}{0},\\
d'_2(t) &= \tfrac{1}{2}(t^2-t-4) = \binom{t}{2} - 2\binom{t}{0},\\
d'_3(t) &= \tfrac{1}{6}(t^3-19t) = \binom{t}{3} + \binom{t}{2} - 3\binom{t}{1},
\end{aligned}
$$

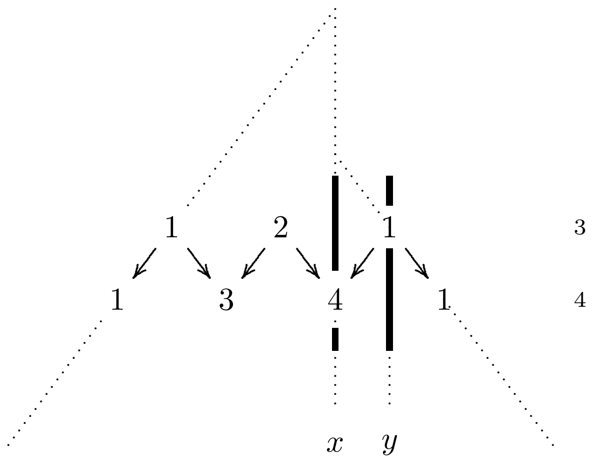
and

$$
\begin{aligned}
d''_0(t) &= 1 = \binom{t}{0},\\
d''_1(t) &= t = \binom{t}{1},\\
d''_2(t) &= \tfrac{1}{2}(t^2+t-8) = \binom{t}{2} + \binom{t}{1} - 4\binom{t}{0},\\
d''_3(t) &= \tfrac{1}{6}(t^3+3t^2-28t-18) = \binom{t}{3} + 2\binom{t}{2} - 4\binom{t}{1} - 3\binom{t}{0}.
\end{aligned}
$$

Again, we do not know a general formula for the polynomials $d'_i(t)$ and $d''_i(t)$.

Finally, the page with the odd-index Fibonacci partition triangle also refers to some Fibonacci modules, namely the modules $R_t = R_t(x, y)$ considered in [FR2]. Again, it should be sufficient to discuss one example in detail, say R_4. On the left, we exhibit the dimension vector of R_4, on the right the corresponding two rows $t = 3$ and $t = 4$ of the "triangle".





Whereas the left picture (copied from [FR2]) tries to put the emphasis on the vertex x as the center, the two pylons in the right picture stress that here we really deal with a distribution of numbers which depend on an edge, thus on two neighboring vertices, say x and y. Note that the two pylons in the right picture mark the positions of x and y, the corresponding edge is indicated in the left picture by the shaded region with the vertical arrow $y \to x$.

The numbers $d_i'(t)$ and $d_i''(t)$ are categorified by the modules $R_n = R_n(x,y)$; they provide the Jordan-Hölder multiplicities of these modules.

First, let us consider the socle of R_t; the composition factors in the socle of R_t are of the form $S(z)$ with $0 \le D(x,z) \le t$, $D(x,z) \equiv t \mod 2$, and $y \notin [x,z]$, or else with $0 < D(x,z) \le t-2$, $D(x,z) \equiv t \mod 2$, and $y \in [x,z]$; here, $[x,z]$ denotes the path between x and z. There is the following multiplicity formula:

$$d_i'(t) = \dim R_t(x,y)_z \qquad \text{for} \quad D(x,z) = \begin{cases} t-2i, & y \notin [x,z], \\ & \text{and} \\ 2i-t, & y \in [x,z], \end{cases}$$

We may reformulate the second case in terms of $d_i''(t)$ as follows:

$$d_i''(t) = \dim R_t(x,y)_z \qquad \text{for} \quad D(x,z) = t-2i-2,\ y \in [x,z].$$

(Remark: In the first displayed line, the condition $D(x,z) = t-2i$ implies that we consider only $2i \le t$, the condition $D(x,z) = t-2i-2$ in the reformulation implies that $2i \le t-2$. Using the convention that $d_i'(t) = 0 = d_i''(t)$ for $i < 0$ and all t, we see again that these rules are valid also for $i < 0$.)

Note that the socle of $R_t(x,y)$ is isomorphic to the top of $R_{t+1}(x,y)$, thus we can interprete $d_i'(t)$ and $d_i''(t)$ also as Jordan-Hölder multiplicities of composition factors in the top of $R_{t+1}(x,y)$,

3. Relations between the two triangles.

Let us outline in which way the two triangles determine each other: in this way, we see that they are intimately connected.

