Gorenstein-projective modules over short local algebras.

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Abstract: Following the well-established terminology in commutative algebra, any (not necessarily commutative) finite-dimensional local algebra A with radical J will be said to be *short* provided $J^3{=}0$. As in the commutative case, we show: If a short local algebra A has an indecomposable non-projective Gorenstein-projective module M, then either A is self-injective (so that all modules are Gorenstein-projective) and then, of course, $|J^2|{\le}1$, or else $|J^2|{=}|J/J^2|{-}1$ and $|JM|{=}|J^2||M/JM|$. More generally, we focus the attention to semi-Gorenstein-projective and ∞-torsionfree modules, even to $\mho$-paths of length 2, 3 and 4. In particular, we show that the existence of a non-projective reflexive module implies that $|J^2|{<}|J/J^2|$ and further restrictions. In addition, we consider exact complexes of projective modules with a non-projective image. Again, as in the commutative case, we see that if such a complex exists, then A is self-injective or satisfies the condition $|J^2|{=}|J/J^2|{-}1$. Also, we show that any non-projective semi-Gorenstein-projective module M satisfies $\operatorname{Ext}^1(M,M){\neq}0$. In this way, we prove the Auslander-Reiten conjecture (one of the classical homological conjectures) for arbitrary short local algebras.

Many arguments used in the commutative case actually work in general, but there are interesting differences and some of our results may be new also in the commutative case.

Key words. Short local algebra, Gorenstein-projective module, semi-Gorenstein-projective module, reflexive module, n-torsionfree module, ∞-torsionfree module, $\mho$-quiver, exact complex of projective modules, Auslander-Reiten conjecture.

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1. Introduction.

1.1. The algebras and their modules

Let A be a finite-dimensional algebra with radical $J = J(A)$. The modules to be considered are left A-modules of finite length (if not otherwise asserted). We denote by $|M|$ the length of the module M. If M is a module, let $p\colon PM \to M$ be a projective cover of M and ΩM the kernel of p. The modules $\Omega^i M$ with $i \geq 1$ are the syzygy modules of M. The module $\operatorname{top} M = M/JM$ will be called the top of M and we write $t(M) = |\operatorname{top} M|$.

All algebras A considered here will be local finite-dimensional k-algebras, where k is a field, and for simplicity, we will assume that $A/J = k$. The module A/J will always be denoted by S; it is the unique simple module. Let $e = e(A) = |J/J^2|$. A local algebra A is said to be *short* provided $J^3 = 0$. Usually, we will assume that A is short and then we write $a = a(A) = |J^2|$ and call $(e(A), a(A))$ the *Hilbert-type* of A.

If M is a module with Loewy length at most 2, we call $\mathbf{dim}\, M = (t(M), |JM|)$ (or its transpose) the *dimension vector* of M (note that $\mathbf{dim}\, M$ is only defined for modules M of Loewy length at most 2; we have $\mathbf{dim}\, S = (1, 0)$ and there is no module with dimension vector $(0, 1)$). We call a module M *bipartite* provided $\operatorname{soc} M = JM$. *A module has Loewy length at most* 2 *if and only if it is the direct sum of a bipartite and a semisimple module.*

1.2. Complexes and A-duality

For any module M, let $M^* = \operatorname{Hom}(M, {}_AA)$ be the *A-dual* of M (it is a right A-module, thus an $A^{\rm op}$-module), and $\phi_M\colon M \to M^{**}$ the canonical map defined by $\phi(m)(\alpha) = \alpha(m)$ for $m \in M$ and $\alpha \in M^*$. A module M is *torsionless* if M is a submodule of a projective module, or, equivalently, if ϕ_M is a monomorphism. The module M is said to be *reflexive* if ϕ_M is bijective. Note that *an indecomposable module which is torsionless and not projective has Loewy length at most* 2.

We will consider exact complexes of projective modules, they are of the form $P_\bullet = (P_i, d_i\colon P_i \to P_{i-1})_i$, thus

$$\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \xrightarrow{d_{-1}} P_{-2} \to \cdots,$$

with projective modules P_i such that $\operatorname{Im} d_i = \operatorname{Ker} d_{i-1}$, for all $i \in \mathbb{Z}$. A module M is said to be an *image in* $P_\bullet$, provided $M = \operatorname{Im} d_i$ for some $i \in \mathbb{Z}$. The exact complex $P_\bullet$ is said to be *minimal* provided that any map d_i maps into the radical of P_{i-1}. Given a complex $P_\bullet$ of projective modules, we may form the A-dual complex $P_\bullet^*$, forming the A-dual of the modules P_i and of the maps d_i.

A module M is *Gorenstein-projective* provided it is an image in an exact complex $P_\bullet$ of projective modules, with $P_\bullet^*$ again being exact; M is *semi-Gorenstein-projective* provided $\operatorname{Ext}^i(M, A) = 0$ for all $i \geq 1$, and M is *∞-torsionfree,* provided $\operatorname{Tr} M$ is semi-Gorenstein-projective, where Tr is Auslander's transpose operator. Note that *a module M is Gorenstein projective iff M is both semi-Gorenstein-projective and ∞-torsionfree.*

1.3. The topics to be considered

The topics to be discussed in this paper (and its sequel [RZ3]) concern the module theory for a short local algebra A. The main results of the present paper are stated in Sections 1.4 to 1.8. The central question concerns the existence of non-projective Gorenstein-projective modules or of related ones, and properties of such modules. The case of A being commutative has been considered before in several papers published between 1980 and 2010 (in particular, see [L, Y, HSV, CV, AIS]). Our aim is to extend the results known for commutative rings to general rings. Some of our observations may be new also in the commutative case.

Two properties of Gorenstein-projective modules are important: Gorenstein-projective modules are reflexive, and they are images in exact complexes of projective modules. Theorems 1.1 and 1.2 announced in Section 1.4 deal with the existence of non-projective reflexive modules. Theorem 1.3 stated in Section 1.5 concerns the existence of non-projective images in exact complexes of projective modules.

1.4. Existence of reflexive modules

We say that a non-zero module M of Loewy length at most 2 is *solid* provided any endomorphism of M is a scalar multiplication on $\operatorname{soc} M$ (as a consequence, any non-invertible endomorphism vanishes on the socle). A solid module is of course indecomposable (a characterization of the solid modules using covering theory will be given in Proposition A.3 of Appendix A).

Theorem 1.1. *Let A be a short local algebra which is not self-injective. If there exists a reflexive module which is not projective, then $2 \le a \le e-1$. Also, ${}_AJ$ and the right module J_A are solid.*

The bound $a \le e-1$ cannot be improved, since Proposition 15.1 shows that for any a with $1 \le a \le e-1$, there exists an algebra of Hilbert type (e, a) with non-projective reflexive modules.

Also note that in general, ${}_AJ$ may be solid, whereas J_A is not solid, as Example 4.8 shows.

Theorem 1.1 can be rephrased: If A is any artin algebra, there exists a non-projective reflexive module iff there exists a non-projective module N with $\operatorname{Ext}^i(N, A) = 0$ for $i = 1, 2$. Namely, there is the following recipe: If N is a non-projective module with $\operatorname{Ext}^i(N, A) = 0$ for $i = 1, 2$, then the module $\Omega^2 N$ is non-projective and reflexive. The reverse construction is given by the agemo-functor $\mho = \operatorname{Tr}\Omega\operatorname{Tr}$: If M is reflexive, then $\operatorname{Ext}^i(\mho^2 M, A) = 0$, for $i = 1, 2$. This recipe is part of general considerations outlined in Section 2, which focus the attention to what we call the $\mho$-paths of A. With reference to $\mho$-paths of A, the existence assumption in Theorem 1.1 just says that there exists an $\mho$-path of length 2.

Theorem 1.1 assumes that there exists a non-projective reflexive module, thus an $\mho$-path of length 2, or equivalently, a non-projective module M with $\operatorname{Ext}^i(M, A) = 0$, for $1 \le i \le 2$. The next theorem shows that the existence of a non-projective module M with $\operatorname{Ext}^i(M, A) = 0$, for $1 \le i \le 4$, yields a stronger assertion. Again using Section 2.4, there are several reformulations. The existence of a non-projective module M with $\operatorname{Ext}^i(M, A) = 0$, for $1 \le i \le 4$ is equivalent to the existence of an $\mho$-path of length 4, and also to the existence of a non-projective reflexive module M with $\operatorname{Ext}^i(M, A) = 0$ for $i = 1, 2$.

