Iyama's finiteness theorem via strongly quasi-hereditary algebras.

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Abstract. Let Λ be an artin algebra and X a Λ-module. Iyama has shown that there exists a module Y such that the endomorphism ring $\Gamma(X)$ of $X \oplus Y$ is quasi-hereditary, with a heredity chain of length $\mathrm{d}(X)$, where $\mathrm{d}(X)$ is the smallest natural number such that the "iterated derivative" $\partial^{\mathrm{d}(X)}X$ is the zero module. As one knows, any quasi-hereditary algebra Γ has finite global dimension: if Γ has a heredity chain of length n, then the global dimension of Γ is at most $2n-2$. Now Iyama asserts that the global dimension of $\Gamma(X)$ is at most $\mathrm{d}(X)$, which is a much better estimate than $2\,\mathrm{d}(X)-2$ (except in case $\mathrm{d}(X)=1$ so that Γ is semisimple). We are going to reformulate the better bound as follows: The endomorphism ring $\Gamma(X)$ is strongly quasi-hereditary with a heredity chain of length $\mathrm{d}(X)$, and any strongly quasi-hereditary algebra with a heredity chain of length n has global dimension at most n. By definition, the strongly quasi-hereditary algebras are the quasi-hereditary algebras with all standard modules of projective dimension at most 1.

Preliminaries. If Λ is an artin algebra, then $\operatorname{mod}\Lambda$ denotes the category of finitely generated left Λ-modules. Morphisms will be written on the opposite side of the scalars, thus if $f\colon X \to Y$ and $g\colon Y \to Z$ are Λ-homomorphisms between Λ-modules, then the composition is denoted by fg.

Recall that the radical rad of $\operatorname{mod}\Lambda$ is defined as follows: If X, Y are Λ-modules and $f\colon X \to Y$, then f belongs to $\operatorname{rad}(X,Y)$ provided for any indecomposable direct summand X' of X with inclusion map $u\colon X' \to X$ and any indecomposable direct summand Y' of Y with projection map $p\colon Y \to Y'$, the composition $ufp\colon X' \to Y'$ is non-invertible.

1. Derivation. Let X be a (left) Λ-module, let $\mathbf{r}$ be the radical of the endomorphism ring of X. We put $\partial X = X\mathbf{r}$, this is a Λ-submodule of X.

Warning 1. Note that the two submodules $\operatorname{rad} X$ and $X\mathbf{r}$ of X usually are incomparable. As an example, consider the Kronecker algebra Λ. Let $X = R[2]$ be a 4-dimensional indecomposable regular Kronecker module with a 2-dimensional regular submodule $R[1]$. Here, $\operatorname{rad} X = \operatorname{soc} X$ is semisimple and of length 2, whereas $X\mathbf{r} = R[1]$ is also of length 2, but indecomposable.

(1_0) *The module X generates the module ∂X.* Proof: Let $\phi_1, \dots, \phi_m$ be a generating set of $\operatorname{rad}(X,X)$, say as a k-module, where k is the center of Λ. Then $\partial X = \sum_i \phi_i(X)$, thus the map $\phi = (\phi_i)_i\colon X^t \to \partial X$ is surjective.

Warning 2. One is tempted to say that X generates ∂X by radical maps, but this is usually not true! For example, let X be the regular representation of the quiver of type A_2. Then ∂X is simple projective and the non-zero maps $X \to \partial X$ are not radical maps. (What is true, is the following: ∂X is generated by X using maps which have the property that when we compose them with the inclusion map $\partial X \subseteq X$, then they become radical maps.)

(2_0) *Any radical map $X \to X$ factors through ∂X.* Proof: This follows directly from the construction.

(3_0) *If $X = \bigoplus_i X_i$, then $\partial X = \bigoplus_i X \operatorname{rad}(X, X_i)$.*

(4_0) *If X is non-zero, then ∂X is a proper submodule of X.* Proof. The ring $\Gamma = \operatorname{End}(X)$ is again an artin algebra and the radical of a non-zero Γ-module is a proper submodule (it is enough to know that Γ is semi-primary).