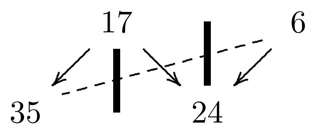
Theorem. For $t \geq 1$

$$\begin{aligned}
&(1) & d'_i(t) &= d_i(t) - d_{i-1}(t-1) && \text{for all } i,\\
&(2) & d''_i(t) &= d_{i+1}(t) - d_{i+1}(t-1) && \text{for all } i,\\
&(3) & d_i(t-1) &= d'_{i+1}(t) - d'_{i+1}(t-1) && \text{for } i \leq \tfrac{t-2}{2},\\
&(4) & d_i(t-1) &= d''_i(t) - d''_{i-1}(t-1) && \text{for } i \leq \tfrac{t-2}{2},\\
&(5) & d_i(t-1) &= d'_i(t) - d''_{i-2}(t-1) && \text{for } i \leq \tfrac{t}{2}.
\end{aligned}$$

Looking at the first four rules (1) to (4), one observes that the differences on the right always concern differences along arrows: one starts with a suitable arrow, say $a \to b$ in one of the two triangles, and with the given additive function g (either $g = d$, or $g = d'$) and looks at the difference $g(b) - g(a)$. In contrast, the rule (5) deals with differences of numbers often quite far apart; of special interest are the cases $t = 2i$ where one deals with the difference along a knight's move (known from chess):

$$d_i(2i) = d'_i(2i+1) \; - d''_{i-2}(2i).$$

Say for $i = 3$, one considers

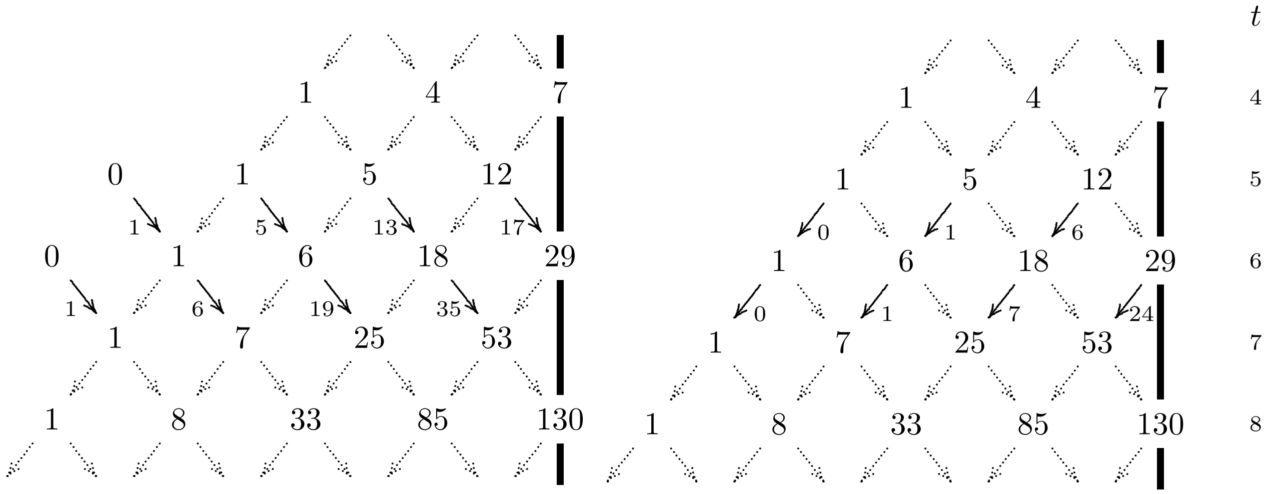


here, $29 = d_3(6) = d'_3(7) - d''_1(6) = 35 - 6$. These differences along knight's moves yield all the numbers on the pylon of the even-index triangle.