Theorem 1.2. *Let A be a short local algebra which is not self-injective. Assume that M is an indecomposable, reflexive and non-projective module with $\operatorname{Ext}^i(M, A) = 0$ for $1 = 1, 2$. Then $2 \le a = e-1$. If $t = t(M)$, then $\mathbf{dim}\, X = (t, at)$ for $X \in \{\Omega^2 M, \Omega M, M, \mho M\}$.*

1.5. Existence of exact complexes of projective modules

Theorem 1.3. *Let A be a short local algebra which is not self-injective, with a non-zero minimal exact complex $P_\bullet = (P_i, d_i)_i$ of projective modules. Then $1 \le a = e-1$.*

Also, $M_i = \operatorname{Im} d_i$ is bipartite for $i \ll 0$. Let $t_i = t(P_i) = t(M_i)$. There is $v \in \mathbb{Z}$ such that for $i \le v$, we have $t_i = t$ and $\mathbf{dim}\, M_i = (t, at)$. And, there are just two possibilities:

Type I. *For all $i \in \mathbb{Z}$, the module M_i is bipartite with $\mathbf{dim}\, M_i = (t, at)$ (thus $t_i = t$).*

Type II. *We can choose v in such a way that first, $t_{i+1} > t_i$ for $i \ge v$, second, the module M_{v+1} is not bipartite, and third, $|JM_i| < at_i$ for $i > v$.*

For commutative rings, Theorem 1.3 is due to Christensen-Veliche [CV]; here, the case $a = 1$ does not occur. But in general, the case $a = 1$ is possible, see Example 9.2. Also, for A commutative, and $P_\bullet$ a complex of type II, all the modules M_i with $i \le v$ are bipartite, whereas we do not know whether this holds true in general. For A commutative, the existence of a non-zero minimal exact complex $P_\bullet$ of projective modules implies that $J^2 = \operatorname{soc} A$, whereas in general, it does neither imply that $J^2 = \operatorname{soc}{}_A A$, nor that $J^2 = \operatorname{soc} A_A$, see Examples 9.2 and 9.3. If $J^2 = \operatorname{soc}{}_A A$, then all the modules M_i with $i \le v$ are bipartite, see Corollary 13.3.

For a typical example of a complex $P_\bullet = (P_i, d_i)$ of type I, see Proposition 10.7: Let A be of Hilbert type $(e, e-1)$, with $e \ge 2$ and $x \in A$ with $x^2 = 0$ and $Jx = J^2$ (a left Conca element). For all i, let $P_i = {}_A A$ and d_i the right multiplication by x. Then all images in $P_\bullet$ are equal to Ax. If x is also right Conca, then also $P_\bullet^*$ is exact, thus Ax is Gorenstein-projective.

Theorem 1.3 describes the structure of a minimal exact complex of projective modules, if A is not self-injective. For A being self-injective, see Corollary A.8 in Appendix A.

1.6. Semi-Gorenstein-projective and ∞-torsionfree modules

Both Theorems 1.2 and 1.3 imply: *If A is a short local algebra which is not self-injective, with a Gorenstein-projective module which is not projective, then $2 \le a = e-1$.* There is the following strengthening.

Theorem 1.4. *Let A be a short local algebra which is not self-injective. Assume that there exists a non-projective indecomposable module M which is semi-Gorenstein-projective or ∞-torsionfree. Then $2 \le a = e-1$ and $J^2 = \operatorname{soc}{}_A A = \operatorname{soc} A_A$. Moreover, let $t = t(M)$. We have in addition:*

(1) *If M is torsionless and semi-Gorenstein-projective, then $\mathbf{dim}\,\Omega^i M = (t, at)$ for all $i \ge 0$.*
(2) *If M is ∞-torsionfree, then $\mathbf{dim}\,\mho^i M = (t, at)$ for all $i \ge 0$.*
(3) *If M is reflexive and semi-Gorenstein-projective, or if M is ∞-torsionfree, then also $\mathbf{dim}\, M^* = (t, at)$.*
(4) *If M is Gorenstein-projective, then $\mathbf{dim}\, X = (t, at)$ for $X = \Omega^i M$ and $X = \mho^i M$, where $i \ge 0$, as well as for $X = M^*$.*

Remark. In general, if A is a short local algebra and M is semi-Gorenstein-projective, its Loewy length may be 3; and if it is 2, we may have $\mathbf{dim}\, M^* \ne \mathbf{dim}\, M$ (see the algebra A mentioned in Example 9.5: the right A-module $M = m_1 A$ is semi-Gorenstein-projective and has $\mathbf{dim}\, M = (1,2)$, whereas $\mathbf{dim}\, M^* = (2,1)$, the right A-module $\mho(m_1 A)$ is also semi-Gorenstein-projective and its Loewy length is 3).

1.7. The Auslander-Reiten conjecture

Using Theorem 1.4 as well as Proposition A,5 in Appendix A we get the following result.

Theorem 1.5. *Let A be a short local algebra and M a non-projective semi-Gorenstein-projective module. Then $\operatorname{Ext}^1(M,M) \ne 0$. Moreover, if A is not self-injective, then $\operatorname{Ext}^i(M,M) \ne 0$ for all $i \ge 1$.*

Recall that the Auslander-Reiten conjecture [AR] for an artin algebra A asserts: *If M is a non-projective semi-Gorenstein-projective module, then $\operatorname{Ext}^i(M,M) \ne 0$ for some $i \ge 1$.* Thus, Theorem 1.5 shows that the Auslander-Reiten conjecture holds true for short local algebras in a stronger from. For A self-injective, Theorem 1.5 is due to Hoshino [Ho1], 1982. For commutative short local rings, a proof of the Auslander-Reiten conjecture was given by Huneke-Şega-Vraciu in 2004.

Let A be a short local algebra which is self-injective. Let M be a non-projective module. Then Theorem 1.5 asserts that $\operatorname{Ext}^1(M,M) \neq 0$ (over a self-injective algebra, all modules are semi-Gorenstein projective). If A is, in addition, commutative, then we even have $\operatorname{Ext}^i(M,M) \neq 0$ for all $i \geq 1$, see Huneke-Şega-Vracio [HSV]. However, for A non-commutative, this is not true: we may have $\operatorname{Ext}^i(M,M) = 0$ for some or even for all $i \geq 2$, see Proposition A.19 in Appendix A.

1.8. Existence of $\mho$-paths of length 3

We have mentioned that for any pair (e,a) with $1 \leq a \leq e-1$, there are short local algebras of Hilbert type (e,a) with a non-projective reflexive module (see Proposition 15.1), thus with an $\mho$-path of length 2, whereas the existence of an $\mho$-path of length 4 implies that $a = e-1$ (see Theorem 1.1). Proposition 15.2 provides an example of a short local algebra of Hilbert type $(6,2)$ with a non-projective 3-torsionfree module, thus with an $\mho$-path of length 3.

We do not know whether for any pair (e,a) with $2 \leq a \leq e-2$ there is a short local algebra which has non-projective 3-torsionfree modules, thus $\mho$-paths of length 3.

1.9. Summary

The short local algebras A with $e \leq 1$ are self-injective Nakayama algebras, thus let us restrict to the short local algebras A with $e \geq 2$. They can be separated as follows:

(1) $a = 0$ (thus A is a radical-square-zero algebra). Reflexive modules and images in exact complexes of projective modules are projective. (Theorems 1.1 and 1.3).
(2) $a = 1$ (this includes the self-injective algebras). There may be $\mho$-paths of arbitrary length. There may be non-projective images in exact complexes of projective modules.
(3) $2 \leq a \leq e-2$. There may be $\mho$-paths of length 3, but never of length 4 (Theorem 1.2). Images in exact complexes of projective modules are projective (Theorem 1.3).
(4) $2 \leq a = e-1$. There may be $\mho$-paths of arbitrary length, and there may be non-projective images in exact complexes of projective modules.
(5) $e \leq a$. Reflexive modules and images in exact complexes of projective modules are projective (Theorems 1.1 and 1.3).

The paper [RZ3] shows a further separation, namely between $a \leq \frac{1}{4}\,e^2$ and $\frac{1}{4}\,e^2 < a$.

It has turned out that several short local algebras with $a = e-1$ are of great interest, see Gasharov-Peeva [GP, 1990], Avramov-Gasharov-Peeva [AGP, 1997], Veliche [V, 2002], Yoshino [Y, 2002], Jorgensen-Şega [JS2, 2006], Christensen-Veliche [CV, 2007], Hughes-Jorgensen-Şega [HJS, 2009], all dealing with commutative rings. A non-commutative algebra A of Hilbert type $(3,2)$ has been analyzed in [RZ1,RZ2]; the construction will be generalized in Section 11.

For $e \geq 3$, Section 11 exhibits a short local algebra Λ with $a = e-1$ which has a non-projective Gorenstein-projective module M, a semi-Gorenstein-projective module M' which is not torsionless, and an ∞-torsionfree module M'' with $\operatorname{Ext}^1(M'',A) \neq 0$. In particular, Λ has complexes of type I and II.

1.10. Outline of the paper

The proofs of Theorems 1.1 and 1.2 are given in Section 4 and 6, respectively. The proofs of Theorems 1.3 and 1.4 can be found in Section 9. The proof of Theorem 1.5 is given in Section 12.

Many arguments used in the commutative case work in general, but there are also some decisive differences. For the convenience of the reader, we will provide complete proofs, the only exceptions are the use of the appendix of [CV], see Lemma 9.1 below, and of basic properties of the $\mho$-quiver and the $\mho$-paths, see Section 2 (here we follow [RZ1]).

Throughout the paper, $L(e)$ denotes the local k-algebra with $J^2 = 0$, $|J| = e$ and $L(e)/J = k$. If A is any local algebra with $e(A) = e$ (and $A/J = k$), then $A/J^2 = L(e)$ and we will interpret the $L(e)$-modules as the A-modules annihilated by J^2, thus as the A-modules of Loewy length at most 2.

Often, we will assume that A is not self-injective. After all, over a self-injective algebra, all modules are Gorenstein-projective. Appendix A provides an overview over the module theory of self-injective (equivalently, Gorenstein) short local algebras and the local radical-square-zero algebras $L(e)$, based on the relationship between these algebras and the Kronecker quivers $K(e)$.

