Generic representations of wild quivers.

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Abstract. Let Δ be a wild connected directed quiver. We show that any generic representation M is the union of its subrepresentations of finite length which are regular. As a consequence, we see that the direct limit closure of the preprojective component does not contain any generic module.

Let Δ be a finite quiver which is connected and directed. We consider representations of Δ with coefficients in the field k, or, what is the same, Λ-modules, where $\Lambda = k\Delta$ is the path algebra of Δ. Note that the path algebra of a quiver is hereditary, and since we assume that Δ is directed, Λ is finite-dimensional.

Given a module M, let $E(M) = \mathrm{End}(M)^{\mathrm{op}}$. We may consider M as an $E(M)$-module. The module M is said to be *endo-finite*, provided it is of finite length when considered as an $E(M)$-module.

Theorem. *Let Δ be a connected directed finite quiver. Let M be an indecomposable endo-finite representation of infinite length. If Δ is wild, then M is the union of its subrepresentations of finite length which are regular.*

Corollary. *Let Δ be a connected directed finite quiver. If Δ is wild, then the direct limit closure of the preprojective component contains no indecomposable endo-finite modules of infinite length.*

This answers a question raised by Henning Krause, see [K] for consequences. Indecomposable endo-finite modules of infinite length are sometimes called *generic* modules. If Δ is representation-finite, then there is no generic module. If Δ is tame, then there is a unique generic module M and $\mathrm{Hom}(R, M) = 0$ for any finite length module R which is regular. In this case, M is in the direct limit closure of the preprojective component.

The main tool for the proof is the endo-length vector $\mathbf{Dim}\, M$ of an endo-finite module M which will be introduced in section 2. In section 3, we first will show that given a generic module M, there is a regular module R with $\mathrm{Hom}(R, M) \neq 0$, and we use this result in order to provide the proof of the Theorem.

1. Preliminaries on finite length modules.

Let Δ_0 be the set of vertices, Δ_1 the set of arrows of the quiver Δ. Given an arrow α, let $s(\alpha)$ be its starting vertex, $t(\alpha)$ its terminal vertex. Let $\Lambda = k\Delta$ and $\operatorname{Mod}\Lambda$ be the category of Λ-modules, and $\operatorname{mod}\Lambda$ the full subcategory of all Λ-modules of finite length. We denote by $K_0(\Lambda)$ the Grothendieck group of $\operatorname{mod}\Lambda$ (with respect to all exact sequences), we may identify it with the set of functions $\Delta_0 \to \mathbb{Z}$, thus with $\mathbb{Z}^n$ if Δ_0 has cardinality n. Given a module M of finite length, the corresponding element in $K_0(\Lambda)$ is called its dimension vector and denoted by $\mathbf{dim}\, M$, the coefficient $(\mathbf{dim}\, M)_i$ for $i \in \Delta_0$ is the Jordan-Hölder multiplicity of the simple module $S(i)$ in M. For $x = (x_i)_i$, $y = (y_i)_i \in K_0(\Lambda)$, one defines

$$\langle x, y \rangle = \sum_{i \in \Delta_0} x_i y_i - \sum_{\alpha \in \Delta_1} x_{s(\alpha)} y_{t(\alpha)},$$

and one obtains in this way an integral bilinear form $\langle -, - \rangle$ on $K_0(\Lambda)$. Since

$$\langle \mathbf{dim}\, X, \mathbf{dim}\, Y \rangle = \dim_k \operatorname{Hom}(X, Y) - \dim_k \operatorname{Ext}^1(X, Y),$$

(and $\operatorname{Ext}^i(X, Y) = 0$ for $i \geq 2$) the form $\langle -, - \rangle$ is called the *Euler form.*

We denote by τ the Auslander-Reiten translation on the category of $\operatorname{mod}\Lambda$. Recall that an indecomposable module X of finite length is called preprojective or preinjective provided $\tau^t X = 0$ or $\tau^{-t} X = 0$, respectively, for some natural number t. A module will be said to be *regular* provided it has finite length and has no indecomposable direct summand which is preprojective or preinjective. We denote by Φ the Coxeter transformation on $K_0(\Lambda)$, it is a linear transformation and $\mathbf{dim}\, \tau X = \Phi\, \mathbf{dim}\, X$ for any indecomposable non-projective module X of finite length.

We say that a module X is *in general position* provided $X = X' \oplus X''$ implies that $\operatorname{Ext}^1(X', X'') = 0$. If x is a dimension vector, then the modules X with $\mathbf{dim}\, X = x$ such that $\dim_k \operatorname{End}(X)$ is minimal, are in general position.

