From representations of quivers via Hall and Loewy algebras to quantum groups

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Let Δ be a symmetric generalized Cartan matrix, and $g = g(\Delta)$ the corresponding Kac–Moody Lie–algebra with triangular decomposition $g = n_- \oplus h \oplus n_+$ (see [**K**]. We denote by $b_+ = b_+(\Delta) = h \oplus n_+$ the Borel subalgebra. Let $U_q(b_+)$ be the quantization of the universal enveloping algebra of b_+, it is defined by generators and relations as we will recall below. For Δ of finite or affine type, we want to survey a construction of $U_q(b_+)$ using the representation theory of quivers, following [**R2**], [**R3**], [**R4**] and [**R5**].

1. Representations of quivers

A (finite) *quiver* $Q = (Q_0, Q_1, s, e)$ is given by two finite sets Q_0, Q_1 and two maps $s, e : Q_1 \longrightarrow Q_0$, the elements of Q_0 will be called vertices or points, those of Q_1 arrows; if α is an arrow, then $s(\alpha)$ is called its start vertex, $e(\alpha)$ its end vertex, and α is said to go from $s(\alpha)$ to $e(\alpha)$, written $\alpha : s(\alpha) \longrightarrow e(\alpha)$. (Thus, a quiver is nothing else than a directed graph with possibly multiple arrows and loops, or a diagram scheme in the sense of Grothendieck; so the concept is old, the denomination "quiver" being due to Gabriel.) Given a quiver $Q = (Q_0, Q_1, s, e)$, there is the opposite quiver $Q^* = (Q_0, Q_1, e, s)$ with the same set of vertices but with the reversed orientation for all the arrows. Recall that an $n \times n$–matrix $(a_{ij})_{ij}$ with $a_{ii} = 2$ and $a_{ij} = a_{ji} \leq 0$ for all $i \neq j$ is called a *symmetric generalized Cartan matrix* [**K**]. Given a symmetric generalized Cartan $n \times n$–matrix $\Delta = (a_{ij})_{ij}$, we associate the following quiver $Q(\Delta)$; its set of vertices is $Q(\Delta)_0 = \{1, 2, \ldots, n\}$, and for $1 \leq i < j \leq n$, we draw $-a_{ij}$ arrows from i to j. The quivers of the form $Q(\Delta)$ have the following property: their vertices may be labelled by $1, \ldots, n$ such that for any arrow α, we have $s(\alpha) < e(\alpha)$: conversely, any quiver with this property is of the form $Q(\Delta)$. The reader should be aware that the construction of $Q(\Delta)$ takes into account the order of the rows and columns of Δ.

Given a quiver Q, a path in Q of length $\ell \geq 1$ is of the form $(x|\alpha_1, \ldots, \alpha_\ell|y)$, where α_i are arrows with $x = s(\alpha_1), e(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i < \ell$, and $e(\alpha_\ell) = y$; a path in Q of length 0 is of the from $(x|x)$ with $x \in Q_0$. A path of the form

$(x|\alpha_1,\ldots,\alpha_\ell|x)$ with $\ell \geq 1$ is called a cyclic path. Note that the quivers of the form $Q(\Delta)$ with Δ a symmetric generalized Cartan matrix are precisely the quivers without a cyclic path.

Let k be a field. The *path algebra* kQ of Q over k is the free vectorspace over k with basis the set of paths in Q, with distributive multiplication given on the basis by

$$(x|\alpha_1,\ldots,\alpha_\ell|y) \cdot (x'|\alpha_1',\ldots,\alpha_{\ell'}'|y') = \begin{cases} (x|\alpha_1,\ldots\alpha_\ell,\alpha_1',\ldots,\alpha_{\ell'}'|y') & \text{if } y = x' \\ 0 & \text{if } y \neq x' . \end{cases}$$

The elements $(x|x)$ with $x \in Q_0$ are primitive and orthogonal idempotents, and $1 = \sum\limits_{x \in Q_0} (x|x)$ is the unit element of kQ. Note that kQ is finite–dimensional if and only if Q has no cyclic path.

Recall that a ring of global dimension ≤ 1 is said to be *hereditary*, a finite-dimensional k–algebra A with radical N is said to be *split basic* provided A/N is a product of copies of k. It is easy to see that the algebras $kQ(\Delta)$ with Δ a symmetric generalized Cartan matrix are precisely the finite–dimensional k–algebras which are hereditary and split basic. Again, we stress that the algebra $kQ(\Delta)$ depends on the given ordering of the rows and columns of Δ; different orderings of the rows and columns usually will lead to algebras which are not isomorphic. For example, the Cartan matrix $\Delta = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ yields the quiver $\overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ}$, and the path algebra of $Q(\Delta)$ is given by $kQ(\Delta) = \begin{bmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{bmatrix}$, the ring of upper triangular matrices, whereas for $\Delta' = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, the corresponding quiver is $\overset{1}{\circ} \longrightarrow \overset{3}{\circ} \longleftarrow \overset{2}{\circ}$, and its path algebra is the algebra $kQ(\Delta') = \begin{bmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$; note that $\dim_k kQ(\Delta) = 6$, whereas $\dim_k kQ(\Delta') = 5$.

A ring A is called an *artin algebra* provided its center C is artinian, and A is a finitely generated C–module. Given an artin algebra A, the A–modules we are interested in usually will be right A–modules of finite length, and we denote the category of these modules by $\mathrm{mod}\, A$. The composition of two A–module maps $f : M \longrightarrow M'$, $g : M' \longrightarrow M''$ will be denoted by $gf : M \longrightarrow M''$ (applying such maps on the opposite side of the scalars).

Given an artin algebra A, we denote by $K(A)$ the Grothendieck group of all (finite–dimensional) A–modules modulo exact sequences. Similarly, let $K(\mathrm{mod}\, A)$

be the Grothendieck group of all (finite–dimensional) A–modules modulo split exact sequences. The theorem of Jordan–Hölder asserts that $K(A)$ is the free abelian group on the set of isomorphism classes of simple A–modules. Given an A–module M, we denote by $\mathbf{dim}\, M$ its equivalence class in $K(A)$. The theorem of Krull–Schmidt asserts that $K(\mathrm{mod}\, A)$ is the free abelian group on the set of isomorphism classes of indecomposable A–modules. If M is an indecomposable A–module, we denote by $u_{[M]} = u_M$ the corresponding element in $K(\mathrm{mod}\, A)$; always, $[M]$ will denote the isomorphism class of a module M. Also note that $K(\mathrm{mod}\, A)$ has a canonical $K(A)$–grading: given $x \in K(A)$, let $K(\mathrm{mod}\, A)_x$ be the subgroup generated by all $u_{[M]}$ where M is indecomposable with $\mathbf{dim}\, M = x$. It follows that for Δ a symmetric generalized Cartan $n \times n$–matrix, and $A = kQ(\Delta)$, we have $K(A) \cong \mathbb{Z}^n$, whereas $K(\mathrm{mod}\, A)$ does not have to be finitely generated. An algebra A is said to be *representation–finite* provided there are only finitely many isomorphism classes of indecomposable A–modules.