The essence of Sections 6 and 7 is: If one is interested in exact complexes of projective modules, or in long $\mho$-paths, then the cases $e \le a$ and $2 \le a \le e-2$ can be discarded, and one has to look at the case $a = e-1$. This case is considered in Sections 7, 10, 11, 12 and in the examples 9.3 and 9.4. In particular, we show in Corollary 10.5 that a commutative short local algebra of Hilbert type $(e, e-1)$ has no complex of type II which involves a projective module of rank 1.

Examples of algebras with or without non-projective modules which are reflexive or are images in exact complexes of projective modules are constructed in Sections 14 and 15. In particular, we show that for any pair (e, a) of integers with $2 \le a \le e-1$, there is an algebra of Hilbert type (e, a) with a non-projective reflexive module. Also, we provide an example of an algebra of Hilbert type $(6, 2)$ with a non-projective 3-torsionfree module.

Sections 3 and 8 are devoted to the simple module S, its syzygies and the $\mho$-component which contains S. We stress that for any local algebra, if S is reflexive or the image in an exact complex of projective modules, then A is self-injective, see Lemma 3.2.

The main tool in the paper will be the use of the transformation ω_a^e on $\mathbb{Z}^2$ as defined in Section 5: It describes for suitable modules M in which way $\mathbf{dim}\, M$ is changed when we apply Ω_A (see the Main Lemma 5.4 and 13.1, but also [RZ3]). The Main Lemma draws the attention to the possible equality $t(\Omega^2 M) = et(\Omega M) - at(M)$. Appendix B is devoted to the numbers $b_n = b(e, a)_n$ defined recursively by the corresponding rule $b_{n+1} = eb_n - ab_{n-1}$, starting with $b_{-1} = 0,\ b_0 = 1$. It presents an explicit formula for these numbers b_n due to Avramov, Iyengar, Şega, provided $a < \frac{1}{4}e^2$.

We hope that the use of two independent numbering systems as suggested by the journal does not lead to confusion: the sections and subsections are numbered consecutively; independently, the assertions, examples and some of the remarks are also numbered consecutively.

2. The $\mho$-quiver and the $\mho$-paths

In this section, A will be an arbitrary artin algebra. We provide a survey on the $\mho$-quiver (and the $\mho$-paths), following [RZ1, Sections 1.5, 4.4 (and also 1.9)].

The $\mho$-quiver was introduced in [RZ1] in order to formalize ideas which are due to Auslander (1968), Auslander-Bridger (1969) and Auslander-Reiten (1996) (and which were further elaborated by many others) building up the realm of Gorenstein-projective modules (for historical references, in particular for the bibliographical data of relevant papers, see [RZ1]). The $\mho$-quiver provides the general frame for several important module theoretical concepts which carry deviating names: torsionless modules, reflexive modules, (semi-)Gorenstein-projective modules, n-torsionfree modules (a module M is said to be *n-torsionfree,* provided $\operatorname{Ext}^i(\operatorname{Tr} M, A) = 0$ for $1 \le i \le n$), ∞-torsionfree modules, and so on; in particular, it explains the wording "totally reflexive" used by Avramov-Martsinkovsky (2002) for the Gorenstein-projective modules. Finally, it highlights the duality between semi-Gorenstein-projective modules and ∞-torsionfree modules.

2.1. The operator $\mho$

Let M be a module. We denote by $\mho M$ the cokernel of a minimal left $\operatorname{add} A$-approximation of M (equivalently, we may define $\mho M = \operatorname{Tr} \Omega \operatorname{Tr} M$, where Tr is Auslander's transpose operator, see [RZ1], Lemma 4.4); the operator $\mho$ is called the *agemo* operator.

Let M be a module.

- *The module $\mho M$ has no indecomposable projective direct summands.*
- *If M is indecomposable, not projective and torsionless, then $\mho M$ is indecomposable (and not projective).*
- *The module M is reflexive iff both M and $\mho M$ are torsionless.*

2.2. The $\mho$-quiver

The vertices of the $\mho$-*quiver* are the isomorphism classes $[X]$ of the indecomposable non-projective modules X and there is an arrow

$$[X] \dashleftarrow [Z]$$

provided $X = \Omega Z$ and $\operatorname{Ext}^1(Z, A) = 0$, or, equivalently, provided X is torsionless and $Z = \mho X$. If X is torsionless (and indecomposable and non-projective), then there is a (uniquely determined) exact sequence $0 \to X \to P \to \mho X \to 0$ with P projective (thus $X \to P$ is a minimal left $\operatorname{add} A$-approximation); such a sequence is called an $\mho$-*sequence.* In this way, the arrows of the $\mho$-quiver just correspond to the $\mho$-sequences. This explains the direction of the arrow $[X] \leftarrow [\mho X]$ used here: The usual convention for using arrows in order to draw attention to short exact sequences $0 \to X \to Y \to Z \to 0$ is to draw an arrow $[X] \leftarrow [Z]$ (and often one uses a dashed arrow).

Paths in the $\mho$-quiver will be called $\mho$-*paths,* the connected components of the $\mho$-quiver will be called $\mho$-*components.*

A decisive feature of the $\mho$-quiver is the following: Any module M is the start of **at most one** arrow in the $\mho$-quiver (and then this arrow ends in ΩM) and also the end of **at most one** arrow in the $\mho$-quiver (and then this arrow starts in $\mho M$). Thus any $\mho$-component is a linearly oriented quiver $\mathbb{A}_n$ with $n \geq 1$ vertices, or an oriented cycle $\widetilde{\mathbb{A}}_n$ with $n{+}1 \geq 1$ vertices, or of the form $-\mathbb{N}$, or $\mathbb{N}$, or $\mathbb{Z}$. (Note that we consider any subset I of $\mathbb{Z}$ as a quiver, with an arrow from z to $z{-}1$ provided that both $z{-}1$ and z belong to I.)

2.3. Dictionary

Let M be an indecomposable non-projective module.

- *M is torsionless iff M is the end of an $\mho$-path of length* 1.
- *M is reflexive iff M is the end of an $\mho$-path of length* 2.
- *M is n-torsionfree iff M is the end of an $\mho$-path of length n.*
- *M is ∞-torsionfree iff M is the end of an infinite $\mho$-path.*
- $\operatorname{Ext}^i(M, A) = 0$ *for* $1 \leq i \leq t$ *iff M is the start of an $\mho$-path of length t.*
- *M is semi-Gorenstein-projective iff M is the start of an infinite $\mho$-path.*
- *M is Gorenstein-projective iff M is the start of an infinite $\mho$-path and the end of an infinite $\mho$-path (thus iff the $\mho$-component containing M is an oriented cycle $\widetilde{\mathbb{A}}_n$, or of the form $\mathbb{Z}$).*

2.4. Some bijections

The operators Ω^2 and $\mho^2$ provide inverse bijections between isomorphism classes as follows:

$$\left\{\begin{matrix}\text{indecomposable}\\ \text{non-projective modules } M\\ \text{which are reflexive}\end{matrix}\right\} \underset{\Omega^2}{\overset{\mho^2}{\rightleftarrows}} \left\{\begin{matrix}\text{indecomposable}\\ \text{non-projective modules } M\\ \text{with } \operatorname{Ext}^i(M,A)=0 \text{ for } i=1,2\end{matrix}\right\}$$

this is the bijection between the end and the start of the $\mho$-paths of length 2.

In the same way, we may look at the $\mho$-paths of length 4. We obtain the following bijections (again, all modules M are assumed to be indecomposable and non-projective) looking at the end, the middle and the start, respectively, of any $\mho$-path of length 4.

$$\left\{\begin{matrix}M\\ \text{4-torsionfree}\end{matrix}\right\} \underset{\Omega^2}{\overset{\mho^2}{\rightleftarrows}} \left\{\begin{matrix}M \text{ reflexive,}\\ \operatorname{Ext}^i(M,A)=0\\ \text{for } i=1,2\end{matrix}\right\} \underset{\Omega^2}{\overset{\mho^2}{\rightleftarrows}} \left\{\begin{matrix}\operatorname{Ext}^i(M,A)=0\\ \text{for } i=1,2,3,4\end{matrix}\right\}$$

2.5. A-duality

Let $0 \to X \to P \to Z \to 0$ be an $\mho$-sequence. If Z is reflexive, then also $0 \to Z^* \to P^* \to X^* \to 0$ is an $\mho$-sequence and X (thus also X^*) is reflexive.

Proof. For the first assertion, see [RZ1], 4.2(b). If Z is reflexive, then $\mho X = Z$ and $\mho^2 X = \mho Z$ both are torsionless, thus X is reflexive. □

In terms of $\mho$-paths, the assumption that Z is reflexive means that there is an $\mho$-path (in $\operatorname{mod} A$) of length 3 as shown below on the left, the conclusion that X^* is reflexive concerns the existence of the $\mho$-path of length 3 in $\operatorname{mod} A^{\mathrm{op}}$ shown on the right.

$$X \dashleftarrow Z \dashleftarrow \circ \dashleftarrow \circ \qquad\qquad Z^* \dashleftarrow X^* \dashleftarrow \circ \dashleftarrow \circ$$

3. The $\mho$-component of S

We collect some general observations concerning finite-dimensional local algebras A which are not necessarily short, mostly well-known.