Lemma 1. *The following conditions are equivalent for $x \in K_0(\Lambda)$.*
 (i) $\Phi^t(x) \geq 0$ *for all $t \in \mathbb{Z}$.*
 (ii) *The finite length modules in general position with dimension vector x are regular.*
(iii) *There exists a regular module with dimension vector x.*

Note that in (iii) we cannot expect that there exists a regular module which is indecomposable, typical examples are the elements in the Φ-orbits of $(1,1,0,1,1)$ and $(1,1,3,1,1)$ for the quiver

$$\circ \rightleftarrows \circ \longleftarrow \circ \longleftarrow \circ \rightleftarrows \circ$$

Let us call $x \in K_0(\Lambda)$ *regular* provided the equivalent conditions of Lemma 1 are satisfied.

Proof of Lemma 1: (i) $\implies$ (ii): Let X be a module in general position and $\mathbf{dim}\, X = x$. Let us assume that X has an non-zero preprojective direct summand, say let $X = X' \oplus \tau^{-t} P$, where P is a non-zero projective module, $t \geq 1$. We can assume that X' has no non-zero direct summand of the form $\tau^{-s} P'$ with P' projective and $0 \leq s \leq t$. With X also $\tau^t X = \tau^t X' \oplus P$ is in general position, thus

$$0 = \operatorname{Ext}^1(\tau^t X', P) = D \operatorname{Hom}(P, \tau^{t+1} X')$$

(here, D denotes the k-duality). This means that $(\mathbf{dim}\,\tau^{t-1}X')_j = 0$ for every vertex j such that the corresponding indecomposable projective module $P(j)$ is a direct summand of P. Now

$$\Phi^{t+1}\,\mathbf{dim}\,X = \Phi^{t+1}\,\mathbf{dim}\,X' - \mathbf{dim}\,\nu P = \mathbf{dim}\,\tau^{t-1}X' - \mathbf{dim}\,\nu P,$$

where ν is the Nakayama functor (it sends $P(i)$ to the corresponding indecomposable injective module $I(i)$). Since P is non-zero, there is a vertex j such that $P(j)$ is a direct summand of P, therefore $I(j)$ is a direct summand of νP. Since $(\mathbf{dim}\,\tau^{t-1}X')_j = 0$ and $(\mathbf{dim}\,\nu P)_j > 0$, we see that $\Phi^{t+1}\,\mathbf{dim}\,X$ has a negative coefficient.

Using duality, we similarly see that X has no indecomposable preinjective direct summand. Therefore X is regular.

(iii) $\implies$ (i) follows from the fact that for a regular module R, also τR and $\tau^- R$ are regular and $\Phi(\mathbf{dim}\,R) = \mathbf{dim}\,\tau R$ and $\Phi^{-1}(\mathbf{dim}\,R) = \mathbf{dim}\,\tau^- R$.

Lemma 2. *Let x be a non-zero regular element of $K_0(\Lambda)$. Then $\langle \Phi^{-t}x, x \rangle > 0$ for $t \gg 0$.*

Proof: Let X be a non-zero regular module with $\mathbf{dim}\,X = x$ and apply Baer [B], Proposition 2.1 for $X = S$.

2. The endo-length vector $\mathbf{Dim}\,M$ of an endo-finite module M.

In [R3], for any endo-finite module M whose endomorphism ring is a division ring an element $\mathbf{Dim}\,M$ in $K_0(\Lambda)$ has been defined (in a similar setting, Lenzing [L] has called such an invariant $\mathbf{Dim}\,M$ the "characteristic class" of M). We extend the definition from [R3] to arbitrary endo-finite modules. Note that for any representation $M = (M_i, M_\alpha)$ of Δ, the vector spaces M_i are $E(M)$-modules. If M is endo-finite, all the $E(M)$-modules M_i have finite length $|_{E(M)}M_i|$ and $|_{E(M)}M| = \sum_{i \in \Delta_0} |_{E(M)}M_i|$. We introduce the *endo-length vector* $\mathbf{Dim}\,M$ as the function $\Delta_0 \to \mathbb{Z}$ with

$$(\mathbf{Dim}\,M)_i = |_{E(M)}M_i|,$$

thus $\mathbf{Dim}\,M$ is an element of $K_0(\Lambda)$.