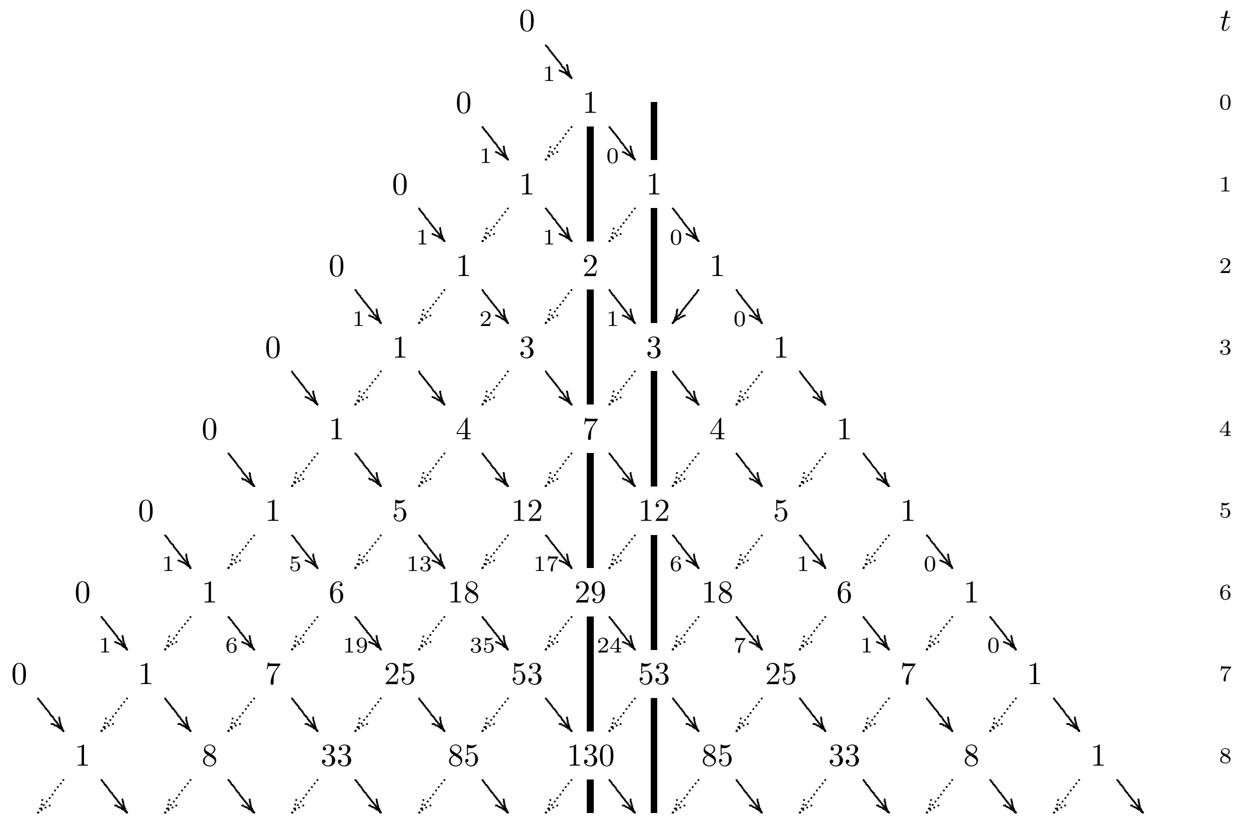
Starting with the even-index triangle, (1) and (2) assert that we obtain the numbers d'_i as differences along the south-west arrows, the numbers d''_i as differences along the south-east arrows. Similarly, the rules (3), (4) and (5) show how to obtain the numbers $d_i(t)$ of the even-index triangle from the odd-index triangle. Those outside of the pylon are obtained in three different ways, twice as differences along the south-west arrows, namely, on the one hand, looking at the arrows left of the second pylon, see (3), and, on the other hand, also by looking at the arrows right of the first pylon, see (4). But all the numbers of the even-index triangle are obtained using the rule (5).

Differences on arrows for the even-index triangle.

For the convenience of the reader, let us visualize the rules (1) and (2) by exhibiting part of the even-index triangle and inserting the difference numbers to some of the arrows:



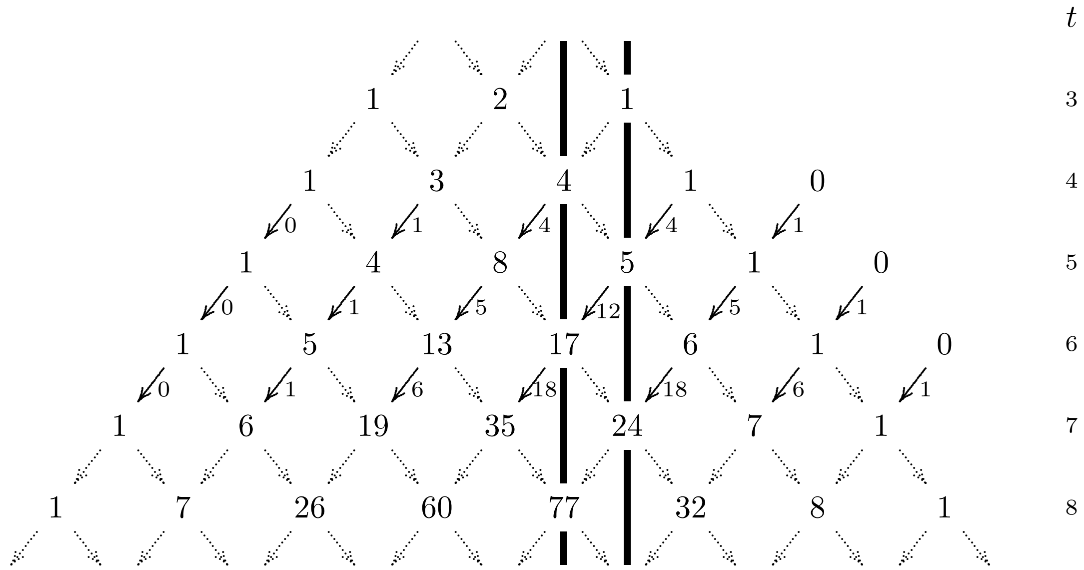
In a slightly different way, we may start with d as an additive function on $\mathbb{N}\Delta^{\text{odd}}$ and consider only the differences along the south-east arrows. (It is quite important to be aware that any additive function g on $\mathbb{N}\Delta^{\text{ev}}$ gives rise to an additive function on $\mathbb{N}\Delta^{\text{odd}}$, also denoted by g, by extending the given function via the rule $g(i,t) = g(t-i,t)$.) In the following display we have inserted the differences for all the south-east arrows ending in the layers $t = 0, 1, 2, 3$ and $t = 6, 7$.