Lemma 3.1. *Let A be a local artinian ring. Any module of finite projective dimension is projective.*

Proof. Let m be the Loewy length of A. Assume that M is a module with finite projective dimension $t \geq 1$. Let

$$0 \to P_t \to \cdots \to P_0 \to M \to 0$$

be a minimal projective resolution, thus $P_t \neq 0$. Now P_t is a submodule of $\operatorname{rad} P_{t-1}$. But P_t has Loewy length m, whereas $\operatorname{rad} P_{t-1}$ has Loewy length $m-1$, impossible. $\square$

Several characterizations of finite dimensional local algebras which are self-injective:

Lemma 3.2. *Let A be a finite-dimensional local algebra. The following assertions are equivalent:*

(i) $\operatorname{soc}{}_A A$ *is simple.*
(ii) A *is self-injective.*
(iii) *All modules are Gorenstein-projective.*
(iv) *All modules are reflexive.*
(v) S *is reflexive.*
(vi) $\mho S$ *has Loewy length at most* $m-1$, *where* m *is the Loewy length of* A.
(vii) *All modules are images in exact complexes of projective modules.*
(viii) S *is the image in an exact complex of projective modules.*
(ix) S *is the kernel of a map* $g \colon P \to P'$ *with* P, P' *projective.*
(x) $I({}_A A)$ *has finite projective dimension.*
(xi) $\operatorname{soc} A_A$ *is simple.*

The left-right-symmetry of the assertions (i) and (xi) means that *we may also add the right module versions of the assertions* (ii) *to* (x).

Proof. (i) $\implies$ (ii): If $\operatorname{soc}{}_A A$ is simple, then the injective envelop of ${}_A A$ is indecomposable. But the indecomposable injective A-module has the same dimension as A, thus ${}_A A$ is injective. (ii) $\implies$ (iii) is well-known. Any Gorenstein-projective module is reflexive and is an image in an exact complex of projective modules. Thus we have (iii) $\implies$ (iv) $\implies$ (v), as well as (iii) $\implies$ (vii) $\implies$ (viii). Of course, there are the obvious implications (viii) $\implies$ (ix), then (v) $\implies$ (ix), and also (ii) $\implies$ (x).

(v) $\implies$ (vi). If M is indecomposable, reflexive and not projective, then $\mho M$ is indecomposable, torsionless and not projective (see Section 2.1), thus there is an embedding $\mho M \subseteq JP$ with P projective. Therefore, the Loewy length of $\mho M$ is at most $m-1$.

(vi) $\implies$ (i). We assume that $\mho S$ has Loewy length at most $m-1$. Let $a = |J^{m-1}|$. By assumption, $a \geq 1$. Since S is torsionless, there is an $\mho$-sequence $0 \to S \xrightarrow{u} P \xrightarrow{p} \mho S \to 0$. Let P be of rank t. Thus $t \geq 1$ and $|J^{m-1}P| = at$. Since $\mho S$ has Loewy length at most $m-1$, $J^{m-1}P$ is contained in the kernel of p, thus $at \leq 1$, and therefore $a = 1$ and $t = 1$.

Assume now that there is a simple submodule U of ${}_AA$ which is not contained in J^{m-1}. Let $v: U \to A$ be the inclusion map. Let $f: S \to U$ be an isomorphism. Since u is a left $\operatorname{add}(A)$-approximation, there is $f': P \to A$ with $f'u = vf$.

Let us assume that f' is not surjective. Then the image of f' is a module of Loewy length at most $m-1$, thus $J^{m-1}P$ is contained in the kernel of f'. We have $J^{m-1} \neq 0$. Since $J^{m-1}P \subseteq \operatorname{Ker}(p) = \operatorname{Im}(u)$ and $\operatorname{Im}(u)$ is simple, we see that $J^{m-1}P = \operatorname{Im}(u)$. It follows that $f'u = 0$ in contrast to $vf \neq 0$.

Thus we see that f' is surjective. There is $f'': \mho S \to A/U$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, "u"] \arrow[d, "f"] & P \arrow[r, "p"] \arrow[d, "f'"] & \mho S \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & U \arrow[r, "v"] & {}_AA \arrow[r] & A/U \arrow[r] & 0.
\end{tikzcd}$$

Since f' is surjective, also f'' is surjective. Since J^{m-1} is not contained in U, the module A/U has Loewy length m. Therefore also $\mho S$ has Loewy length m, a contradiction. This shows that $\operatorname{soc}{}_AA \subseteq J^{m-1}$. Since $a = 1$, it follows that $\operatorname{soc}{}_AA$ is simple.

(ix) $\implies$ (xi). Let S be the kernel of a map $g: P \to P'$ with P, P' projective. Write both P and P' as direct sums of copies of ${}_AA$, thus g is given by a matrix with entries $g_{ij} \in \operatorname{End}({}_AA) = A$ and we can assume that all entries belong to J. But this implies that $\bigoplus \operatorname{soc} A_A$ is contained in the kernel of g. Since the kernel of g is simple, we see that ($P = {}_AA$ and that) $\operatorname{soc} A_A$ is simple.

(xi) $\implies$ (ii). In the previous parts of the proof, we have seen that *ii*) implies (xi). If we apply this to the opposite algebra of A, we see that (xi) implies (ii).

(x) $\implies$ (ii). If $I({}_AA)$ has finite projective dimension, then Lemma 3.2 asserts that $I({}_AA)$ is projective. $\square$

Remark 3.3. We recall that an algebra A is said to be *Gorenstein* provided both modules ${}_AA$ and A_A have finite injective dimension, or, equivalently, provided both modules $I({}_AA)$ and $I(A_A)$ have finite projective dimension. The equivalence of (ii) and (x) shows that *a finite dimensional local algebra is Gorenstein iff it is self-injective* (in commutative algebra, it is customary to refer to these algebras as Gorenstein algebras).

Remark 3.4. Both (vi) and (ix) imply that S is the kernel of a map $g: P \to Z$ with P projective and Z of Loewy length at most $m-1$. However, S may be the kernel of a map $g: P \to Z$ with P projective and Z of Loewy length at most $m-1$, whereas A is not self-injective: Take the algebra $A = k\langle x, y\rangle/\langle x^2, y^2, xy\rangle$ and $Z = A/Ayx$.

Remark 3.5. The implication (v) $\implies$ (ii) has been shown by Marczinzik in [M1], and he used this opportunity to ask whether any finite-dimensional algebra is self-injective provided all simple modules are reflexive. This is not true, see [R2].

Remark 3.6. According to Theorems 1.1 and 1.3, the existence of a non-projective reflexive module or a non-projective image in an exact complex of projective modules, implies severe restrictions on the algebra A, however there do exist many algebras which are not self-injective with such modules. As we see in Lemma 3.2, the situation is different, if S itself is reflexive, or is the image in an exact complex of projective: This can happen only if A is self-injective.

Lemma 3.7. *Let A be a local algebra. The following conditions are equivalent.*
(i) $\operatorname{Ext}^1(S, {}_AA) = 0$.
(ii) *A is self-injective.*

Proof. Of course, (ii) implies (i). Conversely, assume that $\operatorname{Ext}^1(S, {}_AA) = 0$. Then $\operatorname{Ext}^1(M, {}_AA) = 0$ for all A-modules M, thus ${}_AA$ is injective. □

Corollary 3.8. *Let A be a local algebra which is not self-injective. Then the $\mho$-component of A which contains S is of type $\mathbb{A}_2$ with $[S]$ as its sink.*

Proof. Since S is torsionless, there is an arrow ending in S. Since S is not reflexive, there is no $\mho$-path of length 2 ending in S. Since $\operatorname{Ext}^1(S, A) \neq 0$, no arrow starts in S. □

Proposition 3.9. *A short local algebra is self-injective if and only if either $a = 0$ and $e \leq 1$ or else $a = 1$ and $J^2 = \operatorname{soc}{}_AA$.*

Proof. According to Lemma 3.2, A is self-injective if and only if $\operatorname{soc}{}_AA$ is simple. If $a = 0$ and $e \leq 1$ or if $a = 1$ and $J^2 = \operatorname{soc}{}_AA$, then $\operatorname{soc}{}_AA$ is simple, thus A is self-injective. Conversely, assume that A is self-injective. If $J^2 = 0$, and $J \neq 0$, then $\operatorname{soc}{}_AA = J$, thus $a = 0,\ e = 1$; if $J^2 \neq 0$, then $J^2 \subseteq \operatorname{soc}{}_AA$, thus we must have $a = 1$ and $J^2 = \operatorname{soc}{}_AA$. □

4. Reflexive modules and the proof of Theorem 1.1

4.1. Reflexive modules

We assume that A is a short local algebra. We want to analyze the structure of reflexive modules. As we will see, the existence of a reflexive module which is not projective puts severe restrictions on A.