2. Iterated derivative. We define inductively $\partial^0 X = X, \partial^{t+1} X = \partial(\partial^t X)$.

Warning 3. Note that $\partial^2 X$ usually is different from $X\mathbf{r}^2$, a typical example will be given by a serial module with composition factors $1, 1, 2, 1, 1$ (in this order) such that the submodule of length 2 and the factor module of length 2 are isomorphic. Here, $X\mathbf{r}^2 = 0$, whereas $\partial^2 X$ is simple.

(1) *If $i \le j$, then $\partial^i X$ generates $\partial^j X$.*

(2) *Any radical map $\partial^{i-1} X \to \partial^{i-1} X$ factors through $\partial^i X$.*

(3) Let $t \ge 1$. If $\partial^{t-1} X = \bigoplus_i N_i$, then

$$\partial^t X = \bigoplus\nolimits_i \, (\partial^{t-1} X) \operatorname{rad}(\partial^{t-1} X, N_i)$$

(here, $(\partial^{t-1} X) \operatorname{rad}(\partial^{t-1} X, N_i)$ is a sumodule of N_i).

($3'$) In particular, *an indecomposable summand N of $\partial^t X$ is a submodule of an indecomposable summand of $\partial^{t-1} X$*, thus by induction we see: *an indecomposable summand N of $\partial^t X$ is a submodule of an indecomposable summand of X.*

(4) *We have $\partial^{|X|} X = 0$, where $|X|$ denotes the length of X.*

Thus if we define $\mathrm{d}(X)$ as the smallest natural number with $\partial^{\mathrm{d}(X)} X = 0$, then this number exists and $\mathrm{d}(X) \le |X|$.

Let $\mathcal{C}_i = \mathcal{C}_i(X) = \operatorname{add}\{\partial^j X \mid i \le j\}$. Thus we obtain a filtration

$$\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \cdots \supseteq \mathcal{C}_{n-1} \supseteq \mathcal{C}_n = \{0\}.$$

(5) **Main Lemma.** *Let $i \ge 1$. Let N be an indecomposable direct summand of $\partial^{i-1} X$ which is not a direct summand of $\partial^i X$. Let*

$$\alpha(N) = (\partial^{i-1} X) \operatorname{rad}(\partial^{i-1} X, N).$$

Then $\alpha(N)$ is a proper submodule of N and the inclusion map $\alpha(N) \to N$ is a right $\mathcal{C}_i$-approximation (and of course right minimal).

Proof: First, we show that $\alpha(N)$ is a proper submodule of N. Namely, if $\alpha(N) = N$, then (3) shows that N belongs to $\operatorname{add} \partial^i X$. But this is not the case.

Second, observe that $\alpha(N)$ belongs to $\mathcal{C}_i$. Namely, since N is a direct summand of $\partial^{i-1}X$, we see that $\alpha(N) = (\partial^{i-1}X)\operatorname{rad}(\partial^{i-1}X, N)$ is a direct summand of $\partial^i X$, again using (3).

Third, in order to see that the inclusion map $u\colon \alpha(N) \to N$ is a right $\mathcal{C}_i$-approximation, we have to show that any map $g\colon \partial^j X \to N$ with $j \geq i$ factors through u, thus that the image of g is contained in $\alpha(N)$. By (1), there are surjective maps

$$\left(\partial^{i-1}X\right)^t \xrightarrow{\eta} \left(\partial^i X\right)^{t'} \xrightarrow{\eta'} \partial^j X$$

We claim that the composition $\eta\eta' g$

$$\left(\partial^{i-1}X\right)^t \xrightarrow{\eta} \left(\partial^i X\right)^{t'} \xrightarrow{\eta'} \partial^j X \xrightarrow{g} N$$

is a radical map. Otherwise, there is an indecomposable direct summand U of $\left(\partial^{i-1}X\right)^t$ such that the composition

$$U \to \left(\partial^{i-1}X\right)^i \xrightarrow{\eta} \left(\partial^i X\right)^{t'} \xrightarrow{\eta' g} N$$

is an isomorphism, but then N is a direct summand of $\partial^i X$, but this is not the case.