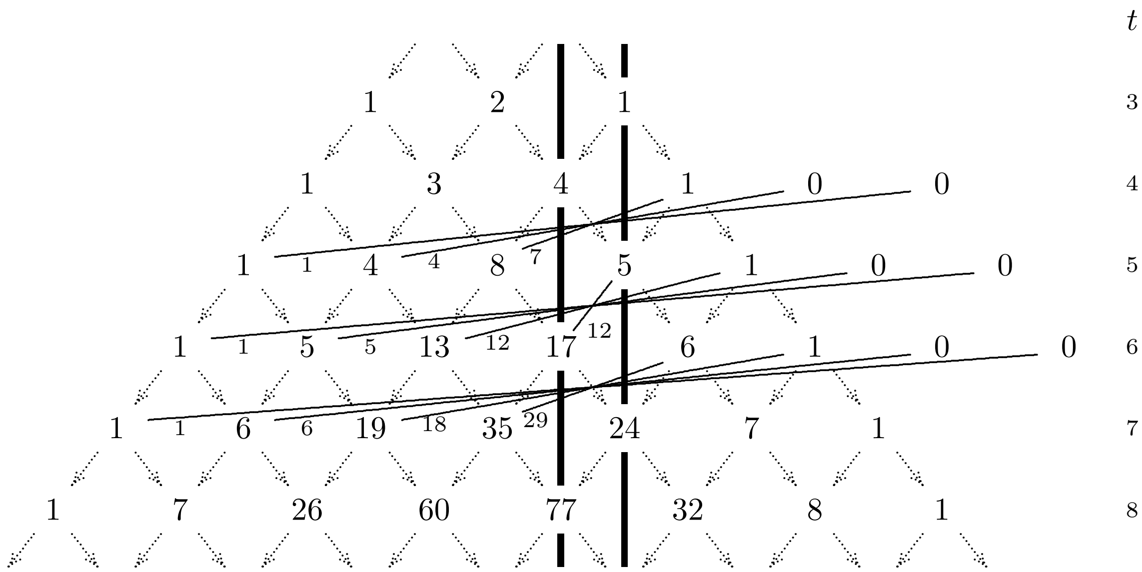
What do we see? Looking at the difference numbers, we obtain precisely the odd-index triangle. The only deviation may concern the position of the pylons; they should be shifted slightly to the left; but we can overcome this problem by inserting the difference number for any arrow $a \to b$ directly at the position b; then also the position of the pylons is correct.

Differences for the odd-index triangle.

In the even-index pictures, we have used differences along the given arrows (as well as additional ones on the left boundary), according to the rules (1) and (2). Now we start with the odd-index triangle, looking at the south-west arrows (as well as the corresponding arrows on the right boundary), see the rules (3) and (4):



The rule (5) deals with the following differences:



As we have mentioned already, of special interest seem to be the knight's moves, which yield the numbers $1, 2, 7, 29, 130, \dots$ on the pylon of the even-index triangle.

4. Consequences.

We see that *for the odd-index triangle, the entries on the left determine those on the right, and conversely, the entries on the right determine those on the left:*

Corollary 3.

$$\begin{aligned} d_i''(t) &= d_{i+2}'(t+1) - 2d_{i+2}'(t) + d_{i+2}'(t-1) \\ d_i'(t) &= d_i''(t+1) - 2d_{i-1}''(t) + d_{i-2}''(t-1) \end{aligned}$$

The first equality holds for $i \le \frac{t-4}{2}$ (and $t \ge 4$), the second for $i < \frac{t}{2}$ (and $t \ge 1$), thus for the vertices not lying on the pylons.

Proof: For the first equality, we use the formulae (2) and (3):

$$\begin{aligned} d_i''(t) &= d_{i+1}(t) - d_{i+1}(t-1) \\ &= \big(d_{i+2}'(t+1) - d_{i+2}'(t)\big) - \big(d_{i+2}'(t) - d_{i+2}'(t-1)\big) \\ &= d_{i+2}'(t+1) - 2d_{i+2}'(t) + d_{i+2}'(t-1). \end{aligned}$$

Similarly, for the second equality, we use (1) and (4):

$$\begin{aligned} d_i'(t) &= d_i(t) - d_{i-1}(t-1) \\ &= (d_i''(t+1) - d_{i-1}''(t)) - (d_{i-1}''(t) - d_{i-2}''(t-1)) \\ &= d_i''(t+1) - 2d_{i-1}''(t) + d_{i-2}''(t-1) \end{aligned}$$

These rules can be reformulated as follows, involving only vertices in two consecutive layers:

$$\begin{aligned} &\text{(N)} & d_i''(t) &= 2d_{i+1}'(t) - d_{i+1}'(t-1) - d_{i+2}'(t) + d_{i+2}'(t-1) \\ &\text{(N}'\text{)} & &= d_{i+2}'(t+1) + d_{i+3}'(t) - d_{i+3}'(t+1). \end{aligned}$$

$$\begin{aligned} &\text{(N)} & d_i'(t) &= 2d_{i-2}''(t) - d_{i-1}''(t+1) - d_{i-1}''(t) + d_i''(t-1) \\ &\text{(N}'\text{)} & &= d_{i-2}''(t-1) + d_i''(t) - d_{i-1}''(t-1). \end{aligned}$$

Remark: Our interest in the vertices of two consecutive layers comes from the fact that the dimension vectors of the modules R_t are obtained by looking at two consecutive layers.