Lemma 4.1. *Let A be a short local algebra. Let M be indecomposable, torsionless and not projective. Then M is bipartite or simple.*

Proof. Since M is torsionless, there is an embedding $u\colon M \to P$ with P projective. Let $P = \bigoplus_i P_i$ with $P_i = {}_AA$ for all i. The composition of u with any projection $P \to P_i$ cannot be surjective, since otherwise it would split and M would have a direct summand isomorphic to ${}_AA$. Thus the image of u is contained in JP and therefore of Loewy length at most 2. It follows that M is the direct sum of a bipartite module and a semi-simple module. Since M is indecomposable, it is bipartite or simple. □

Lemma 4.2. *Let A be a short local algebra which is not self-injective. Let M be a module which is indecomposable, reflexive and not projective. Then both modules M and $\mho M$ are bipartite.*

Proof. According to Lemma 4.1, M is bipartite or simple. According to Lemma 3.2, M cannot be simple, thus M is bipartite. Since M is indecomposable, reflexive, and not projective, $\mho M$ is indecomposable, torsionless, and not projective. Using again 4.1, we see that $\mho M$ has Loewy length at most 2. Since $\operatorname{Ext}^1(\mho M, A) = 0$ (see Section 2.2), Lemma 3.2 asserts that $\mho M$ cannot be simple. Thus also $\mho M$ is bipartite. $\square$

Lemma 4.3. *Let A be a short local algebra which is not self-injective. Then, the module ${}_AB = {}_AA/J^2$ is not reflexive.*

Proof. Any map ${}_AB \to {}_AA$ maps into J, thus $\operatorname{Hom}({}_AB, {}_AA) = \operatorname{Hom}({}_AB, {}_AJ)$, and we can identify $\operatorname{Hom}({}_AB, {}_AJ)$ with J_A, sending $\phi\colon {}_AB \to {}_AJ$ to $\phi(1)$. Thus, ${}_AB^* = J_A$. It is sufficient to show that $\dim\,(J_A)^* = \dim \operatorname{Hom}(J_A, A_A) > 1 + e$, since $\dim B = 1 + e$. But $\operatorname{Hom}(J_A, A_A)$ has the proper subspace $\operatorname{Hom}(\operatorname{top} J_A, \operatorname{soc} A_A)$, and this subspace has dimension at least $2e$, since $\operatorname{soc} A_A$ is not simple. $\square$

4.2. Proof of Theorem 1.1

Let us repeat the assertion.

Let A be a short local algebra which is not self-injective. Let M be a module which is reflexive and not projective. Assume that there exists a reflexive module which is not projective, then $2 \le a \le e - 1$. Also, the module ${}_AJ$ and the right A-module J_A are solid.

Proof. By assumption, there is a reflexive module M which is not projective. In addition, we can assume that M is indecomposable. Let $\mathbf{dim}\, M = (t, s)$ and $z = |\operatorname{top} \mho M|$. Let $0 \to M \xrightarrow{u} P \xrightarrow{p} \mho M \to 0$ be an $\mho$-sequence, where P is projective of rank z, thus also $z = |\operatorname{top} \mho M|$. Of course, we can assume that u is an inclusion map.

In the following, we denote by $X^{(z)}$ the direct sum of z copies of a module X. We have $P = {}_AA^{(z)}$, with $JP = {}_AJ^{(z)}$ and M is a submodule of JP.

(1) *We have $s < et$.*

Proof. Let $B = A/J^2$, thus $B = L(e)$. Since A is not self-injective, we have $e \ge 2$. Since M is bipartite, it is a B-module. Its projective cover as a B-module is of the form $p'\colon P' \to M$ with $\mathbf{dim}\, P' = (t, et)$. Since p' is surjective, we have $s \le et$.

Now assume that $s = et$. Then p' is an isomorphism, thus M is a projective B-module. Since M is indecomposable, M is the projective left B-module of rank 1. However, according to Lemma 4.3, the module ${}_AB$ is not reflexive.

(2) *We have* $\operatorname{soc} M = J^2P$ *and therefore $s = az$.*

Proof. Since $\mho M$ has Loewy length at most 2, we have $J^2P \subseteq \operatorname{Ker}(p) = M$. Since J^2P is semisimple, it follows that $J^2P \subseteq \operatorname{soc} M$. On the other hand, $M \subseteq JP$ implies that $\operatorname{soc} M = JM \subseteq J^2P$, thus $\operatorname{soc} M = J^2P$.

By definition, $a = |J^2|$. Altogether, $s = |JM| = |\operatorname{soc} M| = |J^2P| = |J^2|z = az$. $\square$

(3) *We have $J^2 = \operatorname{soc} {}_AA$ and therefore $a \ge 2$.*

Proof. If $J^2 \neq \operatorname{soc} {}_AA$, there is a simple submodule U of ${}_AA$ which is not contained in J^2. Let $f\colon M \to U$ be a homomorphism with image $f(M) = U$, and $v\colon U \to {}_AA$ the inclusion map. Since $u\colon M \to P$ is a left $\operatorname{add}(A)$-approximation, there is $f'\colon P \to {}_AA$ such that $vf = f'u$. If f' is not surjective, then $f'(P) \subseteq J$, thus $f'(JP) \subseteq J^2$ and therefore

$f'u(M) \subseteq J^2$. But $f'u = vf$ and $vf(M) = v(U) = U$ is not contained in J^2. This shows that f' is surjective. There is the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{u} & P & \xrightarrow{p} & \mho M & \longrightarrow & 0 \\ & & \Big\downarrow f & & \Big\downarrow f' & & \Big\downarrow f'' & & \\ 0 & \longrightarrow & U & \xrightarrow{v} & {}_AA & \longrightarrow & A/U & \longrightarrow & 0. \end{array}$$

Since f' is surjective, also f'' is surjective. Since U is not contained in J^2, the module A/U has Loewy length 3. Thus, also $\mho M$ has Loewy length 3. But $\mho M$ has Loewy length at most 2, see Lemma 4.2. This contradiction shows that $J^2 = \operatorname{soc}{}_AA$.

By definition, $a = |J^2|$, thus $a = |\operatorname{soc}{}_AA|$. Since A is not self-injective, we have $|\operatorname{soc}{}_AA| \geq 2$, see Lemma 3.2. $\square$

(4) *${}_AJ$ is solid.*

Proof. Let ϕ be an endomorphism of ${}_AJ$.

Recall that $P = {}_AA^{(z)}$ with inclusion map $u\colon M \to P$, thus we can write u as the transpose of $[u_1, \ldots, u_z]$, where $u_i\colon M \to {}_AA = A_i$. As we know, $\operatorname{soc} M = J^2P = \bigoplus_{i=1}^z J^2A_i$, thus $J^2A_i = A_i \cap \operatorname{soc} M$.

We denote the inclusion map $JA_1 \subset A_1$ by v_1 and write $u_1 = v_1u_1'$, where $u_1'\colon M \to JA_1$. Let $f\colon M \to {}_AA$ be the composition

$$M \xrightarrow{u_1'} JA_1 \xrightarrow{\phi} JA_1 \xrightarrow{v_1} A_1 = {}_AA.$$

Since u is an $\operatorname{add}({}_AA)$-approximation, there are maps $g_i\colon {}_AA \to {}_AA$ such that $g = [g_1, \ldots, g_z]$ satisfies $f = gu = \sum g_iu_i$. The map $g_1\colon {}_AA \to {}_AA$ is the right multiplication by some element $\lambda \in A$.

Given $x \in A_1 \cap \operatorname{soc} M = J^2A_1$, we consider the element $[x, 0, \ldots, 0] \in M$ and apply the map $f = \sum g_iu_i$ to it. Since $f = v_1\phi u_1'$, we have $f([x, 0 \ldots, 0]) = \phi(x)$. On the other hand, we have $u_i([x, 0, \ldots, 0]) = 0$ for $i \geq 2$, thus $\sum g_iu_i([x, 0, \ldots, 0]) = g_1(x) = x\lambda$. This shows that

$$\phi(x) = f([x, 0 \ldots, 0]) = \sum g_iu_i([x, 0, \ldots, 0]) = x\lambda$$

for all $x \in J^2A_1$. Now J^2A_1 is annihilated from the right by J, thus $x\lambda = \overline{\lambda}x$, where $\overline{\lambda} = \lambda + J$ is an element of $A/J = k$. This shows that the restriction of ϕ to $J^2A_1 = J^2$ is the scalar multiplication by $\overline{\lambda}$. By (3), $J^2 = \operatorname{soc}{}_AA$. It follows that ${}_AJ$ is solid. $\square$

(5) *We have $ez \geq at$.*

Proof. We use again the decomposition $P = {}_AA^{(z)}$. We have $JP = {}_AJ^{(z)}$ and M is a submodule of JP. Let $u'\colon M \to J^{(z)}$, $v\colon J \to A$ and $w\colon J^2 \to A$ be the canonical inclusion maps. Thus $u = v^{(z)}u'$. Given $a \in A$, we denote by $r(a)\colon {}_AA \to {}_AA$ the right multiplication by c. If $c \in J$, then $r(c)$ maps J into J^2 and the map $r(c)\colon J \to J^2$ depends only on the residue class $\overline{c}$ of c modulo J^2. Thus we may write $r(\overline{c}) = r(c)\colon J \to J^2$ and there is the following commutative diagram

$$\begin{array}{ccc} J & \xrightarrow{v} & A \\ {\scriptstyle r(\overline{c})}\Big\downarrow & & \Big\downarrow{\scriptstyle r(c)} \\ J^2 & \xrightarrow{w} & A \end{array}$$

In this way, we obtain the following linear map

$$\Phi\colon (J/J^2)^{(z)} \to \operatorname{Hom}(M, J^2), \quad \text{defined by} \quad \Phi(\overline{c}_1, \dots, \overline{c}_z) = [r(\overline{c}_1), \dots, r(\overline{c}_z)]u'.$$

Let us show that Φ is surjective. Let $f\colon M \to J^2$ be any homomorphism. By assumption, the inclusion map $u = v^{(z)}u'\colon M \to A^{(z)}$ is a left $\operatorname{add}({}_AA)$-approximation. Thus, there is $f'\colon A^{(z)} \to {}_AA$ such that $wf = f'u$. We write f' as $[r(c_1), \dots, r(c_z)]$ with elements $c_i \in A$. Since f vanishes on $\operatorname{soc} M = (J^2)^{(z)}$, we have $(J^2)c_i = 0$, thus $c_i \in J$, for all $1 \le i \le z$.