It follows that the image of $\eta\eta' g$ is contained in $(\partial^{i-1}X)\operatorname{rad}(\partial^{i-1}X, N) = \alpha(N)$. Since $\eta\eta'$ is surjective, it follows that the image of g itself is contained in $\alpha(N)$.

(6) *Let N be an indecomposable summand of $\partial^{i-1}X$ which is not a direct summand of $\partial^i X$. Then N is not a direct summand of $\partial^j X$ for any $j \geq i$.* Proof: If N would be a direct summand of $\partial^j X$ for some $j \geq i$, then we can factor the identity map $N \to N$ through the inclusion map $\alpha(N) \to N$. But then $\alpha(N) = N$ and N is a direct summand of $\partial^i X$, a contradiction.

3. Theorem. *Let X be a module. Write $\mathcal{C}_i = \mathcal{C}_i(X)$. Choose a module M such that $\mathcal{C}_0 = \operatorname{add} M$ and let $\Gamma = \operatorname{End}(M)$. Let N be indecomposable in $\mathcal{C}_{i-1} \setminus \mathcal{C}_i$ for some $i \geq 1$. Then the minimal right $\mathcal{C}_i$-approximation $u\colon \alpha(N) \to N$ yields an exact sequence*

$$0 \to \operatorname{Hom}(M, \alpha(N)) \xrightarrow{\operatorname{Hom}(M,u)} \operatorname{Hom}(M, N) \to \operatorname{Hom}(M, N)/\langle \mathcal{C}_i \rangle \to 0$$

of Γ-modules.

The module $R(N) = \operatorname{Hom}(M, \alpha(N))$ is projective, and the composition factors of $\operatorname{top} R(N)$ are of the form $S(N'')$ with $N'' \in \mathcal{C}_i$.

The endomorphism ring of $\Delta(N) = \operatorname{Hom}(M, N)/\langle \mathcal{C}_i \rangle$ is a division ring and any composition factor of $\operatorname{rad} \Delta(N)$ is of the form $S(N')$ where N' is an indecomposable Λ-module in $\mathcal{C}_0 \setminus \mathcal{C}_{i-1}$.

Proof: Since u is injective, also $\operatorname{Hom}(u, -)$ is injective. Now $\alpha(N)$ belongs to $\mathcal{C}_i$, thus $\operatorname{Hom}(M, \alpha(N))$ is mapped unter u to a set of maps $f: M \to N$ which factor through $\mathcal{C}_i$. But since u is a right $\mathcal{C}_i$-approximation, we see that the converse also is true: any map $M \to N$ which factors through $\mathcal{C}_i$ factors through u. This shows that the cokernel of $\operatorname{Hom}(M, u)$ is $\operatorname{Hom}(M, N)/\langle \mathcal{C}_i \rangle$.

Of course, $R(N)$ is projective. If we decompose $\alpha(N)$ as a direct sum of indecomposable modules N'', then $\operatorname{Hom}(M, \alpha(N))$ is a direct sum of the corresponding projective Γ-modules $\operatorname{Hom}(M, N'')$ with N'' indecomposable and in $\mathcal{C}_i$, and $\operatorname{top} R(N)$ is the direct sum of the corresponding simple Γ-modules $S(N'')$.

Now we consider $\Delta(N)$. Let N' be an indecomposable direct summand of M such that $S(N')$ is a composition factor of $\Delta(N)$. This means that there is a map $f: N' \to N$ which does not factor through $\mathcal{C}_i$. In particular, N' itself does not belong to $\mathcal{C}_i$, thus N' is a direct summand of $\partial^{j-1}X$ with $j \le i$. Also N is a direct summand of $\partial^{i-1}X$. Now, according to (2) any radical map $\partial^{i-1}X \to \partial^{i-1}X$ factors through $\partial^i X$, thus f has to be invertible. This shows that for $N' \in \mathcal{C}_i$, the only possibility is that N' is isomorphic to N and that the composition factor of $\Delta(N)$ given by the map f is the top composition factor. Thus, $S(N)$ appears exactly once as composition factor of $\Delta(N)$, namely at the top: this shows that the endomorphism ring of $\Delta(N)$ is a division ring. Also we have shown that the remaining composition factors of $\Delta(N)$, thus those of $\operatorname{rad} \Delta(N)$ are of the form $S(N')$ with N' indecomposable and in $\mathcal{C}_{i-1}$.