Proof of the first N-rule:

$$\begin{aligned} d_i''(t) &= d_{i+2}'(t+1) - 2d_{i+2}'(t) + d_{i+2}'(t-1) \\ &= \big(2'd_{i+1}(t) + d_{i+2}'(t) - d_{i+1}'(t-1)\big) - 2d_{i+2}'(t) + d_{i+2}'(t-1) \\ &= 2d_{i+1}'(t) - d_{i+1}'(t-1) - d_{i+2}'(t) + d_{i+2}'(t-1) \end{aligned}$$

Proof of the first N′-rule:

$$\begin{aligned} d''_i(t) &= d'_{i+2}(t+1) - 2d'_{i+2}(t) + d'_{i+2}(t-1) \\ &= d'_{i+2}(t+1) - 2d'_{i+2}(t) + \big(2d'_{i+2}(t) + d'_{i+3}(t) - d'_{i+3}(t+1)\big) \\ &= d'_{i+2}(t+1) + d'_{i+3}(t) - d'_{i+3}(t+1) \end{aligned}$$

The remaining rules are shown in the same way.

There are also the following two summation formulae, adding up the values along a sequence of south-east arrows starting at the left boundary. In the case of the even-index triangle, the sequence has to stop before the pylon. There is no such restriction in the case of the odd-index triangle.

Corollary 4. (Summation formulae)

(a) *For all* $0 \le i \le \frac{t-1}{2}$

$$\sum_{0\le j\le i} d_j(t-i+j) = d''_i(t+1).$$

(b) *For all* $0 \le i$

$$\sum_{0\le j\le i} d'_j(t-i+j) = d_i(t).$$

In particular, since $d_i(t) = d_{t-i}(t)$, we see that

$$\sum_{0\le j\le i} d'_j(t-i+j) = \sum_{0\le j\le t-i} d'_j(t-i+j).$$

Proof: (a) If $i \le \frac{t-1}{2}$, and $0 \le j \le i$, then $j \le \frac{t-1}{2} \le \frac{t-i+j-1}{2}$, thus, according to (4),

$$d_j(t-i+j) = d''_j(t-i+j+1) - d''_{j-1}(t-i+j),$$

and therefore

$$\begin{aligned} \sum_{0\le j\le i} d_j(t-i+j) &= \sum_{0\le j\le i} \big(d''_j(t-i+j+1) - d''_{j-1}(t-i+j)\big) \\ &= d''_i(t+1) - d''_{-1}(t-i) = d''_i(t+1). \end{aligned}$$

Similarly, for (b) we use the formula (1).

5. Proof of Theorem.

First, let us note that the rules (1) and (2) are actually equivalent, due to the fact that $d_i(t) = d_{t-i}(t)$ and $d''_i(t) = d'_{t-i-1}(t)$. Namely, assume (1) is satisfied. Then

$$\begin{aligned} d''_i(t) &= d'_{t-i-1}(t) \\ &= d_{t-i-1}(t) - d_{t-i-2}(t-1) \\ &= d_{t-(t-i-1)}(t) - d_{(t-1)-(t-i-2)}(t-1) \\ &= d_{i+1}(t) - d_{i+1}(t-1), \end{aligned}$$

thus (2) holds. In the same way, one sees the opposite implication.

We are going to show that the theorem is a direct consequence of Proposition 4.1 in [FR2]. We fix some $t \geq 1$. If t is even, we assume that x is a sink, otherwise that x is a source. We choose the corresponding bipartite orientation on the 3-regular tree, this is the quiver whose representations will be considered.

First, let us consider the rule (1). Let y be a neighbor of x. According to [FR2], there is an exact sequence

$$0 \to P_{t-1}(y) \to P_t(x) \to R_t(x,y) \to 0,$$

thus for any vertex z of the quiver, there is an exact sequence of vector spaces

$$0 \to P_{t-1}(y)_z \to P_t(x)_z \to R_t(x,y)_z \to 0.$$

This means that

$$\dim R_t(x,y)_z = \dim P_t(x)_z - \dim P_{t-1}(y)_z.$$

The only vertices z to be considered are those with $D(x,z) \equiv t \mod 2$. We denote by $[x,z]$ the path between x and z. We have to distinguish whether y belongs to $[x,z]$ or not.