Thus, we have the following diagram.

$$\begin{tikzcd}
M \arrow[r, "u'"] \arrow[d, "f"'] & J^{(z)} \arrow[r, "v^{(z)}"] \arrow[dl, "{[r(\overline{c}_1),\dots,r(\overline{c}_z)]}"] & {}_AA^{(z)} \arrow[d, "{f' = [r(c_1), \dots, r(c_z)]}"] \\
J^2 \arrow[rr, "w"] & & {}_AA
\end{tikzcd}$$

Here, the outer rectangle commutes by the choice of f'. Since $c_i \in J$, we have $r(c_i)v = wr(\overline{c}_i)$, thus $[r(c_1), \dots, r(c_z)]v^{(z)} = w[r(\overline{c}_1), \dots, r(\overline{c}_z)]$. Since w is a monomorphism, it follows that also the triangle on the left commutes: $f = [r(\overline{c}_1), \dots, r(\overline{c}_z)]u'$. Thus, we see that

$$f = [r(\overline{c}_1), \dots, r(\overline{c}_z)]u' = \Phi(\overline{c}_1, \dots, \overline{c}_z).$$

In this way, we see that Φ is surjective, thus $\dim\,(J/J^2)^{(z)} \ge \dim \operatorname{Hom}(M, J^2)$

Now, $\dim\,(J/J^2)^{(z)} = ez$. Second, any map $M \to J^2$ factors through the projection $M \to \operatorname{top} M$, thus $\dim \operatorname{Hom}(M, J^2) = \dim \operatorname{Hom}(\operatorname{top} M, J^2) = ta$. Therefore $ez \ge ta$. □

(6) *We have $a < e$.*

Proof. Assume for the contrary, that $e \le a$. Using (2) and (1), we have $az = s < et \le at$, and therefore $z < t$. Using (5), we have $at \le ez \le az$, thus $t \le z$. Thus, we obtain a contradiction. □

(7) *The right A-module J_A is solid.*

Proof. If M is a reflexive and non-projective module, then M^* is a reflexive and non-projective A^{op}-module. Thus (3) asserts that J_A is solid. □

The assertions (4), (6) and (3) and (7) are as required. This completes the proof. □

Corollary 4.4. *Let A be a short local algebra which is not self-injective. If there exists a reflexive module which is not projective, then both modules ${}_AJ$ and J_A are solid and of Loewy length* 2. *In particular,* $\operatorname{soc} {}_AA = J^2 = \operatorname{soc} A_A$.

Proof. Theorem 1.1 asserts that ${}_AJ$ is solid, and that $a \ge 2$, thus ${}_AJ$ is bipartite and of Loewy length 2. It follows that $\operatorname{soc} {}_AA = \operatorname{soc} {}_AJ = J^2$. If M is reflexive and not-projective, then M^* is a reflexive and non-projective A^{op}-module, thus A^{op} satisfies also the assumptions of Theorem 1.1. □

Remark 4.5. Note that an element $c \in J$ belongs to $\operatorname{soc} A_A = \operatorname{soc} J_A$ if and only if $cJ = 0$. As a consequence, $J^2 = \operatorname{soc} A_A$ *if and only if* ${}_AJ$ *is a faithful* A/J^2*-module.*

4.3. Some Examples

Example 4.6. *A short local algebra with* $J^2 = \operatorname{soc}{}_AA \subset \operatorname{soc} A_A$. Since our general assumption is $J^3 = 0$, we always have $J^2 \subseteq \operatorname{soc}{}_AA$ as well as $J^2 \subseteq \operatorname{soc} A_A$. We may have $J^2 = \operatorname{soc}{}_AA$ and $J^2 \neq \operatorname{soc} A_A$ as the following example shows. Let A be the k-algebra with radical generators x, y and relations

$$yx,\ y^2,\ x^3,\ x^2y. \qquad \begin{tikzcd} {}_AJ & x \arrow[d, "x"'] & y \arrow[d, "x"'] \\ & x^2 & xy \end{tikzcd}$$

Here, $J^2 = Ax^2 + Axy = \operatorname{soc}{}_AA$ is of length 2, whereas $\operatorname{soc} A_A = x^2A + yA + xyA$ is of length 3.

Examples 4.7. *Short local algebras with* ${}_AJ$ *indecomposable, but not solid.* First example: Here, ${}_AJ$ has a non-zero nilpotent endomorphism.

Let A be generated by x, y, z with relations

$$z^2,\ xy,\ yx,\ yz,\ zy,\ zx - xz,\ y^2 - xz,\ x^3.$$

$$\begin{tikzcd} {}_AJ & x \arrow[d, "x"'] \arrow[dr, "z"] & y \arrow[d, "y"] & z \arrow[dl, "x"'] \\ & x^2 & y^2 & \end{tikzcd}$$

There is the endomorphism f of ${}_AJ$ given by $f(y) = f(z) = 0$ and $f(x) = z$.

Second example: Here we exhibit an $\mathbb{R}$-algebra such that $\operatorname{End}({}_AJ) \sim \mathbb{C}$. We consider the $\mathbb{R}$-algebra with generators x, y, and the relations are

$$xy - yx,\ x^2 + y^2. \qquad \begin{tikzcd} {}_AJ & x \arrow[d, "x"'] \arrow[dr, "y"] & y \arrow[d, "x"] \arrow[dl, dashed, "y"'] \\ & x^2 & y^2 \end{tikzcd}$$

(Note that the 2-Kronecker module $\widetilde{J}$ as mentioned in Section A.2 of Appendix A is $(\mathbb{C}, \mathbb{C}; 1, i)$, where we write 1 for the identity map $\mathbb{C} \to \mathbb{C}$ and $i\colon \mathbb{C} \to \mathbb{C}$ for the multiplication by i; of course, $\operatorname{End}(\mathbb{C}, \mathbb{C}; 1, i) = \mathbb{C}$.)

Note that both algebras are commutative.

Example 4.8. *A short local algebra with* ${}_AJ$ *solid, whereas* J_A *is not solid.* Let A be generated by x, y, z with relations

$$x^2,\ y^2,\ z^2,\ yx,\ yz,\ zx - xy,\ zy - xz.$$

$$\begin{tikzcd} {}_AJ & x \arrow[dr, "z"] & & y \arrow[dl, "x"'] \arrow[dr, "z"] & & z \arrow[dl, "x"'] \\ & & zx & & xz & \end{tikzcd} \qquad \begin{tikzcd} J_A & x \arrow[dr, "y"'] \arrow[drrr, "z"] & & y & & z \arrow[dlll, "x"'] \arrow[dl, "y"] \\ & & zx & & xz & \end{tikzcd}$$

Here, ${}_AJ$ is solid, whereas J_A is the direct sum of a module with dimension vector $(2,2)$ and a simple module (generated by y). Note that ${}_AJ$ is solid, but not faithful.

Note that Theorem 1.1 and its Corollary 4.4 assert that the algebras exhibited in Examples 4.6, 4.7 and 4.8 do not have non-projective reflexive modules, thus all semi-Gorenstein-projective and all ∞-torsionfree modules are projective.

4.4. Short local algebras A with $a \le 1$

Recall that a module of finite length is said to be *uniform* provided it has a simple socle. If the module M has Loewy length at most 2, then JM is simple if and only if M is the direct sum of a uniform module and a semisimple module. Thus, if A is a short local algebra, then $a \le 1$ and $e \ge 1$ if and only if ${}_AJ$ is the direct sum of a uniform module and a semisimple module.