Remark. If M is an indecomposable module of finite length, then both $\mathbf{dim}\,M$ and $\mathbf{Dim}\,M$ are defined, and are multiples of each other, namely we have $\mathbf{dim}\,M = \dim_k \underline{\mathrm{End}}M \cdot \mathbf{Dim}\,M$, where $\underline{\mathrm{End}}M = \mathrm{End}\,M/\mathrm{rad}\,\mathrm{End}\,M$. Of course, in case k is an algebraically closed field, $\underline{\mathrm{End}}M = k$ for all indecomposable modules M, thus $\mathbf{dim}\,M = \mathbf{Dim}\,M$. But if k is not algebraically closed, then already for the Kronecker quiver there exist indecomposable modules of finite length with $\dim_k \underline{\mathrm{End}}M > 1$.

Recall that Bernstein-Gelfand-Ponomarev [BGP] have defined reflection functors. If i is a vertex of Δ, the quiver $\sigma_i\Delta$ is obtained from Δ by changing the orientation of all the arrows starting or ending in i. Given a sink i of the quiver, there is the reflection functor $\sigma_i \colon \mathrm{Mod}\,k\Delta \to \mathrm{Mod}\,k(\sigma_i\Delta)$, it provides an equivalence between the full subcategory of $\mathrm{Mod}\,k\Delta$ of all $k\Delta$-modules without direct summands $S(i)$ and the full subcategory of

Mod $k(\sigma_i \Delta)$ of all $k(\sigma_i \Delta)$-modules without direct summands $S(i)$. Similarly, for i a sink, there is the corresponding reflection functor $\sigma_i \colon \operatorname{Mod} k\Delta \to \operatorname{Mod} k(\sigma_i \Delta)$. In addition, we also denote by σ_i the reflection $\sigma_i \colon \mathbb{Z}^n \to \mathbb{Z}^n$ such that $\mathbf{dim}\, \sigma_i M = \sigma_i \mathbf{dim}\, M$ for any finite length module M without a direct summand of the form $S(i)$ (and i a sink or a source).

Lemma 3. *Let M be a generic module. If i is a sink of the quiver Δ, then $\mathbf{Dim}\, \sigma_i M = \sigma_i \mathbf{Dim}\, M$. If i is a source of the quiver Δ, then $\mathbf{Dim}\, \sigma_i^- M = \sigma_i \mathbf{Dim}\, M$.*

Proof: We only discuss the case of i being a sink. By definition, $(\sigma_i M)_i$ is the kernel of the map

$$\bigoplus_{t(\alpha)=i} M_{s(\alpha)} \to M_i$$

whose restriction to $M_{s(\alpha)}$ is M_α. Since M has no direct summand of the form $S(i)$, this map is surjective. Also, this is an $E(M)$-module homomorphism. Altogether, we see that we deal with the exact sequence

$$0 \to (\sigma_i M)_i \to \bigoplus_{t(\alpha)=i} M_{s(\alpha)} \to M_i \to 0$$

of $E(M)$-modules. Looking at the length of these $E(M)$-modules, we have

$$|_{E(M)}(\sigma_i M)_i| = \sum_{t(\alpha)=i} |_{E(M)} M_{s(\alpha)}| - |_{E(M)} M_i| = (\sigma_i \mathbf{Dim}\, M)_i.$$

This shows that $\mathbf{Dim}\, \sigma_i(M) = \sigma_i \mathbf{Dim}\, M$.

If we label the vertices of Δ as $\Delta_0 = \{1, 2, \ldots, n\}$ so that there is no arrow $i \to j$ for $i \le j$, then the composition $\Phi = \sigma_n \cdots \sigma_1$ of the reflection functors σ_i is defined and is called *Coxeter functor*, the corresponding composition $\Phi = \sigma_n \cdots \sigma_1$ of the reflections $\sigma_i \colon \mathbb{Z}^n \to \mathbb{Z}^n$ is the Coxeter transformation mentioned already.

Lemma 4. *Let M be a generic module. Then ΦM, $\Phi^{-1} M$ are generic modules and*

$$\mathbf{Dim}\, \Phi M = \Phi \mathbf{Dim}\, M, \quad \mathbf{Dim}\, \Phi^{-1} M = \Phi^{-1} \mathbf{Dim}\, M.$$

Proof. This follows immediately from the fact that M has no non-zero projective direct summands.

Lemma 5. *Let M be a generic module. Then $\mathbf{Dim}\, M$ is a non-zero regular element of $K_0(\Lambda)$.*

Proof: Let $x = \mathbf{Dim}\, M$. According to Lemma 4, all the vectors $\Phi^t x$ are endo-length vectors of non-zero modules, thus non-negative.