First, consider the case $y \notin [x,z]$. Let $i = \frac{1}{2}(t - D(x,z))$, thus $2i \leq t$ (since $t - 2i = D(x,z) \geq 0$) and

$$d'_i(t) = \dim R_t(x,y)_z = \dim P_t(x)_z - \dim P_{t-1}(y)_z.$$

As we know, $\dim P_t(x)_z = d_i(t)$, thus it remains to calculate $\dim P_{t-1}(y)_z$. Since $y \notin [x,z]$, the path from y to z runs through x, thus

$$D(y,z) = 1 + D(x,z) = 1 + t - 2i = (t-1) - 2(i-1),$$

and consequently

$$\dim P_{t-1}(y)_z = d_{i-1}(t-1).$$

Next, consider the case $y \in [x,z]$ and let $i = \frac{1}{2}(D(x,z) + t)$, thus $D(x,z) = 2i - t$ and therefore

$$d'_i(t) = \dim R_t(x,y)_z.$$

Since $y \in [x,z]$, we have

$$D(y,z) = -1 + D(x,z) = -1 + 2i - t,$$

therefore

$$D(y,z) = |-1 + 2i - t| = |t - 2i + 1| = |(t-1) - 2(i-1)|,$$

but this means again that

$$\dim P_{t-1}(y)_z = d_{i-1}(t-1).$$

In both cases, we have shown that

$$d'_i(t) = \dim R_t(x,y)_z = \dim P_t(x)_z - \dim P_{t-1}(y)_z = d_i(t) - d_{i-1}(t).$$

This completes the proof of (1).

Next we show the three rules (3), (4), (5).

Let y, y', y'' be the neighbors of x. According to [FR2], there is an exact sequence

$$0 \to P_{t-1}(y') \to R_t(x,y) \to R_{t-1}(y'',x) \to 0,$$

thus for any vertex z of the quiver, there is an exact sequence of vector spaces

$$0 \to P_{t-1}(y')_z \to R_t(x,y)_z \to R_{t-1}(y'',x)_z \to 0,$$

This means that

$$(*) \qquad \dim P_{t-1}(y')_z = \dim R_t(x,y)_z - \dim R_{t-1}(y'',x)_z.$$

Again, we consider only the vertices z with $D(x,z) \equiv t \mod 2$.

For the proof of (3) and (4), we consider vertices z with $y' \notin [x,z]$, say with $D(x,z) = t-2i-2$. We claim that

$$\dim P_{t-1}(y')_z = d_i(t-1).$$

In order to verify this equality, we have to show that $D(y',z) = |t-1-2i|$. But $D(y',z) = 1 + D(x,z) = 1 + t - 2(i+1) = t-1-2i$.

In order to establish the rule (3), we start with a vertex z such that either $z = x$ or else $y'' \in [x,z]$, thus in both cases $y \notin [x,z]$. Let $i = \frac{1}{2}(t - D(x,z)) - 1$, thus $D(x,z) = t-2i-2$ and

$$d'_{i+1}(t) = \dim R_t(x,y)_z.$$

Next, let us show that

$$d'_{i+1}(t-1) = \dim R_{t-1}(y'',x)_z$$

This holds true in case $z = x$: namely, then $x \in [y'',z]$ and $D(y'',x) = 1 = 2(i+1) - (t-1)$ (since $0 = D(x,z) = t - 2(i+1)$). If $z \neq x$, then $x \notin [y'',z]$ and $D(y'',z) = D(x,z) - 1 = t - 2(i+1) - 1 = (t-1) - 2(i+1)$.

Altogether, we see that $(*)$ yields the required equality

$$d_i(t-1) = d'_{i+1}(t) - d'_{i+1}(t-1).$$

Let us discuss the rule (4). The case $i = \frac{t-2}{2}$ follows from (3), since for this value of i, we have both $d''_i(t) = d'_{i+1}(t)$ as well as $d''_{i-1}(t-1) = d'_{i+1}(t-1)$. Thus, we only have to consider the cases $i < \frac{t-2}{2}$.

We start with a vertex z such that $y \in [x,z]$. Let $i = \frac{1}{2}(t - D(x,z) - 2)$, thus $D(x,z) = t-2i-2$ and

$$d''_i(t) = \dim R_t(x,y)_z.$$

Note that $x \in [y'',z]$ and $D(y'',z) = 1 + D(x,z) = t-2i-1 = (t-1) - 2(i-1) - 2$, thus

$$d''_{i-1}(t-1) = \dim R_{t-1}(y'',x)_z.$$

It remains to show that $\dim P_{t-1}(y')_z = d_i(t-1)$.

Since $D(x,z) = t-2i-2$, we know that $\dim P_{t-1}(y')_z = d_i(t-1)$, thus $(*)$ yields the required equality

$$d_i(t-1) = d''_i(t) - d''_{i-1}(t-1).$$

The final considerations concern the rule (5). The case $i = \frac{t}{2}$ of (5) coincides with the case $j = \frac{t-2}{2}$ of (3) and (4), thus we only have to establish the cases $i < \frac{t}{2}$.

We consider a vertex z such that $y' \in [x,z]$ and let $i = \frac{1}{2}(t - D(x,z))$, thus $D(x,z) = t-2i$ and $D(y',z) = -1 + D(x,z) = t-2i-1$. Then

$$\dim P_{t-1}(y')_z = d_i(t-1),$$

since $D(y',z) = t-1-2i$. We have

$$d'_i(t) = \dim R_t(x,y)_z,$$

since $y \notin [x,z]$ and $D(x,z) = t-2i$. And we have

$$d''_{i-2}(t-1) = \dim R_{t-1}(y'',x)_z,$$

since $x \in [y'',z]$ and $D(y'',z) = 1 + D(x,z) = t-2i+1 = (t-1)-2(i-2)-2$. Altogether, we see again that $(*)$ yields the required equality, namely now

$$d_i(t-1) = d'_i(t) - d''_{i-2}(t-1).$$

This completes the proof.