Let Q be a quiver and k a field. A *representation* $V = (V_x, V_\alpha)$ of Q over k is given by a family of (finite–dimensional) vectorspaces V_x $(x \in Q_0)$, and a family of linear maps $V_\alpha : V_{s(\alpha)} \longrightarrow V_{e(\alpha)}$ $(\alpha \in Q_1)$. Given two representations V, V', a map $f = (f_x) : V \longrightarrow V'$ is given by linear maps $f_x : V_x \longrightarrow V'_x$ such that for any $\alpha \in Q_1$, we have $V'_\alpha f_{s(\alpha)} = f_{e(\alpha)} V_\alpha$. Note that the category $\mathrm{mod}\, kQ$ is equivalent to the category of representations of the opposite quiver Q^* of Q, and we may (and will) identify these categories. Given a representation V, its dimension vector $\mathbf{dim}\, V$ has, by definition, coordinates $(\mathbf{dim}\, V)_x = \dim_k V_x$, for $x \in Q_0$; it belongs to $\mathbb{Z}^{Q_0}$, and $\sum_{x \in Q_0} (\mathbf{dim}\, V)_x$ is called the dimension of V. For any vertex $x \in Q_0$, we will consider the one–dimensional representation $S(x)$ of Q^* defined by $S(x)_x = k$, $S(x)_y = 0$ for $y \neq x \in Q_0$ and $S(x)_\alpha = 0$ for $\alpha \in Q_1$. Assume now that Q has no cyclic path, say let $Q = Q(\Delta)$ with $\Delta = (a_{ij})_{ij}$ a symmetric generalized Cartan $n \times n$–matrix. Then $S(1), \ldots, S(n)$ are the simple $kQ(\Delta)$–modules, and we have

$$\mathrm{Ext}^1(S(i), S(j)) = 0 \quad \text{for} \quad i \geq j,$$

and

$$\dim_k \mathrm{Ext}^1(S(i), S(j)) = -a_{ij} \quad \text{for} \quad i < j.$$

(For, let $I(i)$ be the subspace of $kQ(\Delta)$ generated by all paths in $Q(\Delta)$ different from $(i|i)$, and $u(i)$ the inclusion map of $I(i)$ into $kQ(\Delta)$. Clearly, $I(i)$ is a twosided ideal, and $kQ(\Delta)/I(i)$ and $S(i)$ are isomorphic right $kQ(\Delta)$–modules. It follows that $\mathrm{Ext}^1(S(i), S(j))$ is isomorphic as a k–space to the cokernel of $\mathrm{Hom}_{kQ(\Delta)}(u(i), S(j))$, and therefore to the k–dual of the subspace of $kQ(\Delta)$ generated by the paths of the form $(i|\alpha|j)$.) In

particular, we see how to recover Δ from $kQ(\Delta)$. Also, we note the following: Given a representation V of Q, and $x \in Q_0$, the k–dimension of V_x is just the Jordan–Hölder multiplicity of $S(x)$ in V, thus the two definitions of $\mathbf{dim}\, V$ given above coincide; in particular, $M \mapsto \mathbf{dim}\, M$ yields the canonical identification of $K(kQ(\Delta))$ with $\mathbb{Z}^n$.
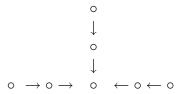
Let us write down at least a few representations explicitly. First, consider the quiver Q

$$
\begin{array}{ccccccccc}
 & & & & \overset{5}{\circ} & & & & \\
 & & & & \downarrow & & & & \\
\underset{1}{\circ} & \to & \underset{2}{\circ} & \to & \underset{6}{\circ} & \leftarrow & \underset{4}{\circ} & \leftarrow & \underset{3}{\circ}
\end{array}
$$

of type E_6, and its representation V

$$
\begin{array}{ccccccccc}
 & & & & U & & & & \\
 & & & & \downarrow & & & & \\
k0^2 & \to & k^2 0 & \to & k^3 & \leftarrow & 0k^2 & \leftarrow & 0^2 k
\end{array}
$$

with $U = V_5$ the 2–dimensional subspace of k^3 generated by (110) and (011). It is an indecomposable representation and its dimension vector is

$$
\begin{array}{ccccc}
 & & 2 & & \\
1 & 2 & 3 & 2 & 1,
\end{array}
$$

where we arrange its coordinates according to the shape of the quiver. The reader should observe that dealing with this particular quiver Q, we obtain a lot of representations by considering (as in the example above) a vectorspace V_6 endowed with five subspaces $V_1, \dots, V_5$ such that $V_1 \leq V_2$, and $V_3 \leq V_4$. Actually, one may observe without difficulties that all but seven isomorphism classes of indecomposable representations of Q are obtained in this way. Second, for the quiver

$$
\begin{array}{ccccccc}
 & & & \circ & & & \\
 & & & \downarrow & & & \\
 & & & \circ & & & \\
 & & & \downarrow & & & \\
\circ & \to & \circ & \to & \circ & \leftarrow & \circ & \leftarrow & \circ
\end{array}
$$

of type $\tilde{E}_6$, we obtain a 1–parameter family of indecomposable representations

$$
\begin{array}{ccccccccc}
 & & & & U_\lambda & & & & \\
 & & & & \downarrow & & & & \\
 & & & & U & & & & , \\
 & & & & \downarrow & & & & \\
k0^2 & \to & k^2 0 & \to & k^3 & \leftarrow & ok^2 & \leftarrow & 0^2 k
\end{array}
$$

where U is as above, and U_λ is the 1–dimensional subspace of k^3 generated by $(1, 1 + \lambda, \lambda)$, with λ a fixed element of k.

We recall that in 1972, Gabriel has shown that the path algebra of a quiver Q is representation–finite if and only if Q is the disjoint union of quivers of the form $Q(\Delta)$, with Δ of finite type (thus of type A_n, D_n, E_6, E_7, or E_8), and that for $Q = Q(\Delta)$, with Δ of finite type, **dim** furnishes a bijection between the set $\mathcal{B}_0$ of isomorphism classes of indecomposable representations of Q and the set Φ^+ of positive roots for Δ. We can reformulate this result as follows: $K(\operatorname{mod} kQ(\Delta)) \otimes \mathbb{C}$ is the free $\mathbb{C}$–space with basis $(u_{[M]})_{[M] \in \mathcal{B}_0}$, whereas for the semisimple complex Lie algebra $g = g(\Delta)$ with triangular decomposition $g = n_- \oplus h \oplus n_+$, the $\mathbb{C}$–space n_+ has a canonical decomposition $n_+ = \bigoplus_{\alpha \in \Phi^+} g_\alpha$ into weight spaces g_α, and all g_α are 1–dimensional. Thus,

$$K(\operatorname{mod} kQ(\Delta)) \otimes \mathbb{C} \cong n_+$$

as $\mathbb{C}$–spaces, and in fact as $\mathbb{Z}^n$–graded $\mathbb{C}$–spaces: here, $K(\operatorname{mod} kQ(\Delta))$ is graded by $\mathbb{Z}^n = K(kQ(\Delta))$, whereas n_+ is graded by the root lattice.

We may ask whether the representation theory of quivers may be used to endow $K(\operatorname{mod} kQ(\Delta)) \otimes \mathbb{C}$, or even $K(\operatorname{mod} kQ(\Delta))$ itself with a Lie structure so that this isomorphism becomes an isomorphism of Lie algebras.

2. Hall algebras

Let R be a finite ring. We denote by $\operatorname{mod} R$ the category of finitely generated R–modules. Note that an R–module is finitely generated if and only if it is of finite length if and only if it has finitely many elements, so we call these R–modules just the *finite* R–modules. Given finite R–modules $M, N_1, \dots, N_t$ let $F^M_{N_1 \dots N_t}$ be the number of filtrations $M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_t = 0$ of M such that $M_{i-1}/M_i \cong N_i$, for $1 \leq i \leq t$. The Hall algebra $\mathcal{H}(R)$ is, by definition, the free abelian group with basis $(u_{[M]})_{[M] \in \mathcal{B}}$ indexed by the set of $\mathcal{B}$ of isomorphism classes $[M]$ of finite R–modules M, with multiplication given by

$$u_{[N_1]} u_{[N_2]} = \sum_{[M]} F^M_{N_1 N_2} u_{[M]};$$

note that we have $F^M_{N_1 N_2} \neq 0$ only in case **dim** $M = $ **dim** $N_1 + $ **dim** N_2, and that the number of isomorphism classes of modules M with given **dim** M is finite, thus

the sum considered above always is a finite sum. We note that the multiplication is associative, in fact we have

$$u_{[N_1]}u_{[N_2]}u_{[N_3]} = \sum_{[M]} F^M_{N_1 N_2 N_3} u_{[M]},$$

and $u_{[0]}$ is the unit element for this multiplication, thus $\mathcal{H}(R)$ is a ring.