4. Strongly quasi-hereditary algebras. Let Γ be an artin algebra. Let $\mathcal{S} = \mathcal{S}(\Gamma)$ be the set of isomorphism classes of simple Γ-modules. For any module M, let $P(M)$ be the projective cover of M.

We say that Γ is (left) *strongly quasihereditary with n layers* provided there is a function $l: \mathcal{S} \to \{1, 2, \dots, n\}$ (the layer function) such that for any $S \in \mathcal{S}(\Gamma)$, there is an exact sequence

$$0 \to R(S) \to P(S) \to \Delta(S) \to 0$$

with the following two properties: First of all, if S' is a composition factor of $\operatorname{rad} \Delta(S)$, then $l(S') < l(S)$. And second, $R(S)$ is a direct sum of projective modules $P(S'')$ with $l(S'') > l(S)$.

Proposition. *If Γ is strongly quasi-hereditary with n layers, then Γ is quasi-hereditary and the global dimension of Γ is at most n.*

Proof. The top of $R(S)$ is given by simple modules S' with $l(S') > l(S)$, thus $\Delta(S)$ is the maximal factor module with composition factors S' such that $l(S') \le l(S)$. Since S does not occur as composition factor of $\operatorname{rad} \Delta(S)$, we see that the endomorphism ring of $\Delta(S)$ is a division ring.

It remains to be shown that $P(S)$ has a Δ-filtration for all S. This we show by decreasing induction on $l(S)$. If $l(S) = n$, then $P(S) = \Delta(S)$. Assume we know that all $P(S)$ with $l(S) > i$ have a Δ-filtration. Let $l(S) = i$. Then $R(S)$ is a direct sum of projective modules $P(S')$ with $l(S') > l(S)$, thus it has a Δ-filtration. Then also $P(S)$ has a Δ-filtration. This shows that Γ is quasi-hereditary (see for example [DR]).

Now we have to see that the global dimension of Γ is at most n. We show by induction on $l(S)$ that $\operatorname{proj.dim} S \leq l(S)$. We start with $l(S) = 1$. In this case, $\Delta(S) = S$, thus there is the exact sequence $0 \to R(S) \to P(S) \to S \to 0$ with $R(S)$ projective. This shows that $\operatorname{proj.dim} S \leq 1$. For the induction step, consider some $i \geq 2$ and assume that $\operatorname{proj.dim} S \leq l(S)$ for all S with $l(S) < i$. Now there is the exact sequence

$$0 \to R(S) \to \operatorname{rad} P(S) \to \operatorname{rad} \Delta(S) \to S \to 0.$$

All the composition factors S' of $\operatorname{rad} \Delta(S)$ satisfy $l(S') < i$, thus $\operatorname{proj.dim} S' < i$. Also, $R(S)$ is projective, thus $\operatorname{proj.dim} R(S) = 0 < i$. This shows that $\operatorname{rad} P(S)$ has a filtration whose factors have projective dimension less than i, and therefore $\operatorname{proj.dim} \operatorname{rad} P(S) < i$. As a consequence, $\operatorname{proj.dim} S \leq i$.

Since all the simple Γ-modules have layer at most n, it follows that all the simple modules have projective dimension at most n, thus the global dimension of Γ is bounded by n.