Lemma 4.9. *Let A be a short local algebra with $a \le 1$. The following assertions are equivalent:*
(i) *A is self-injective and $J \neq 0$.*
(ii) *There exists a non-projective reflexive module.*
(iii) *${}_AJ$ is solid.*
(iv) *${}_AJ$ is indecomposable.*
(v) *${}_AJ$ is uniform.*
(vi) *${}_AJ$ is simple or bipartite.*
(vii) *Either $a = 0$ and $e = 1$, or else $a = 1$ and $J^2 = \operatorname{soc}{}_AA$.*

The proof is straightforward: (i) $\implies$ (ii): If $J \neq 0$, then there are non-projective modules. For A self-injective, all modules are reflexive. (ii) $\implies$ (iii): See Theorem 1.1. (iii) $\implies$ (iv): Solid modules are indecomposable. (iv) $\implies$ (v): An indecomposable module M with $|JM| \le 1$ is uniform. (v) $\implies$ (vi): Clear. (vi) $\implies$ (vii): If ${}_AJ$ is simple, then $a = 0,\ e = 1$. Otherwise J^2 is the socle of ${}_AJ$, and thus $a = 1$. (vii) $\implies$ (i): See Lemma 3.2. □

4.5. Short local algebras A with $e \le 2$

Lemma 4.10. *Let A be a short local algebra with $e \le 2$. The following assertions are equivalent:*
(i) *A is self-injective and $J \neq 0$.*
(ii) *There exists a non-projective reflexive module.*
(iii) *${}_AJ$ is uniform.*
(iv) *Either $a = 0$ and $e = 1$, or else $a = 1$ and $J^2 = \operatorname{soc}{}_AJ$.*

Again, the proof is straightforward: (i) $\implies$ (ii): If $J \neq 0$, then there are non-projective modules. For A self-injective, all modules are reflexive. (ii) $\implies$ (iii): Since there exists a non-projective reflexive module, $e \ge 1$. If $e = 1$, then $a = 0$ or $a = 1$ and in both cases ${}_AJ$ is of course uniform. Thus, according to Theorem 1.1, we can assume that $a < e = 2$ and that $M = {}_AJ$ is solid. Since M is indecomposable, it follows that $a \neq 0$. But $|JM| = a = 1$ implies that $M = {}_AJ$ is uniform. (iii) $\implies$ (iv): Assume that ${}_AJ$ is uniform. Either ${}_AJ$ is simple, then $a = 0$ and $e = 1$, or else $J^2 = \operatorname{soc}{}_AJ$ and $a = |J^2| = 1$. (iv) $\implies$ (i): See Lemma 3.2. □

Example 4.11. The algebra $A = k[x,y]/(x,y)^3$ is a short local algebra with $e = 2$ such that ${}_AJ$ is solid, thus indecomposable, but (of course) not uniform.

5. Bipartite modules

5.1. $\mho$-sequences over short local algebras

First, let us apply the observations of Section 3 to $\mho$-sequences over short local algebras.

Corollary 5.1. *Let A be a short local algebra and $0 \to X \to P \to Z \to 0$ an $\mho$-sequence.*

(a) *If A is self-injective, then either X is bipartite, or else X is simple and then $Z = A/\operatorname{soc}{}_AA$.*

(b) *If A is not self-injective, and Z has Loewy length at most 2, then Z is bipartite, and either X is also bipartite or else X is simple and $a = 0,\ e \geq 2$.*

Proof. (a) The module X is indecomposable and of Loewy length at most 2. Thus, if X is not simple, then X is bipartite. If $X = S$ is simple, then $Z = A/\operatorname{soc}{}_AA$.

(b) Both X and Z are indecomposable modules of Loewy length at most 2. Now Z cannot be simple, since otherwise Lemma 3.2 asserts that A is self-injective. Since X is indecomposable, it is either bipartite or simple. If $X = S$ is simple, then Lemma 3.1 shows that the Loewy length of A cannot be 3 (since we assume that $Z = \mho S$ has Loewy length at most 2). Thus $a = 0$. Since A is not self-injective, we have $e \geq 2$. $\square$

Let us add also the following observation.

Lemma 5.2. *Let A be a short local algebra. If M is a reflexive module which is bipartite, then also M^* is (reflexive and) bipartite.*

Proof. We can assume that M is indecomposable. If M is projective, then $M = {}_AA$ implies that A has Loewy length 2, thus also $M^* = A_A$ is bipartite. Thus, we assume that M is not projective. Of course, M^* is torsionless. If M^* would be projective, also M would be projective. Thus M^* has Loewy length at most 2. Also M^* cannot be simple, since otherwise A is self-injective and also M is simple. $\square$

Proposition 6.1 will provide more information on the A-dual M^* of a bipartite reflexive module M.

Example 5.3. *If M is torsionless and bipartite, then M^* has Loewy length at most 2, but does not have to be bipartite.*

Namely, if M is bipartite, then M is annihilated by J^2, thus any map $f\colon M \to {}_AA$ maps into J. If $x \in J^2$, then the right multiplication $r(x)\colon {}_AA \to {}_AA$ by x sends J to 0, thus $r(x)f = 0$. Thus shows that M^* has Loewy length at most 2.

A typical example is given by the algebra $A = \Lambda_0$ considered in Section 11 (and before in [RZ1]), namely the **right** A-module $m_1A = (x-y)A$, as discussed in [RZ1]. Of course, m_1A is torsionless and bipartite, but $(m_1A)^* = M(q)^{**} = \Omega M(1)$ (see 6.5 (8) and Theorem 1.6 in [RZ1]) has a simple direct summand.

5.2. The Main Lemma

Given arbitrary integers a and e, let

$$\omega_a^e = \begin{bmatrix} e & -1 \\ a & 0 \end{bmatrix}.$$

If M is a module of Loewy length at most 2, we write $\omega_a^e\,\mathbf{dim}\,M$ for multiplying ω_a^e with (the transpose of) $\mathbf{dim}\,M$.

Lemma 5.4. *If M is a module of Loewy length at most 2, then there is a natural number $w \geq 0$ such that*

$$\mathbf{dim}\,\Omega M = \omega_a^e\,\mathbf{dim}\,M + (w, -w),$$

and such that ΩM has a direct summand of the form S^w. In particular, if ΩM is bipartite, then

$$\mathbf{dim}\,\Omega M = \omega_a^e\,\mathbf{dim}\,M.$$

Proof. Let $M' = \Omega M$. There is an exact sequence $0 \to M' \to P \to M \to 0$ with P projective and we can assume that the map $M' \to P$ is an inclusion map. Let $U = J^2P$. Since M has Loewy length at most 2, U is mapped under $P \to M$ to zero, thus $U \subseteq M'$. Since U is semisimple, we have $U \subseteq \operatorname{soc} M'$. Also, M' is a submodule of JP, thus M'/U is a submodule of JP/J^2P and therefore semisimple. This shows that $JM' \subseteq U$. Let $w = |U/JM'|$. Then

$$\mathbf{dim}\,M' = (|M'/JM'|, |JM'|) = (|M'/U| + w, |U| - w) = (|M'/U|, |U|) + (w, -w).$$

It remains to calculate $|U|$ and $|M'/U|$. Let $\mathbf{dim}\,M = (t, s)$. Then $P = {}_AA^t$, thus $|U| = |J^2P| = at$. Also, $|M'/U| = |JP/J^2P| - |JM| = et - s$. This shows that $(|M'/U|, |U|) = \omega_a^e\,\mathbf{dim}\,M$. This completes the proof of the first formula.

Write $M' = X \oplus Y$ with X bipartite and Y semisimple. Then $\operatorname{soc} M' = \operatorname{soc} X \oplus \operatorname{soc} Y = JX \oplus Y$ (here we use that X is bipartite), and $JM' = JX \oplus JY = JX \oplus 0 = JX$. Since $JM' \subseteq U \subseteq \operatorname{soc} M'$, the direct decomposition $\operatorname{soc} M' = JX \oplus Y$ yields $U = JX \oplus (Y \cap U)$. As a submodule of Y, the module $Y \cap U$ is a direct sum of copies of S. Since $Y \cap U$ is isomorphic to U/JM', we have $|Y \cap U| = |U/JM'| = w$, thus $Y \cap U$ is isomorphic to S^w. Since Y is semisimple, the submodule $Y \cap U$ is a direct summand of Y, thus a direct summand of M'. This shows that M' has a direct summand of the form S^w, namely $Y \cap U$.

It remains to show the second assertion: If ΩM is bipartite, then ΩM has no direct summand isomorphic to S, thus $w = 0$. □

Remark 5.5. The Main Lemma 5.4 focuses the attention to a direct summand of ΩM which is of the form S^w. However, we should stress that S^w may not be the largest semisimple direct summand of ΩM, as already the case $e = 1,\ a = 0$ and $M = S$ shows: here is $\Omega M = S$ and $w = 0$, thus $S^w = 0$ (see also Remark 13.4). Section 13 is devoted to a discussion of ΩM and its semisimple direct summands.

5.3. Aligned modules

Let A be a short local algebra of Hilbert type (e, a). We say that a module M of Loewy length at most 2 is *aligned* provided $\mathbf{dim}\,\Omega M = \omega_a^e\,\mathbf{dim}\,M$. Note that *if M is aligned, then* $|J\Omega M| = a \cdot t(M)$. Here is a reformulation of part of the Main Lemma 5.4.

Corollary 5.6. *Let A be a short local algebra and M a module of Loewy length at most 2. If ΩM is bipartite, then M is aligned.* $\square$

The converse is not true: We have $\Omega S = {}_AJ$, thus the module S is always aligned (since $\omega_a^e\,\mathbf{dim}\,S = \omega_a^e(1,0) = (e,a) = \mathbf{dim}\,{}_AJ$), whereas ΩS is bipartite iff $J^2 = \operatorname{soc}{}_AA$. In particular, for $a = 1$, ${}_AJ$ is bipartite iff A is self-injective (as mentioned already in 4.10).

Corollary 5.7. *Let A be a short local algebra which is not self-injective. Let M be indecomposable, reflexive and not projective. Then $\mho M$ is aligned.*

Proof. According to 4.2, M has Loewy length at most 2 and $\mho M$ is bipartite. Since M is torsionless, $\Omega(\mho M) = M$. $\square$

Remark 5.8. The subsequent paper [RZ3] will provide several characterizations of the aligned modules.