More generally, we may start with a ring R having the property that given two simple R–modules S_1, S_2 with only finitely many elements, then also $\operatorname{Ext}^1_R(S_1, S_2)$ has only finitely many elements, and take for $\mathcal{B}$ the set of isomorphism classes of R–modules with finitely many elements. If $R = \mathbb{Z}_{(p)}$ then $\mathcal{B}$ is just the set of isomorphism classes of finite abelian p–groups. It is this case which has been considered in detail in the literature, first in 1900, by Steinitz [S], then in 1959 by Ph. Hall [H]: he called $\mathcal{H}(\mathbb{Z}_{(p)})$ the "algebra of partitions", since the isomorphism classes of finite abelian p–groups correspond bijectively to the partitions. The main reference for this particular Hall algebra is a book by Macdonald [M]. (The Hall algebra $\mathcal{H}(\mathbb{Z}_{(p)})$ can also be interpreted as a Hecke algebra: Let G^+_n be the set of $n \times n$–matrices over $\mathbb{Z}_{(p)}$ which are invertible over $\mathbb{Q}$, and K_n the subset of those elements which are invertible over $\mathbb{Z}_{(p)}$. The canonical embedding of G^+_n into G^+_{n+1} yields a semigroup $G^+ = \bigcup_n G^+_n$, with a subgroup $K = \bigcup_n K_n$, and $\mathcal{H}(\mathbb{Z}_{(p)})$ may be identified with the Hecke algebra for the inclusion of K into G^+ (note that the double cosets of K in G^+ correspond bijectively again to the partitions), see [M].)

If R_1, R_2 are finite rings, then $\mathcal{H}(R_1 \times R_2) = \mathcal{H}(R_1) \otimes \mathcal{H}(R_2)$, thus for studying Hall algebras $\mathcal{H}(R)$, we may assume that R is connected, or, equivalently, that the center $C = C(R)$ of R is a local ring, and we denote by $q = q(R)$ the cardinality of $C/\operatorname{rad}C$, and by $l_C(M)$ the length of the C–module M. Since $C/\operatorname{rad}C$ is the only simple C–module, we have $|M| = q^{l_C(M)}$ for any (finite) C–module M. Note that if X is an R–module, then $X, \operatorname{End}(X)$, and $\operatorname{rad}\operatorname{End}(X)$ all are C–modules.

The Hall algebra $\mathcal{H}(R)$ usually is non–commutative, in contrast to the classical case of $R = \mathbb{Z}_{(p)}$. For example, let $R = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$, the ring of upper triangular 2×2–matrices over the finite field k. Then there are two simple R–modules S_1, S_2, one of them, say S_1, is injective, the other one, S_2, is projective. It follows that in $\mathcal{H}(R)$, we have

$$u_{S_2}u_{S_1} = u_{S_1 \oplus S_2},$$
$$u_{S_1}u_{S_2} = u_{S_1 \oplus S_2} + u_P,$$

where P is the unique indecomposable R–module of length 2. In particular,

$$u_P = u_{S_1}u_{S_2} - u_{S_2}u_{S_1} = [u_{S_1}, u_{S_2}],$$

so the basis element u_P corresponding to the indecomposable R–module P is expressed as a commutator of other basis elements (corresponding to other indecomposable modules). This is a rather general feature and it demonstrates the usefulness of $\mathcal{H}(R)$: we obtain large indecomposable modules (or better, their counterparts in $\mathcal{H}(R)$) by forming iterated commutators or related algebraic operations, starting with small indecomposable modules, say simple, or at least serial ones. Note that by, definition, $K(\mathrm{mod}\,R)$ is a subgroup of $\mathcal{H}(R)$, namely the subgroup generated by the elements $u_{[M]}$, with M indecomposable. We claim that up to multiples of $q-1$, the subgroup $K(\mathrm{mod}\,R)$ is closed under commutators:

Proposition 1. *Let R be a finite connected ring, and $q = q(R)$. Let N_1, N_2 be indecomposable R–modules, let M be a decomposable R–module. Then $q - 1$ divides $F^M_{N_1 N_2} - F^M_{N_2 N_1}$.*

For example, consider the "3–subspace quiver"

$$
\begin{array}{ccccc}
 & & \overset{2}{\circ} & & \\
 & & \downarrow & & \\
\underset{1}{\circ} & \rightarrow & \underset{0}{\circ} & \leftarrow & \underset{3}{\circ}
\end{array}
$$

We denote by $P(i)$ the indecomposable projective, by $Q(i)$ the indecomposable injective module corresponding to the vertex i. For $1 \leq i \leq 3$, let $M(i)$ be the maximal submodule of $Q(0)$ with $Q(0)/M(i) \cong Q(i)$, and let M be the unique indecomposable module with dimension vector $\begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}$. Then

$$u_{P(0)} u_{Q(0)} = u_{P(0) \oplus Q(0)},$$

$$u_{Q(0)} u_{P(0)} = q u_{P(0) \oplus Q(0)} + (q-1) \sum_{i=1}^{3} u_{P(i) \oplus M(i)} + (q-2) u_M,$$

thus

$$[u_{P(0)}, u_{Q(0)}] \equiv u_M \bmod (q-1).$$

Proof of Proposition 1. We can assume that $N_1 \not\cong N_2$. Let $M = M' \oplus M''$ with M' indecomposable and $M'' \neq 0$. Let $\mathcal{U}$ be the set of submodules U of M with $U \neq M'$, $U \neq M''$, $U \cong N_2$ and $M/U \cong N_1$. We claim that $q - 1$ divides $|\mathcal{U}|$. Note that for $U \in \mathcal{U}$, we have that $U \cap M'$ is a proper submodule of M'. Otherwise, $M' \subseteq U$, thus $U = M' \oplus (U \cap M'')$; but U is indecomposable, therefore $U = M'$, contrary to our assumption. Let $G = \mathrm{Aut}\, M'$, the automorphism group of M'.

Then $|G| = |\operatorname{End} M'| - |\operatorname{rad} \operatorname{End} M'| = q^r(q^e - 1)$, where $r = \ell_C(\operatorname{rad} \operatorname{End} M')$ and $e = \ell_C(\operatorname{End} M'/\operatorname{rad} \operatorname{End} M')$. We identify G with a subgroup of $\operatorname{Aut} M$ via $g \mapsto g \oplus 1_{M''}$, thus G also acts on $\mathcal{U}$ via $g * U = \{(gu', u'') | (u', u'') \in U\}$ where $g \in G$ and $U \in \mathcal{U}$. To compute the stabilizer G_U of U in G, note first that we may consider $\operatorname{Hom}(M', U \cap M')$ as a subset of $\operatorname{rad} \operatorname{End} M'$. Now, if $f \in \operatorname{Hom}(M', U \cap M')$, then $(1 + f) * U \subseteq U$, since for $(u', u'') \in U$, also $((1 + f)u', u'') = (u', u'') + (fu', 0) \in U$, thus $(1 + f) * U = U$. Conversely, if $g * U = U$, then for all $(u', u'') \in U$ also $(gu', u'') \in U$, thus $(gu' - u', 0) \in U$, consequently $g - 1 \in \operatorname{Hom}(M', U \cap M')$, thus $g \in 1 + \operatorname{Hom}(M', U \cap M')$. It follows that $|G_U| = |\operatorname{Hom}(M', U \cap M')| = q^{r'}$ for some $r' \leq r$, and therefore $|G/G_U|$ is divisible by $q^e - 1$. Since $|G/G_U|$ is the orbit length of U, and U was arbitrary, we see that $q^e - 1$ divides $|\mathcal{U}|$, so $q - 1$ divides $|\mathcal{U}|$.

In order to finish the proof, we have to distinguish to cases. If let $M \not\approx N_1 \oplus N_2$, then any submodule U of M with $U \cong N_2$, $M/U \cong N_1$ satisfies in addition $U \neq M', U \neq M''$, thus $F^M_{N_1 N_2} = |\mathcal{U}|$, and therefore $q - 1 | F^M_{N_1 N_2}$. The same argument shows $q - 1 | F^M_{N_2 N_1}$.

Otherwise, we can assume $M' = N_1$ and $M'' = N_2$. Then, besides the elements of $\mathcal{U}$, there is just one additional submodule U of M with $U \cong N_2$, $M/U \cong N_1$, namely $U = M''$. Therefore $F^M_{N_1 N_2} \equiv 1 \mod(q - 1)$, and similarly, $F^M_{N_2 N_1} \equiv 1 \mod(q - 1)$.