Lemma 6. *Let X be of finite length and M endo-finite. Then there is an exact sequence of $E(M)$-modules*

$$0 \to \operatorname{Hom}(X, M) \to \bigoplus_i \operatorname{Hom}_k(X_i, M_i) \xrightarrow{\delta_{XM}} \bigoplus_\alpha \operatorname{Hom}_k(X_{s(\alpha)}, M_{t(\alpha)}) \to \operatorname{Ext}^1(X, M) \to 0.$$

Proof. Let us refer to [R1] where the special case of both X, M being of finite length has been considered, but without taking into account the $E(M)$-module structure. As in the special case, the map δ_{XM} is defined in general as follows: it sends an element $f = (f_i)_i$ with $f_i \in \mathrm{Hom}_k(X_i, M_i)$ to $\delta_{XM}(f)$ with components $(\delta_{XM}(f))_\alpha = f_{t(\alpha)} X_\alpha - M_\alpha f_{s(\alpha)}$. It is clear that this map is an $E(M)$-homomorphism. Now it is trivial to verify that the kernel of δ_{XM} is just $\mathrm{Hom}(X, M)$, thus only the assertion that the cokernel of δ_{XM} is equal to $\mathrm{Ext}^1(X, M)$ has to be shown. The proof given in [R1] remains true in our more general setting.

As an immediate consequence we obtain:

Lemma 7. *Let X be of finite length and M endo-finite. Then*

$$\langle \mathbf{dim}\, X, \mathbf{Dim}\, M \rangle = |_{E(M)}\, \mathrm{Hom}(X, M)| - |_{E(M)}\, \mathrm{Ext}^1(X, M)|.$$

Proof: Let $\mathbf{dim}\, X = x$. Note that for any d-dimensional vector space V, we have $|_{E(M)}\, \mathrm{Hom}_k(V, M_i)| = d|_{E(M)} M_i|$. The direct sum decompositions $\bigoplus_i \mathrm{Hom}_k(X_i, M_i)$ and $\bigoplus_\alpha \mathrm{Hom}_k(X_{s(\alpha)}, M_{t(\alpha)})$ are direct sums of $E(M)$-modules, thus, using Lemma 6, we have

$$
\begin{aligned}
|_{E(M)}\, \mathrm{Hom}(X, M)| &- |_{E(M)}\, \mathrm{Ext}^1(X, M)| \\
&= \sum_i |_{E(M)}\, \mathrm{Hom}_k(X_i, M_i)| - \sum_\alpha |_{E(M)}\, \mathrm{Hom}_k(X_{s(\alpha)}, M_{t(\alpha)})| \\
&= \sum_i x_i |_{E(M)} M_i| - \sum_\alpha x_{s(\alpha)} |_{E(M)} M_{t(\alpha)}| \\
&= \langle \mathbf{dim}\, X, \mathbf{Dim}\, M \rangle
\end{aligned}
$$

This implies:

Lemma 8. *Let X be of finite length and M endo-finite. If $\langle \mathbf{dim}\, X, \mathbf{Dim}\, M \rangle > 0$, then $\mathrm{Hom}(X, M) \neq 0$.*

We should note that for Lemma 8, we only need the trivial part of Lemma 6: that the kernel of δ_{XM} is equal to $\mathrm{Hom}(X, M)$. Namely, if $\langle \mathbf{dim}\, X, \mathbf{Dim}\, M \rangle > 0$, then δ_{XM} cannot be a monomorphism, thus the kernel of δ_{XM} is non-zero.

3. Proof of Theorem.

Lemma 9. *A generic module M has indecomposable regular submodules, but no indecomposable preinjective submodules.*

Proof. If X is an indecomposable preinjective module and N is any indecomposable module with $\mathrm{Hom}(X, N) \neq 0$, then also N is preinjective (see for example [R2]). This shows that a generic module M has no indecomposable preinjective submodule

Let $x = \mathbf{Dim}\, M$. Then x is a non-zero regular element of $K_0(\Lambda)$, according to Lemma 5. According to Lemma 2, we have $\langle \Phi^{-t} x, x \rangle > 0$ for some $t \geq 0$. With x also $\Phi^{-t} x$ is a regular element of $K_0(\Lambda)$, thus according to Lemma 1, there is a regular module R with

$\mathbf{dim}\, R = \Phi^{-t}x$. According to Lemma 8, we have $\mathrm{Hom}(R,M) \neq 0$. Let $\phi\colon R \to M$ be a non-zero homomorphism and let X be the image of ϕ. As a factor module of R, the module X is a direct sum of indecomposable modules which are preinjective or regular. But as we have noted, M has no preinjective submodules, thus X is regular and of course non-zero.