6. Questions and remarks.

6.1. The south-west arrows of the odd-index triangle. The relationship between the two triangles was established using differences between numbers, mostly along arrows. In this way, all the arrows of the even-index triangle, and all the south-west arrows of the odd-index triangle have been used.

It seems to be of interest to understand also the meaning of the differences along the sout-east arrows in the odd-index triangle. Here are the numbers $d'_i(t) - d'_{i-1}(t-1)$, for $0 \le t \le 12$:

1										
1										
1										
1	1									
1	2	2								
1	3	5	1							
1	4	9	9	1						
1	5	14	22	7	1					
1	6	20	41	42	8	1				
1	7	27	67	102	40	9	1			
1	8	35	101	195	202	50	10	1		
1	9	44	144	330	490	217	61	11	1	
1	10	54	197	517	955	995	289	73	12	1

It should be a challenge to find some interpretation of this set of numbers.

6.2. The difference operator for sequences. Some of the results presented here, may be reformulated using the following operators Δ and E. Given a function $u\colon \mathbb{N}_0 \to \mathbb{R}$, we define new functions on $\mathbb{N}_0$ as follows

$$
\begin{aligned}
(\Delta u)(t) &= u(t+1) - u(t) \\
(Eu)(t) &= u(t+1),
\end{aligned}
$$

the operator Δ is called the difference operator, E the shift operator (see [E]). For example, the sequence f of Fibonacci numbers satisfies the condition $\Delta E f = f$. Let us stress that we have used the difference operator in the proof of the Corollaries. Here we note that the assertions (2) and (3) concern the difference and the shift operators, as follows:

$$
\begin{aligned}
Ed''_i &= \Delta d_{i+1}, \\
d_i &= \Delta d'_{i+1}.
\end{aligned}
$$

The first assertion of Corollary 3 concerns the following identity:

$$Ed''_i = \Delta d_{i+1} = \Delta^2 d'_{i+2}.$$

6.3. Group actions on quivers and valued quivers.

The valued quivers which have been considered in the paper are derived from group actions an the 3-regular tree T. If x, y is a pair of neighboring vertices, we may consider the following groups of automorphisms of T. Let G_x denote the group of automorphisms which fix x, let G_{xy} denote the group of automorphisms which fix both x and y. Then the functions on T_0 which are G_x-invariant may be identified with the functions on Δ^{ev}, whereas the functions on T_0 which are G_{xy}-invariant may be identified with the functions on Δ^{odd}.

6.4. Left hammocks.

We should mention that the additive functions d, d' exhibited here are typical left hammock functions, see [RV].

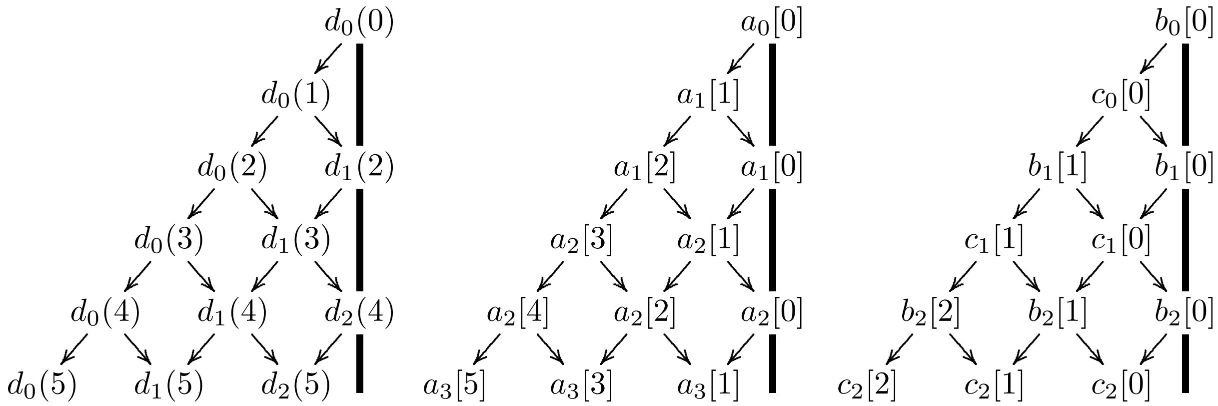
6.5. Delannoy paths.

Hirschhorn [H] has shown that the second last column of the even-index triangle (with numbers $1, 3, 12, 53, 247, 1192, \dots$) is just the sequence A110122 in Sloane's On-Line Encyclopedia of Integer Sequences, it counts the number of the Delannoy paths from $(0,0)$ to (n,n) which do not cross horizontally the diagonal $x = y$ (we recall that a Delannoy path is a sequence of steps $(1,0), (1,1), (0,1)$ in the plane (thus going north, northeast and east); and such a path is said to cross the diagonal $x = y$ horizontally provided it contains a subpath of the form $(m-1, m) \to (m, m) \to (m+1, m)$).