Given a specific R–module M, it usually is not easy to decide whether M is indecomposable or not. Proposition 1 yields an effective procedure for certain cases. For example, consider the E_6–quiver and its representation V with $\mathbf{dim}\, V = \begin{smallmatrix} & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}$, exhibited in section 1. There is a unique submodule V' with dimension vector $\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 2 & 1 & 1 \end{smallmatrix}$, namely

$$
\begin{array}{ccccc}
& & \{(x0x)\} & & \\
& & \downarrow & & \\
k0^2 \to k0^2 \to & & k0k & \leftarrow 0^2k \leftarrow & 0^2k
\end{array},
$$

V' is indecomposable, and V/V' is isomorphic to the indecomposable representation

$$
\begin{array}{ccccc}
& & k & & \\
& & \downarrow & & \\
0 \to k \to & & k & \leftarrow k \leftarrow & 0.
\end{array}
$$

Thus $F^V_{V/V', V'} = 1$. On the other hand, we clearly have $\operatorname{Hom}(V/V', V) = 0$, therefore $F^V_{V', V/V'} = 0$. So Proposition 1 asserts that V is indecomposable. (Of

course, we can apply Proposition 1 only in case k is a finite field, and there are other methods which may be used for the example above in case k is arbitrary: We have $\mathrm{Hom}(V', V/V') = 0$, therefore V has to be indecomposable!)

We will see in the next section that for special rings R we can calculate $F^M_{N_1 N_2}$ modulo $q - 1$ for all indecomposable R–modules N_1, N_2, M.

In order to deal with presentations of Hall algebras, we mention the following fundamental relations, where $\begin{bmatrix} n \\ t \end{bmatrix}_T = \frac{\varphi_n(T)}{\varphi_t(T)\varphi_{n-t}(T)}$ is the Gauss polynomial, with $\varphi_n(T) = (1 - T) \cdots (1 - T^n)$.

Proposition 2. *Let R be a finite ring, let S_i, S_j be simple R–modules with* $\mathrm{Ext}^1(S_i, S_j) = 0$. *Let* $q_i = |\mathrm{End}(S_i)|$, $q_j = |\mathrm{End}(S_j)|$, *and let*

$$a_{ij} = -\dim{}_{\mathrm{End}(S_i)}\mathrm{Ext}^1(S_j, S_i), \quad a_{ji} = -\dim \mathrm{Ext}^1(S_j, S_i)_{\mathrm{End}(S_j)}.$$

Let $u_i = u_{[S_i]}, u_j = u_{[S_j]}$. *If* $\mathrm{Ext}^1(S_i, S_i) = 0$, *then*

$$\sum_{t=0}^{n}(-1)^t \begin{bmatrix} n \\ t \end{bmatrix}_{q_i} q_i^{\binom{t}{2}} u_i^t u_j u_i^{n-t} = 0 \quad with \quad n = 1 - a_{ij},$$

if $\mathrm{Ext}^1(S_j, S_j) = 0$, *then*

$$\sum_{t=0}^{n}(-1)^t \begin{bmatrix} n \\ t \end{bmatrix}_{q_j} q_j^{\binom{t}{2}} u_j^{n-1} u_i u_j^t = 0 \quad with \quad n = 1 - a_{ji}.$$

These relations look rather similar to those used by Drinfeld and Jimbo in order to define quantizations of Lie algebras, this similarity will be discussed in the next section. But the reader may observe already here that the polynomials

$$\rho_n^+(T, X, Y) = \sum_{t=0}^{n}(-1)^t \begin{bmatrix} n \\ t \end{bmatrix}_T T^{\binom{t}{2}} X^t Y X^{n-t}$$

$${}^+\rho_n(T, X, Y) = \sum_{t=0}^{n}(-1)^t \begin{bmatrix} n \\ t \end{bmatrix}_T T^{\binom{t}{2}} X^{n-t} Y X^t$$

both yield for $T = 1$ an iterated commutator:

$$\rho_n^+(1, X, Y) = (\mathrm{ad}\ X)^n Y,$$
$${}^+\rho_n(1, X, Y) = (-1)^n (\mathrm{ad}\ X)^n Y.$$

The proof of Proposition 2 (see [**R4**]) is not difficult:

We assume that $\text{Ext}^1(S_i, S_i) = 0$. By the definition of the multiplication in $\mathcal{H}(R)$, the elements $u_i^t u_j u_i^{n-t}$ are linear combinations of elemens $u_{[M]}$ where M is an R–module with $\dim M = \dim S_j + n \dim S_i$. Since $\text{Ext}^1(S_i, S_i) = 0$, $\text{Ext}^1(S_i, S_j) = 0$, such a module M is the direct sum of an indecomposable module M' with $M'/\text{rad } M' = S_j$, and several copies of S_i, say $M = M' \oplus dS_i$. Since $\dim_{\text{End}(S_i)} \text{Ext}^1(S_j, S_i) < n$, it follows that $d \geq 1$. The coefficient of $u_{[M]}$ in $u_i^t u_j u_i^{n-t}$ just counts the number of composition series of M of the form

$$M = M_0 \supset M_1 \supset \ldots \supset M_{n+1} = 0$$

with $M_t/M_{t+1} \cong S_j$. If $t > d$, then there is no such composition series, if $t \leq d$, then the number of such composition series is $\alpha_i(q_i)$, where

$$\alpha_t(T) = \frac{\varphi_d(T)\varphi_{n-t}(T)}{(1-T)^n \varphi_{d-t}(T)}$$

and

$$\sum_{t=0}^{d} (-1)^t \begin{bmatrix} n \\ t \end{bmatrix}_T T^{\binom{t}{2}} \alpha_t = 0.$$

The ring $\mathcal{H}(R)$ is $K(R)$–graded: for $x \in K(R)$, let $\mathcal{H}(R)_x$ be the subgroup generated by the elements $u_{[M]}$ with $\dim M = x$; clearly, for $x, y \in K(R)$, we have $\mathcal{H}(R)_x \cdot \mathcal{H}(R)_y \subseteq \mathcal{H}(R)_{x+y}$. As a consequence, any element $d \in K(R)^* = \text{Hom}(K(R), \mathbb{Z})$ gives rise to a derivation $\delta_d : \mathcal{H}(R) \longrightarrow \mathcal{H}(R)$, defined by $\delta_d(u_{[M]}) = d(\dim M) \cdot u_{[M]}$. In particular, let $S_1, \ldots, S_n$ be the simple R–modules, thus $\dim S_1, \ldots, \dim S_n$ is a basis of $K(R)$, and we denote by $d_1, \ldots, d_n$ the dual basis of $K(R)^*$; thus $d_i(\dim M) = (\dim M)_i$ is the Jordan Hölder multiplicity of S_i in M. In this way, we obtain derivations $\delta_i = \delta_{d_i}$ of $\mathcal{H}(R)$, and we may form the skew polynomial ring

$$\mathcal{H}'(R) = \mathcal{H}(R)[T_i, \delta_i]_i,$$

where $T_1, \ldots, T_n$ are indeterminates satisfying the commutation rules

$$[T_i, T_j] = 0$$

$$[T_i, u_{[M]}] = \delta_i(u_{[M]}) = (\dim M)_i u_{[M]}.$$

We call $\mathcal{H}'(R)$ the *extended Hall algebra* of R.

For $e \in \mathbb{N}_0$, let $\psi_e(T) = \frac{(1-T)\cdots(1-T^e)}{(1-T)^e}$.