Here is now the proof of Theorem. Let M be an indecomposable endo-finite module of infinite length. Let $R(M)$ be the sum of all regular submodules (since the sum of two regular submodules is again regular, this actually is the union of all regular submodules). According to Lemma 9, we know that $R(M) \neq 0$. Of course, $R(M)$ is a Λ-submodule: if M' is a regular submodule and ϕ an endomorphism of M, then also $\phi(M)$ is regular. Let us consider the factor module $M/R(M)$. Since $R(M)$ is both a Λ-submodule as well as an $E(M)$-submodule, it follows that $M/R(M)$ is endo-finite.

We claim that any indecomposable submodule of $M/R(M)$ of finite length is preprojective. Consider a finite length submodule U of $M/R(M)$ which is regular or preinjective. Then there is a finite length submodule M' of M such that the canonical map $M' \subset M \to M/R(M)$ maps onto U. Since $R(M)$ is the filtered union of regular modules, and $M' \cap R(M)$ is a finite length submodule of $R(M)$, there is a regular submodule M'' of $R(M)$ which contains $M' \cap R(M)$. It follows from $M' \cap R(M) \subseteq M'' \subseteq R(M)$ that $M' \cap M'' = M' \cap R(M)$, thus

$$(M' + M'')/M'' \simeq M'/(M' \cap M'') = M'/(M' \cap R(M) \simeq U.$$

This shows that $M' + M''$ is an extension of the regular module M'' by the finite length module U which is regular or preinjective, thus $M' + M''$ is a direct sum of regular and preinjective modules. Clearly, M has no non-zero preinjective submodules, thus $M' + M'' \subseteq R(M)$. But $M' \subseteq R(M)$ implies that the image U of the canonical map $M' \subset M \to M/R(M)$ is zero.

Since $M/R(M)$ is endo-finite, it is a direct sum of copies of a finite number of indecomposable endo-finite modules, say $N_1, \ldots, N_t$. As we have shown, none of the modules N_i can be regular or preinjective. Also, if N_i has infinite length, then according to Lemma 9 the module N_i has a non-zero regular submodule, but this is impossible. This shows that all the modules N_i are preprojective. But this implies that $R(M)$ is a direct summand of M, see for example [R2]. Since M is indecomposable, we conclude that $M = R(M)$.

As Krause has pointed out, $R(M)$ is the torsion submodule of M for a torsion pair and $R(M)$ is a pure submodule of M. Thus, if M is indecomposable and endo-finite, then either $R(M) = 0$ or $R(M) = M$.

4. Remarks.

(1) A question. It seems to be of interest to determine the set of endo-length vectors of the generic modules. One may conjecture that it is the set of all positive imaginary roots which are not proper multiples of zero roots. (Here, we call a root x a *zero root* provided $\langle x, x \rangle = 0$.)

(2) Hereditary artin algebras. The assertions of Theorem and its Corollary are true in the more general setting of dealing with an arbitrary hereditary artin algebra Λ, and not just the path algebras of finite directed quivers. Namely: *If Λ is a wild connected hereditary artin algebra, then any generic module is the union of its regular submodules and the limit closure of the preprojective component does not contain any generic module.*

The proof of the general result follows the given one, step by step, only few alterations are necessary. The actual calculations depend on the decision which kind of dimension vectors for finite length modules one wants to use. There are two obvious choices. Let Λ be a k-algebra, where k is a commutative artinian ring such that Λ is a finite length k-module, and let M be a Λ-module of finite length. First of all, one may look at the equivalence class $\mathbf{dim}\, M$ of M in $K_0(\Lambda)$, this means that the coefficient $(\mathbf{dim}\, M)_i$ is just the Jordan-Hölder multiplicity of $S(i)$ in M. But one may also take the vector $\mathbf{dim}_k\, M$ with coefficient $(\mathbf{dim}_k\, M)_i$ being the length of M_i as a k-module. Of course, for the path algebra $\Lambda = k\Delta$ of a directed quiver Δ, we have $\mathbf{dim} = \mathbf{dim}_k$, but in general $\mathbf{dim}_k$ is obtained from $\mathbf{dim}$ by a non-trivial diagonal linear transformation: we have to multiply the coefficient $(\mathbf{dim}\, M)_i$ by $\dim_k S(i)$.

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