Hirschhorn's proof is computational and does not provide an intrinsic relationship between the Delannoy paths in question and say suitable elements of the Fibonacci modules. It seems to be of interest to establish a direct relationship.

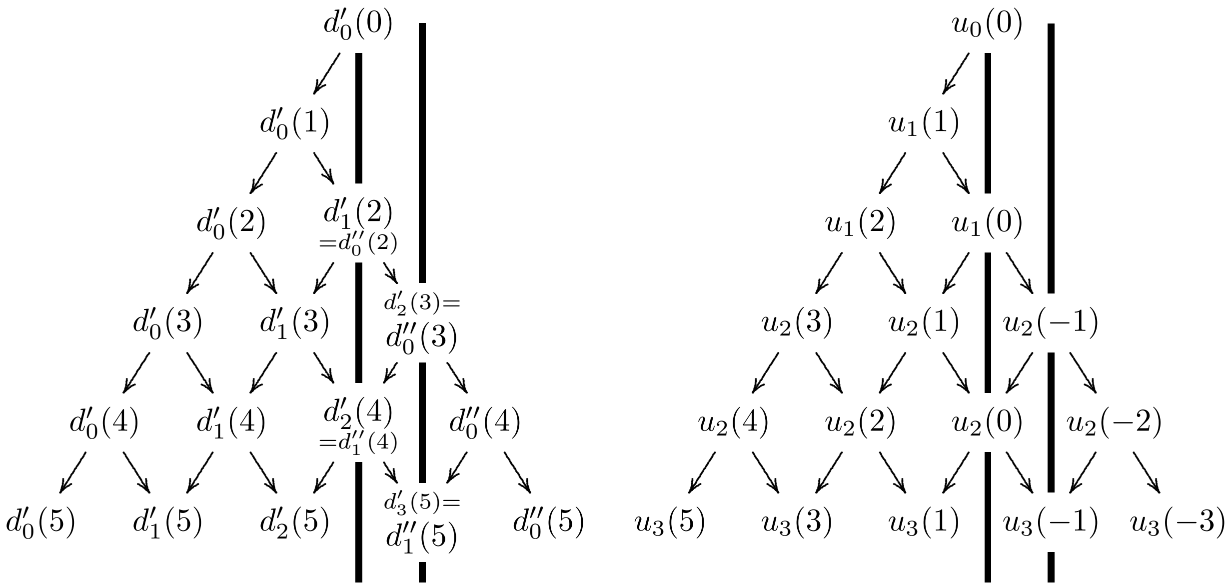
6.6. Concordance.

As we have mentioned, the numbers displayed in the even-index triangle have been considered already in the paper [FR1], as well in subsequent publications by other authors. In [FR1], these numbers have also been denoted by $b_i[j]$ and $c_i[j]$; for the convenience of the reader, let us present the different notations for the numbers of the triangle:



present notation — section 2 of [FR1] — section 4 of [FR1]

Similarly, let us consider the odd-index triangle. Again, for the convenience of the reader, let us compare the new notation (left) with the notation used in the paper [FR2] (right):



7. References

[E] S. N. Elaydi: An Introduction to Difference Equations. Springer (1996).

[F] Ph. Fahr. Infinite Gabriel-Roiter measures for the 3-Kronecker quiver. Dissertation Bielefeld. (2008).

[FR1] Ph. Fahr, C.M. Ringel: A partition formula for Fibonacci numbers, J. Integer Sequences, 11 (2008), Paper 08.1.4.

[FR2] Ph. Fahr, C.M. Ringel: Categorification of the Fibonacci Numbers Using Representations of Quivers. Journal Integer Sequences (to appear).

[H] M.D. Hirschhorn: On Recurrences of Fahr and Ringel Arising in Graph Theory. J. Integer Sequences, 12 (2009), Paper 09.6.8.

[K] Th. Koshy: Fibonacci and Lucas Numbers with Applications. John Wiley. (2001).

[HPR] D. Happel, U. Preiser, C.M. Ringel: Vinberg's characterization of Dynkin diagrams using subadditive functions with applications to DTr-periodic modules. In: Representation Theory II (ed V. Dlab, P. Gabriel), Springer LNM 832 (1980), 280-294.

[RV] C.M. Ringel, D. Vossieck: Hammocks. Proc. London Math. Soc. (3) 54 (1987), 216-246.

[W] Wikipedia: Fibonacci number. http://en.wikipedia.org/wiki/Fibonacci_number

Philipp Fahr
Brehmstr. 51
D-40239 Düsseldorf, Germany
e-mail: philfahr@gmx.net

Claus Michael Ringel
Fakultät für Mathematik, Universität Bielefeld
POBox 100 131, D-33 501 Bielefeld, Germany, and
King Abdulaziz University, P O Box 80200
Jeddah, Saudi Arabia
e-mail: ringel@math.uni-bielefeld.de