Proposition 3. *Let $X_1, \ldots, X_m$ be indecomposable modules with $\mathrm{End}(X_i)$ the field with q_i elements. Assume that $\mathrm{Ext}^1(X_i, X_j) = 0$ for $i \leq j$ and that $\mathrm{Hom}(X_i, X_j) = 0$ for $i > j$. Let $a \in \mathbb{N}_0^m$. Then*

$$u_{[X_1]}^{a(1)} \cdots u_{[X_m]}^{a(m)} = \left(\prod_{i=1}^m \psi_{a(i)}(q_i)\right) \cdot u_{[\oplus a(i)X_i]},$$

and

$$u_{[a(1)X_1]} \cdots u_{[a(m)X_m]} = u_{[\oplus a(i)X_i]}.$$

Proof: First, we show the second formula. Given M with a filtration

$$M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_m = 0$$

such that $M_{i-1}/M_i \cong a(i)X_i$, then all M_i are direct summands and $M \cong \overset{m}{\underset{i=1}{\oplus}} a(i)X_i$, since $\mathrm{Ext}^1(X_i, X_j) = 0$ for $i \leq j$. Thus, let $M = \overset{m}{\underset{i=1}{\oplus}} a(i)X_i$. Using the condition $\mathrm{Hom}(X_i, X_j) = 0$ for $i > j$, we conclude that $M_i = \underset{j>i}{\oplus} a(j)X_j$ is uniquely determined, and therefore M has precisely one filtration with the prescribed factors $a(i)X_i$.

It remains to show that

$$u_{[X_i]}^{a(i)} = \psi_{a(i)}(q_i) \cdot u_{[a(i)X_i]}.$$

However, since $\mathrm{End}(X_i)$ is a field, the filtrations of $a(i)X_i$ with factors X_i correspond bijectively to the complete flags in the $a(i)$–dimensional $\mathrm{End}(X_i)$–space, but the number of such flags is $\psi_{a(i)}(q_i)$.

We call R *representation–directed* provided R has only finitely many indecomposable modules and they can be ordered as $X_1, \ldots, X_m$ such that $\mathrm{Hom}(X_i, X_j) = 0$ for $i > j$. Clearly, such an ordering satisfies in addition $\mathrm{Ext}^1(X_i, X_j) = 0$ for $i \leq j$.

3. Generic Hall algebras

We return to the path algebra of a quiver, and consider the representation–finite case. Thus, *let Δ be a symmetric Cartan matrix* (therefore of type A_n, D_n, E_6, E_7, or E_8), and let $R = kQ(\Delta)$. Recall that Φ^+ denotes the set of positive roots

for Δ. Given $\alpha \in \Phi^+$, let $M(\alpha) = M(\alpha, R)$ be the indecomposable R–module with $\mathbf{dim}\, M(\alpha) = \alpha$. Given a function $a : \Phi^+ \longrightarrow \mathbb{N}_0$, let $M(a) = M(a, R) = \underset{\alpha \in \Phi^+}{\oplus}\, a(\alpha) M(\alpha)$. In this way, we obtain a bijection between the set of functions $\Phi^+ \longrightarrow \mathbb{N}_0$ and $\mathcal{B}$.

Proposition 4.   *Given $a, b, c : \Phi^+ \longrightarrow \mathbb{N}_0$, there exists a polynomial $\varphi_{ac}^b \in \mathbb{Z}[T]$ with the following property: if k is a finite field, $q = |k|$, then*

$$F_{M(a,R),M(c,R)}^{M(b,R)} = \varphi_{ac}^b(q).$$

For the proof, we refer to [**R2**]. The polynomials φ_{ac}^b occurring in this way will be called *Hall polynomials*. This proposition allows us to introduce generic Hall algebras. Before we do this, let us write down several of these ploynomials explicitly. If $\alpha \in \Phi^+$, we will denote the corresponding characteristic function $\Phi^+ \longrightarrow \mathbb{N}_0$ also by α. If $\alpha, \gamma \in \Phi^+$, we have to distinguish the two possible additions: the addition inside the root lattice will be denoted by $+$, the addition of functions $\Phi^+ \longrightarrow \mathbb{N}_0$ may be denoted by $\oplus$, but we will not need the notation.

The symmetric generalized Cartan matrix Δ (with its fixed ordering of rows and columns) determines a usually non–symmetric bilinear form $\langle -, - \rangle$ on the root lattice $\mathbb{Z}^n$ as follows: Let $\overline{\Delta}$ be the (uniquely determined) lower triangular matrix with $\Delta = \overline{\Delta} + \overline{\Delta}^t$, thus $\overline{\Delta} = (\overline{a}_{ij})_{ij}$, with $\overline{a}_{ii} = 1, \overline{a}_{ij} = a_{ij}$ for $i > j$, and $\overline{a}_{ij} = 0$ for $i < j$, and let $\langle \alpha, \beta \rangle = \alpha \overline{\Delta} \beta^t$. The importance of this bilinear form for the representation theory of $Q(\Delta)^*$ is well–known: given representations V, V' of $Q(\Delta)^*$, we have

$$\langle \mathbf{dim}\, V, \mathbf{dim}\, V' \rangle = \dim_k \mathrm{Hom}(V, V') - \dim_k \mathrm{Ext}^1(V, V').$$

Proposition 5.   *Let $\alpha, \gamma \in \Phi^+$ with $\langle \alpha, \gamma \rangle < 0$. Let $\beta = \alpha + \gamma$. Then $\beta \in \Phi^+$,*

and $\varphi_{\alpha\gamma}^\beta$ is as follows:

$\langle\gamma,\alpha\rangle$	$\varphi_{\alpha\gamma}^\beta$
0	1
1	$T - 2$
2	$(T - 2)^2$
3	$(T - 2)^3$ *or* $T^3 - 5T^2 + 10T - 7$
4	$(T - 2)(T^3 - 4T^2 + 8T - 6)$
5	$T^5 - 6T^4 + 15T^3 - 23T^2 + 25T - 13$

The proof is rather tedious, see [**R3**].

Corollary. *Let $\alpha, \gamma \in \Phi^+$ with $\langle\alpha,\gamma\rangle < 0$. Let $\beta = \alpha + \gamma$. Then $\varphi_{\alpha\gamma}^\beta(1) = (-1)^{\langle\gamma,\alpha\rangle}$, whereas $\varphi_{\gamma\alpha}^\beta = 0$.*

The first assertion is a direct consequence of Proposition 5. The second assertion is shown as follows: From $\langle\alpha,\gamma\rangle < 0$ we conclude that $\mathrm{Ext}^1(M(\alpha,R), M(\gamma,R)) \neq 0$, and therefore $\mathrm{Ext}^1(M(\gamma,R), M(\alpha,R)) = 0$, for $R = kQ(\Delta)$. But this implies that $F_{M(\gamma,R)M(\alpha,R)}^{M(\beta,R)} = 0$, and therefore $\varphi_{\gamma\alpha}^\beta = 0$, by Proposition 3.

Let us change the notation as follows: we will denote by q instead of T the indeterminate we are dealing with, thus $\mathbb{Z}[q]$ is the polynomial ring over $\mathbb{Z}$ in one variable, and given $a, b, c : \Phi^+ \longrightarrow \mathbb{N}_0$, then $\varphi_{ac}^b \in \mathbb{Z}[q]$. Let Λ be the completion of the polynomial ring $\mathbb{C}[q]$ at the maximal ideal generated by $q - 1$. In Λ we may consider $h = \ln q$, thus Λ is a complete discrete valuation ring and h (or also $q - 1$) generates its radical.

As before, Δ is a symmetric Cartan $n \times n$–matrix. The *generic Hall algebra* $\mathcal{H}(\Delta)$ is the free $\mathbb{Z}[q]$–module with basis $(u_a)_a$ indexed by the set of functions $a : \Phi^+ \longrightarrow \mathbb{N}_0$, with multiplication

$$u_a u_c = \sum_b \varphi_{ac}^b u_b.$$

Again, $\mathcal{H}(\Delta)$ is a ring, its identity element is the zero function. Also. $\mathcal{H}(\Delta)$ is generated by the root lattice $\mathbb{Z}^n$, thus, as before, we can form the *extended generic*