5. Theorem. *Let X be a Λ-module. Then there is a Λ-module Y such that $\Gamma = \operatorname{End}(X \oplus Y)$ is strongly quasi-hereditary with $\mathrm{d}(X)$ layers. In particular, the global dimension of Γ is at most $\mathrm{d}(X)$.*

In addition, we record:

- $\mathrm{d}(X) \leq |X|$
- The construction of Y yields a module with the following property: *Any indecomposable direct summand of the module Y is a submodule of an indecomposable direct summand of X.*

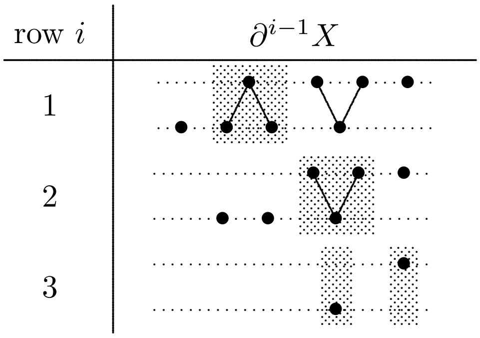
Proof: By definition, $\partial^{\mathrm{d}(X)} X = 0$. Take $Y = \bigoplus_{i=1}^{\mathrm{d}(X)-1} \partial^i X$, and $M = X \oplus Y$ with endomorphism ring $\Gamma = \operatorname{End}(M)$. Also, let $\mathcal{C}_i = \mathcal{C}_i(X)$. If N is an indecomposable module in $\mathcal{C}_{i-1} \setminus \mathcal{C}_i$, with $i \geq 1$, we define the layer $l(S(N)) = i$. Thus we obtain a layer function with values in $\{1, 2, \dots, n\}$. According to theorem, Γ is left strongly quasi-hereditary with n layers, thus the global dimension of Γ is bounded by n, according to section 4.

The additional information comes from (4) and (3$'$) in section 2.

6. Corollary. *The representation dimension of Λ is at most $2|\Lambda|$.*

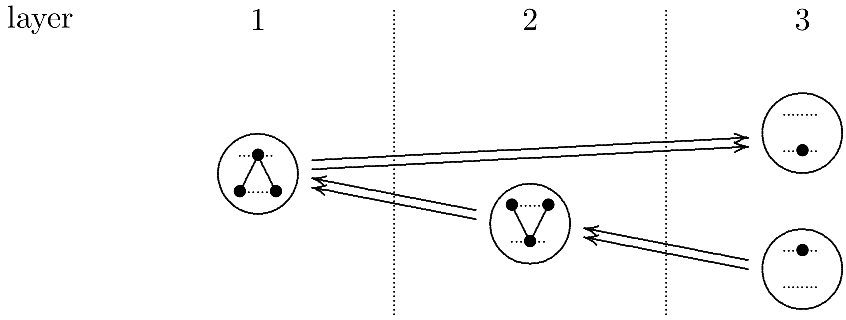
Proof: Consider the module $X = \Lambda \oplus D\Lambda$. Its length is $n = |\Lambda \oplus D\Lambda| = 2|\Lambda|$. Let $M = X \oplus Y$ as in Theorem. By construction, M is a generator-cogenerator, thus the representation dimension of Λ is bounded by n.

7. Example. Let us consider in detail the minimal generator-cogenerator $X = \Lambda \oplus D\Lambda$ for the Kronecker algebra Λ.



In row i ($1 \leq i \leq 3$) we have exhibited the indecomposable direct summands N of the module $\partial^{i-1}X$ by specifying a suitable basis of N using bullets; these bullets are connected by arrows pointing downwards (we draw just line segments) which indicate scalar multiplications by some elements of Λ. The modules in $\mathcal{C}_{i-1} \setminus \mathcal{C}_i$ are shaded.

The quiver of Γ with its layer structure looks as follows:



If we denote the simple Γ-modules by $1, 2, 3, 3'$, where $1, 3$ correspond to the projective Λ-modules, $2, 3$ to the injective Λ-modules and $3, 3'$ to the simple Λ-modules, then the indecomposable projective Γ-modules look as follows

$$\begin{smallmatrix} & 1 & \\ 3 & & 3 \end{smallmatrix} \qquad \begin{smallmatrix} & 2 & \\ 1 & & 1 \\ & 3 & \end{smallmatrix} \qquad 3 \qquad \begin{smallmatrix} & 3' & \\ 2 & & 2 \\ & 1 & \end{smallmatrix}$$

References.

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[DR] Dlab, V., Ringel C. M.: The module theoretical approach to quasi-hereditary algebras. In: Representations of Algebras and Related Topics. London Math. Soc. Lecture Note Series 168 (1992), 200-224.