Hall algebra $\mathcal{H}'(\Delta) = \mathcal{H}(\Delta)[T_i, \delta_i]_i$, where $T_1, \ldots, T_n$ are indeterminates satisfying the commutation rules

$$[T_i, T_j] = 0,$$

$$[T_i, u_a] = \delta_i(u_a) = (\mathbf{dim}\, a)_i u_a,$$

where $(\mathbf{dim}\, a)_i = \sum_{\alpha \in \Phi^+} a(\alpha)\alpha_i$ with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Phi^+ \subseteq \mathbb{Z}^n$. The ring we are interested in is

$$\widehat{\mathcal{H}'}(\Delta) := \lim_{\substack{\leftarrow \\ m}} \mathcal{H}'(\Delta) \underset{\mathbb{Z}[q]}{\otimes} \mathbb{C}[q]/(q-1)^m,$$

it is a complete Λ–algebra containing $\mathcal{H}'(\Delta)$ as a subring. Instead of dealing with the elements $T_i, u_i, 1 \le i \le n$, let us consider the elements

$$H_i := \sum_{j=1}^{n} a_{ij} T_j$$

$$X_i := \exp\left(-\frac{1}{2} \sum_{j=1}^{i-1} a_{ij} T_j \ln q\right) \cdot u_i.$$

Theorem 1. *Let Δ be a symmetric Cartan matrix. The algebra $\widehat{\mathcal{H}'}(\Delta)$ is, as a complete Λ–algebra, generated by $H_1, \ldots, H_n, X_1, \ldots, X_n$ with the relations*

$$[H_1, H_j] = 0, \quad [H_i, X_j] = a_{ij} X_j, \quad \text{for all} \quad i, j,$$

$$\sum_{t=0}^{n} (-1)^t \begin{bmatrix} n \\ t \end{bmatrix}_q q^{-\frac{t(n-t)}{2}} X_i^t X_j X_i^{n-t} = 0 \quad \text{with } n = 1 - a_{ij}, \text{ and } i \ne j.$$

This shows that $\widehat{\mathcal{H}'}(\Delta) \cong U_q(b_+(\Delta))$, the quantization of $U(b_+(\Delta))$ in the sense of Drinfeld and Jimbo [**D**].

It is not difficult to derive the relations mentioned in the theorem from the known relations for T_i, u_i: since $[T_i, T_j] = 0$ for all i, j, also $[H_i, H_j] = 0$ for all i, j; since $[T_i, u_i] = u_i$ and $[T_i, u_j] = 0$ for $i \ne j$, it follows that $[H_i, u_j] = a_{ij} u_j$, and therefore also $[T_i, X_j] = a_{ij} X_j$. Finally, one easily verifies that for $c \in \mathbb{C}$

$$\exp(cT_i \ln q)u_i = q^c \cdot u_i \exp(cT_i \ln q),$$

whereas for $i \neq j$, the elements $\exp(cT_i \ln q)$ and u_j do commute. Using this information one observes without difficulties (see [**R4**]) that the last relation just rewrites the relations

$$\rho_n^+(q, u_i, u_j) = 0, \quad {}^+\rho_n(q, u_j, u_i) = 0$$

which are valid for $i < j$. Note that the only difference between ${}^+\rho_n, \rho_n^+$ and the new relation are the factors $q^{\binom{n}{i}}$ and $q^{-\frac{t(n-1)}{2}}$. Whereas the relations ${}^+\rho_n, \rho_n^+$ depend on the ordering of rows and columns of Δ, thus on the orientation of $Q(\Delta)$, the Drinfeld–Jimbo relation is independent of this ordering. Of course, the definition of X_i in terms of u_i also takes into account the ordering of Δ.

Starting with the isomorphism $\widehat{\mathcal{H}'}(\Delta) \cong U_q(b_+(\Delta))$, and factoring out on both sides the ideals generated by $q - 1$, we obtain the following:

Corollary.

$$\mathcal{H}'(\Delta) \underset{\mathbb{Z}[q]}{\otimes} \mathbb{C}[q]/(q-1) \cong U(b_+(\Delta)), \quad and$$
$$\mathcal{H}(\Delta) \underset{\mathbb{Z}[q]}{\otimes} \mathbb{C}[q]/(q-1) \cong U(n_+(\Delta)).$$

Note that $\mathcal{H}(\Delta)_1 = \mathcal{H}(\Delta) \underset{\mathbb{Z}[q]}{\otimes} \mathbb{Z}[q]/(q-1)$ is the free abelian group with basis $(u_a)_a$ indexed by the functions $a : \Phi^+ \longrightarrow \mathbb{N}_0$, with multiplication

$$u_a u_c = \sum_b \varphi_{ac}^b(1) u_b,$$

we may call it the *degenerate Hall algebra*. Proposition 1 implies that the subgroup generated by all $u_\alpha, \alpha \in \Phi^+$, actually is a Lie subalgebra of $\mathcal{H}(\Delta)_1$. Of course, this subgroup is canonically identified with $K(\text{mod } kQ(\Delta))$, with u_α corresponding to $u_{M(\alpha, kQ(\Delta))}$. Thus, we have found a Lie structure on $K(\text{mod } kQ(\Delta))$ which is derived from the representation theory of the quiver $Q(\Delta)$, namely

$$[u_{M(\alpha)}, u_{M(\gamma)}] = \sum_\beta (\varphi_{\alpha\gamma}^\beta(1) - \varphi_{\gamma\alpha}^\beta(1)) u_{M(\beta)},$$

and

$$K(\text{mod } kQ(\Delta)) \otimes \mathbb{C} \quad \text{and} \quad n_+$$

are isomorphic as Lie algebras. More is true: $K(\mathrm{mod}\,kQ(\Delta))$ is a $\mathbb{Z}$–form of $K(\mathrm{mod}\,kQ(\Delta)) \otimes \mathbb{C} \cong n_+$, and in fact it is a Chevalley $\mathbb{Z}$–form of n_+, as the corollary to Proposition 5 shows. Recall that a finite–dimensional semisimple complex Lie algebra has several Chevalley $\mathbb{Z}$–forms; in order to write down the corresponding structure constants, the subtle point are the signs. In [**T**], Tits gave a complete description of all possible sign choices. The signs which we obtain using the representation theory of quivers were first exhibited by Frenkel and Kac [**FK**].

Finally, let us mention that $\mathcal{H}(\Delta)_1$ itself may be considered as a corresponding Kostant $\mathbb{Z}$–form of $U(n_+)$: we can order the positive roots $\alpha_1, \ldots, \alpha_m$ in such a way that for $R = kQ(\Delta)$ we have $\mathrm{Hom}(M(\alpha_i, R), M(\alpha_i, R))$ only in case $i \le j$. Thus we can apply Proposition 3 in order to conclude that for $u_i = u_{\alpha_i}$ and $a(i) \in \mathbb{N}_0$ we have

$$u_1^{a(1)} \ldots u^{a(m)} = \prod_{i=1}^{m} \psi_{a(i)} \cdot u_a$$

in $\mathcal{H}(\Delta)$. Now $\psi_{a(i)}(1) = a(i)!$, thus, in $\mathcal{H}(\Delta)_1$, we have

$$u_1^{a(1)} \ldots u_m^{a(m)} = \prod_{i=1}^{m} a(i)! \cdot u_a,$$

and we can rewrite this as

$$u_a = \frac{u_1^{a(1)}}{a(1)!} \ldots \frac{u_m^{a(m)}}{a(m)!}.$$

The right side is a typical generator in the Kostant $\mathbb{Z}$–form, on the left we just have the basis element of $\mathcal{H}(\Delta)_1$ corresponding to the module $M(a, R)$; note that all R–modules are of the form $M(a, R)$, so the rather technical looking generators in the Kostant $\mathbb{Z}$–form do correspond just to modules.

For a full proof of Theorem 1 we refer to [**R3**] and [**R4**]. This proof first establishes the corollary and derives from this the theorem. The proof of the corollary presented in [**R3**] relies on the actual calculation of the Hall polynomials as stated in Proposition 5. A more direct proof will be given in [**R5**].

4. Composition algebras and Loewy algebras

In order to deal with symmetric generalized Cartan matrices which are not of finite type we have to change our view point slightly. For general symmetric

generalized Cartan matrices, there does not exist a generic Hall algebra: the idea of constructing a generic Hall algebra relies on the fact that the set of finite–dimensional $kQ(\Delta)$–modules, for Δ of finite type, is indexed by a fixed set (namely the set of function $\Phi^+ \longrightarrow \mathbb{N}_0$) independent of k, and that on this set we have various multiplications using different finite fields k. However, if Δ is not of finite type, then already the set of finite–dimensional $kQ(\Delta)$–modules will depend on k. Of course, there are classes of $kQ(\Delta)$–modules which can be indexed without reference to k, for example the simple, or more generally, the semisimple modules.

Let R be a finite ring. The subring $\mathcal{C}(R)$ of $\mathcal{H}(R)$ generated by the elements $u_{[S]}$ with S simple is called the *composition algebra* of R, the subring $\mathcal{L}(R)$ of $\mathcal{H}(R)$ generated by the elements $u_{[M]}$ with M semisimple will be called the *Loewy algebra* of R, since it encodes the possible Loewy series of modules (a *Loewy series* of a module is by definition a filtration with semisimple factors).

The rings $\mathcal{C}(R)$ and $\mathcal{L}(R)$ may be constructed also as follows: Let $S_1, \ldots, S_n$ be a complete set of simple R–modules, we denote the isomorphism class of S_i by $s_i = [S_i]$, and we define $S(s_i) = S_i$. We use the set $\mathcal{S} = \{s_1, \ldots, s_n\}$ as a set of letters in order to form words. Thus, let $\mathcal{W} = \mathcal{W}(\mathcal{S})$ be the set of words in $\mathcal{S}$ (there is the empty word 1, the remaining words are of the form $w = a_1 a_2 \ldots a_t$ with $t \geq 1$ and all $a_i \in \mathcal{S}$). Of course, words are multiplied using juxtaposition, thus $\mathcal{W}$ is a semigroup. Given $w = a_1 a_2 \ldots a_t \in \mathcal{W}$, and M and R–module, let $F_w^M = F_{S(a_1)\ldots S(a_t)}^M$, and, of course, $F_1^M = 0$ for $M \neq 0$, and $F_1^0 = 1$. Note that F_w^M counts the number of composition series of M of tye w, the *type* of the composition series $M = M_0 \supset M_1 \supset \ldots \supset M_t = 0$ of M being the word $[M_0/M_1] \ldots [M_{t-1}/M_t] \in \mathcal{W}$. Note that the free $\mathbb{Z}$–algebra $\mathcal{A}(\mathcal{S})$ generated by $\mathcal{S}$ is just the semigroup algebra $\mathbb{Z}(\mathcal{W}(\mathcal{S}))$. In $\mathcal{A}(\mathcal{S})$, we consider the set $\mathcal{I}(R)$ of all finite linear combinations $\sum_i \lambda_i w_i$ with $\lambda_i \in \mathbb{Z}$, $w_i \in \mathcal{W}$ such that $\sum_i \lambda_i F_{w_i}^M = 0$ for all R–modules M. Clearly, $\mathcal{I}(R)$ is an ideal, it is just the kernel of the canonical surjection $\mathcal{A}(\mathcal{S}) \longrightarrow \mathcal{C}(R)$ which sends s_i onto $u_{[S(s_i)]}$. Thus,

$$\mathcal{C}(R) \cong \mathcal{A}(\mathcal{S})/\mathcal{I}(R).$$

Similarly, given $d \in \mathbb{N}_0^n$, let $s_d = [\oplus d(i)S_i]$, and $S(s_d) = \oplus d(i)S_i$. Again, we may consider the set $\tilde{\mathcal{W}}$ of words in $\tilde{\mathcal{S}} = \{s_d | d \in \mathbb{N}_0^n\}$. Given a word $a_1 \ldots a_t \in \tilde{\mathcal{W}}$ with $a_i \in \tilde{\mathcal{S}}$, and M an R–module, let $F_{a_1\ldots a_t}^M = F_{S(a_1)\ldots S(a_t)}^M$, this counts the number of filtrations $M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_t = 0$ with $a_i = [M_{i-1}/M_i]$, for $1 \leq i \leq t$. Thus, here we deal with Loewy series. As before, we denote by $\mathcal{A}(\tilde{\mathcal{S}})$ the free $\mathbb{Z}$–algebra generated by $\tilde{\mathcal{S}}$, and denote by $\tilde{\mathcal{I}}(R)$ the set of finite linear

combinations $\sum \lambda_i w_i$ with $\lambda_i \in \mathbb{Z}$, $w_i \in \tilde{\mathcal{W}}$ such that $\sum \lambda_i F^M_{w_i} = 0$ for all R–modules M. Then the canonical map $\mathcal{A}(\tilde{\mathcal{S}}) \longrightarrow \mathcal{L}(R)$, which sends s_d onto $u_{[S(s_d)]}$ yields an isomorphism

$$\mathcal{L}(R) \cong \mathcal{A}(\tilde{\mathcal{S}})/\tilde{\mathcal{I}}(R).$$

Proposition 6. *If $S_1, \ldots, S_n$ are the simple R–modules and $\operatorname{Ext}^1_R(S_i, S_j) = 0$ for $i \geq j$, then $\mathcal{C}(R) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = \mathcal{L}(R) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. If R is representation–directed, then $\mathcal{L}(R) = \mathcal{H}(R)$.*

Proof: We assume $\operatorname{Ext}^1_R(S_i, S_j) = 0$ for $i \geq j$, and let $u_i = u_{[S_i]}$. For $d \in \mathbb{N}_0^n$, let $S(d) = \oplus d(i) S_i$, and $u_d = u_{[S(d)]}$. We can apply Proposition 4 and conclude that

$$u_d = u_{[S(d)]} = \prod \frac{1}{\psi_{d(i)}(q_i)} \cdot u_1^{d(1)} \cdots u_n^{d(n)},$$

where $q_i = |\operatorname{End}(S_i)|$. This yields the first assertion.

Assume now that R is representation–directed, with indecomposable modules $X_1, \ldots, X_m$ such that $\operatorname{Hom}(X_i, X_j) = 0$ for $i > j$. Note that this implies that $\operatorname{End}(X_i)$ is a division ring, thus a field, and that $\operatorname{Ext}^1(X_i, X_j) = 0$ for $i \leq j$. Thus, again we can use Proposition 4 and conclude that for $a \in \mathbb{N}^m$,

(*) $$u_{[\oplus a(i) X_1]} = u_{[a(1) X_1]} \cdots u_{[a(m) X_m]}.$$

It remains to see that for X indecomposable, $b \in \mathbb{N}_0$, the element $u_{[bX]}$ belongs to $\mathcal{L}(R)$. We use induction on the length of bX. Given $d \in \mathbb{N}_0^n$, we have

$$u_{[d(n) S_n]} \cdots u_{[d(1) S_1]} = \sum_{\mathbf{dim}\, M = d} u_{[M]}.$$

For, since S_n is projective, any module M with $\mathbf{dim}\, M = d$ has a unique submodule M_1 with $M/M_1 \cong d(n) S_n$, thus for any module M with $\mathbf{dim}\, M = d$, we have $F^M_{d(n) S_n, \ldots, d(1) S_1} = 1$. Therefore for $d = \mathbf{dim}\, bX$, we have

$$u_{[bX]} = u_{[d(n) S_n]} \cdots u_{[d(1) S_1]} - \sum_{\substack{\mathbf{dim}\, M = d \\ [M] \neq [bX]}} u_{[M]}.$$

Consider a module M with $\mathbf{dim}\, M = \mathbf{dim}\, bX$ and $[M] \neq [bX]$. It is well–known that such a module M has at least two non–isomorphic indecomposable

direct summands, thus $M \cong \overset{m}{\underset{i=1}{\oplus}} a(i)X_i$, and all $a(i)X_i$ are proper submodules. By induction, $u_{[a(i)X_i]} \in \mathcal{L}(R)$, and therefore $u_{[M]} \in \mathcal{L}(R)$, according to (*). It follows that $u_{[bX]} \in \mathcal{L}(R)$. This completes the proof.

Generic Loewy algebras

Let Δ be a symmetric generalized Cartan matrix. For $1 \leq i \leq n$, and k a field, we denote by $S(i, kQ(\Delta))$ the simple $kQ(\Delta)$–module corresponding to the vertex i of $Q(\Delta)$. For $d \in \mathbb{N}_0^n$, let $S(d, kQ(\Delta)) = \underset{i}{\oplus} d(i)S(i, kQ(\Delta))$. We consider the product

$$\prod_k \mathcal{H}(kQ(\Delta))$$

where k runs through all (isomorphism classes of) finite fields, its elements are of the form $x = (x^{(k)})_k$, where $x^{(k)} \in \mathcal{H}(kQ(\Delta))$. In particular, we will have to consider the element $q = (q^{(k)})_k$, where $q^{(k)} = |k| = |k|u_{[0]}$, for $1 \leq i \leq n$ the elements $u_i = (u_i^{(k)})_k$, and for $d \in \mathbb{N}_0^n$ the elements $u_d = (u_d^{(k)})_k$, where $u_i^{(k)} = u_{[S(i,kQ(\Delta))]}$, $u_d^{(k)} = u_{[S(d,kQ(\Delta))]}$. Note that q is a central element of $\prod \mathcal{H}(kQ(\Delta))$.

The *generic composition algebra* $\mathcal{C}(\Delta)$ is the subring of $\prod \mathcal{H}(kQ(\Delta))$ generated by q and u_i, where $1 \leq i \leq n$. The *generic Loewy algebra* $\mathcal{L}(\Delta)$ is the subring generated by q and u_d, where $d \in \mathbb{N}_0^n$. Note that we have $\mathcal{C}(\Delta) \otimes \mathbb{Q} = \mathcal{L}(\Delta) \otimes \mathbb{Q}$, by propositon 4.

In order to define the corresponding extended algebras, we have to be aware that the generalized Cartan matrix $\Delta = (a_{ij})_{1 \leq i,j \leq n}$ may be singular, so that we have to deal with a realisation of Δ (see [**K**]). Let l be the rank of Δ, let $n' = 2n - l$, and let $\Delta' = (a'_{ij})_{1 \leq i,j \leq n'}$ be a non–singular $n' \times n'$–matrix with integer coefficients such that $a'_{ij} = a_{ij}$ for all $1 \leq i, j \leq n$. For $1 \leq i \leq n'$, let $\gamma_i = \sum_{j=1}^n a'_{ij}\delta_j$, this is a derivation of $\mathcal{H}(kQ(\Delta))$. We form the skew polynomial ring

$$\mathcal{H}'(kQ(\Delta)) = \mathcal{H}(kQ(\Delta))[H_i, \gamma_i]_{1 \leq i \leq n'},$$

thus $H_1, \ldots, H_{n'}$ are indeterminates satisfying the commutation rules

$$[H_i, H_j] = 0, \quad \text{for } 1 \leq i, j \leq n',$$

$$[H_i, u_j] = a'_{ij}u_j, \quad \text{for } 1 \leq i \leq n', 1 \leq j \leq n.$$

In $\prod \mathcal{H}'(kQ(\Delta))$ we denote the element $(H_i^{(k)})_k$ with $H_i^{(k)} = H_i \in \mathcal{H}'(kQ(\Delta))$ again by H_i. Let $\mathcal{C}'(\Delta)$ be the subring of $\prod \mathcal{H}'(kQ(\Delta))$ generated by q and all

u_i, H_j, where $1 \leq i \leq n$, $1 \leq j \leq n'$, and let $\mathcal{L}'(\Delta)$ be its subring generated by q, and all u_d, H_j, where $d \in \mathbb{N}_0^n$, and $1 \leq j \leq n'$. There is the following alternative description. The derivations γ_i of the various $\mathcal{H}(kQ(\Delta))$ yield a derivation, again denoted by γ_i, of $\prod \mathcal{H}(kQ(\Delta))$, and both $\mathcal{C}(\Delta)$ and $\mathcal{L}(\Delta)$ are mapped into itself under γ_i, thus we can form the corresponding skew polynomial rings and we have

$$\mathcal{C}'(\Delta) = \mathcal{C}(\Delta)[H_i, \gamma_i]_{1 \leq i \leq n'}, \quad \mathcal{L}'(\Delta) = \mathcal{L}(\Delta)[T_i, \gamma_i]_{1 \leq i \leq n'}.$$

Note that instead of starting with $\prod \mathcal{H}(kQ(\Delta))$, we may consider the $\mathbb{Z}^n$–graded subring

$$\bigoplus_d \prod_k \mathcal{H}(kQ(\Delta))_d,$$

since both $\mathcal{C}(\Delta)$ and $\mathcal{L}(\Delta)$ lie inside this subring.

Since q is a central element, we will consider $\mathcal{C}(\Delta), \mathcal{L}(\Delta), \mathcal{C}'(\Delta)$ and $\mathcal{L}'(\Delta)$ as $\mathbb{Z}[q]$–algebras, and we form

$$\widehat{\mathcal{L}'(\Delta)} := \lim_{\substack{\leftarrow \\ m}} \mathcal{L}'(\Delta) \underset{\mathbb{Z}[q]}{\otimes} \mathbb{C}[q]/(q-1)^m$$

(since $\mathcal{C}'(\Delta) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = \mathcal{L}'(d) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, there is no need to introduce also the notation $\widehat{\mathcal{C}'(\Delta)}$). Let $(b_{ij})_{1 \leq i,j \leq n'} = (\Delta')^{-1}$, and define

$$T_i = \sum_{j=1}^{n'} b_{ij} H_j,$$

for $1 \leq i \leq n$. Then, clearly,

$$[T_i, u_i] = u_i, \quad \text{and} \quad [T_i, u_j] = 0 \quad \text{for } i \neq j,$$

since $\sum_{t=1}^{n'} b_{it} a'_{tj} = 1$ for $i = j$, and zero otherwise. As before, we consider instead of u_i the element

$$X_i := \exp(-\frac{1}{2} \sum_{j=1}^{i-1} a_{ij} T_j \ln q) u_i.$$

The elements X_i, H_j with $1 \leq i \leq n$, $1 \leq j \leq n'$ generate $\widehat{\mathcal{L}'(\Delta)}$ as a complete Λ–algebra. By the definition of H_i, and X_i the following relations are satisfied

(*) $$[H_i, H_j] = 0, \quad \text{for } 1 \leq i, j \leq n',$$

(**) $$[H_i, X_j] = a'_{ij} X_j, \quad \text{for } 1 \le i \le n', 1 \le j \le n.$$

According to Proposition 2, we have

$$\rho_n^+(q, u_i, u_j) = 0, \ {}^+\rho_n(q, u_j, u_i) = 0 \text{ for } i < j, \text{ where } n = 1 - a_{ij},$$

and, as before, this translates to

(***) $$\sum_{t=0}^{n} (-1)^t \begin{bmatrix} n \\ t \end{bmatrix}_q q^{-\frac{t(n-t)}{2}} X_i^t X_j X_i^{n-t} = 0 \ \text{ with } n = 1 - a_{ij}, \text{ and } i \ne j.$$

Altogether, we see:

Theorem 2. *Let Δ be a symmetric generalized Cartan matrix. Then $\widehat{\mathcal{L}'(\Delta)}$ is generated, as a complete Λ–algebra, by $H_1, \ldots, H_{2n-l}, X_1, \ldots, X_n$, and these elements satisfy the relations (*), (**), and (***).*

It follows that there is a surjective ring homomorphism

$$\eta_\Delta : U_q(b_+(\Delta)) \longrightarrow \widehat{\mathcal{L}'(\Delta)},$$

since, by definition, $U_q(b_+(\Delta))$ is the free complete Λ–algebra, with generators $X_1, \ldots, X_n, H_1, \ldots, H_{2n-l}$ and relations (*), (**) and (***).

We have seen in Theorem 1 that η_Δ is an isomorphism for Δ of finite type. We want to announce the following result:

Theorem 3. *If Δ is of affine type, then η_Δ is an isomorphism.*

The proof uses the structure theory of $\text{mod } kQ(\Delta)$, it will be given in [**R5**].

6. Symmetrizable generalized Cartan matrices

The results presented above all generalize to the case of a symmetrizable generalized Cartan matrix. In this case, we have to deal with species instead of quivers, see [**R2**], [**R3**], [**R4**] and [**R5**]. The actual calculation of the corresponding Hall polynomials as in Proposition 5 above yields direct sign choices for the Chevalley $\mathbb{Z}$–forms of the Lie algebras of type B_n, C_n, F_4, G_2 which seem to be new.

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