5.4. The module class $\mathcal{Z}(q)$, where q is a rational number

If A is a short local algebra with $a \geq 1$, and q is a non-negative rational number, let $\mathcal{Z}(q) = \mathcal{Z}_A(q)$ be the class of all indecomposable modules M with Loewy length at most 2 such that $|JM| = q \cdot a \cdot t(M)$. Note that $\mathcal{Z}(0) = \{S\}$.

Lemma 5.9. *If $M \in \mathcal{Z}(q)$ is aligned, then $\Omega M \in \mathcal{Z}(\frac{1}{e-qa})$.*

Proof. If $\mathbf{dim}\,M = (t, qat)$, then $\mathbf{dim}\,\Omega M = (et - qat, at) = ((e-qa)t, at)$ and thus $|J(\Omega M)| = at = \frac{1}{e-qa} \cdot a \cdot t(\Omega M)$. $\square$

5.5. Bipartite sequences and bipartite syzygy modules

We say that an exact sequence

$$\epsilon\colon \quad 0 \to X \to P \xrightarrow{p} Z \to 0$$

is *bipartite,* provided P is projective, both X, Z have Loewy length at most 2 and X is bipartite, or, equivalently, provided Z has Loewy length at most 2, p is a projective cover, and S is not a direct summand of X. Note that if M has Loewy length at most 2, then ΩM is bipartite if and only if the projective cover $p\colon PM \to M$ yields a bipartite sequence $0 \to \Omega M \to PM \xrightarrow{p} M \to 0$.

Starting with a module M of Loewy length at most 2, we look at all its syzygy modules $\Omega^i M$ with $i \geq 1$. Of particular interest will be the case that the modules $\Omega^i M$ with $1 \leq i \leq n$ are bipartite (thus S is not a direct summand of $\Omega^i M$ for all $1 \leq i \leq n$).

Corollary 5.10. *Let M be of Loewy length at most 2 and assume that there is $n \geq 1$ such that the modules $\Omega^i M$ with $1 \leq i \leq n$ are bipartite. Then*

$$\mathbf{dim}\,\Omega^n M = (\omega_r^e)^n\,\mathbf{dim}\,M.$$

$\square$

5.6. The recursion formula

Let M be a module. Recall that we write $t(M) = |\operatorname{top} M|$. For $i \in \mathbb{N}$, let $\beta_i(M) = t(\Omega^i M)$. As in commutative algebra [BH, L], one may call these numbers $\beta_i(M)$ the *Betti numbers* of M.

Proposition 5.11. *Let M be of Loewy length at most 2. If the modules M and ΩM are aligned, then*

$$\beta_2(M) = e\beta_1(M) - a\beta_0(M);$$

thus either $a = 0$, or else $\beta_0(M) = \frac{1}{a}(e\beta_1(M) - \beta_2(M))$.

In particular, if M is a module such that both modules ΩM and $\Omega^2 M$ are bipartite, then M and ΩM are aligned.

Proof. We write $t_i = \beta_i(M) = t(\Omega^i M)$ for $0 \leq i \leq 2$. Let $s_1 = |J\Omega M|$. Since M is aligned, $s_1 = at_0$. Since ΩM is aligned, $t_2 = et_1 - s_1$. Thus $t_2 = et_1 - s_1 = et_1 - at_0$.

The last sentence follows from Corollary 1 in 5.5. $\square$

Remark 5.12. In Lescot [L], modules with Loewy length at most 2 such that the modules $\Omega^i M$ with $1 \leq i \leq n$ are bipartite, are called "n-exceptional" modules; the modules which are n-exceptional for all $n \geq 1$ are called "exceptional". See [RZ3] for a further discussion of these "exceptional" modules.

6. More on reflexive modules and the proof of Theorem 1.2

6.1. The module class $\mathcal{Z}(q)$

Recall that for $q \in \mathbb{Q}$, the class $\mathcal{Z}(q)$ consists of all the indecomposable modules M of Loewy length at most 2 such that $|JM| = q \cdot a \cdot t(M)$.

Proposition 6.1. *Let A be a short local algebra of Hilbert type (e, a). Let M be a reflexive bipartite module with $\mathbf{dim}\, M = (t, s)$. Then a divides s and $\mathbf{dim}\, M^* = (s/a, at)$. Thus, if $M \in \mathcal{Z}(q)$, then $M^* \in \mathcal{Z}(q^{-1})$.*

Proof. We have seen already in the proof of Theorem 1.1 that a divides s; the essential assertion is the formula for $\mathbf{dim}\, M^*$ (but it implies, of course, that a divides s).

Since there exists a non-projective reflexive module M, we know that ${}_AJ$ is a solid A-module. Since M is not simple, we also know that $a \geq 1$. Let $\mathcal{H}$ be the set of homomorphisms $f \colon M \to {}_AA$ with semi-simple image (thus, these are the homomorphisms with image in J^2, and also the homomorphisms with kernel containing the socle of M). If $g \colon {}_AA \to {}_AA$ is the right multiplication by some element from J, then $gf = 0$. This shows that $\mathcal{H}$ is contained in the socle of M^*. Of course, $|\mathcal{H}| = at$. On the other hand, if $f \colon M \to {}_AA$ is any element of M^*, then $gf(M) \subseteq g(J) \subseteq J^2$ shows that gf belongs to $\mathcal{H}$. This shows that $M^*/\mathcal{H}$ is a semi-simple right A-module. Now M^* is indecomposable and has no simple direct summand, thus $\mathcal{H} = \operatorname{soc} M^*$.

Let $u_i \colon M \to A_i = {}_AA$ be maps such that $u = [u_1, \ldots, u_z] \colon M \to \bigoplus_{i=1}^{z} A_i$ is a minimal left add(A)-approximation of M. We can assume that u is an inclusion map. Since the cokernel of u has Loewy length at most 2, we know that J^2P is contained in the socle of M and actually equal to $\operatorname{soc} M$. It follows that $s = |\operatorname{soc} M| = az$. In particular, s is divisible by a.

We claim that $u_1, \dots, u_z$ is a basis of $M^*/\mathcal{H}$. First, we show the linear independence. Thus, let us assume that there are scalars $\lambda_i \in k$ such that $f = \sum_i \lambda_i u_i$ belongs to $\mathcal{H}$. We have to show that $\lambda_i = 0$ for all i. Thus, assume that some λ_i is non-zero, say let $\lambda_1 \neq 0$. Let $0 \neq x \in J^2 A_1$. We apply f to $[x, 0, \dots, 0]$ and get $f([x, 0, \dots, 0]) = \lambda_1 x \neq 0$. But this means that f does not vanish on $\operatorname{soc} M$, thus $f \notin \mathcal{H}$, a contradiction.

Second, we have to show that $u_1, \dots, u_z$ generate M^* modulo $\mathcal{H}$. Let $f\colon M \to {}_AA$ be any homomorphism. Since u is a left $\operatorname{add}(A)$-approximation, there are maps $f_i\colon {}_AA \to {}_AA$ such that $f = \sum_i f_i u_i$. Write $f_i = \lambda_i \cdot 1_M + g_i$ where $\lambda_i \in k$ and g_i maps into J. Then $f = \sum_i f_i u_i = \sum_i \lambda_i u_i + g$, with $g = \sum_i g_i u_i$. The image of any u_i is contained in J, thus the image of $g_i u_i$ is contained in J^2. This shows that $g \in \mathcal{H}$.

Altogether, we see that $u_1, \dots, u_z$ is a basis of $M^*/\mathcal{H}$. Since M^* is bipartite, $\operatorname{top} M^* = M^*/\operatorname{soc} M^* = M^*/\mathcal{H}$. Therefore $t(M^*) = |M^*/\mathcal{H}| = z = s/a$.

Since M is not simple, we have $s \neq 0$. We write $q = \frac{s}{at}$, so that $M \in \mathcal{Z}(q)$. Then $\mathbf{dim}\, M^* = (s/a, at)$ shows that $M^* \in \mathcal{Z}(q^{-1})$. $\square$

Corollary 6.2. *Let A be a short local algebra of Hilbert type (e, a). Let M be a reflexive bipartite module with $\mathbf{dim}\, M = (t, at)$. Then $\mathbf{dim}\, M^* = \mathbf{dim}\, M$.* $\square$

Proposition 6.3. *Let A be a short local algebra which is not self-injective. Let M be indecomposable, reflexive, not projective, with $\operatorname{Ext}^1(M, A) = 0$. Then*

$$\Omega M \in \mathcal{Z}\Big(\frac{a+1}{e}\Big) \quad \text{and} \quad M \in \mathcal{Z}\Big(\frac{e}{a+1}\Big).$$

We may add that we also have $\mho M \in \mathcal{Z}(\frac{e^2-a-1}{ae})$.

Proof. Since A is not self-injective, the modules ΩM and M are not simple. Also, we know that $a \geq 2$ according to Theorem 1.1.

Let $\mathbf{dim}\, M = (z, ay)$. Therefore $\mathbf{dim}\, \Omega M = (ez - ay, az)$, according to Lemma 5.4. By Proposition 6.1, we have $\mathbf{dim}\, M^* = (y, az)$ and $\mathbf{dim}(\Omega M)^* = (z, aez - a^2 y)$. According to Section 2.5, the A-dual of the $\mho$-sequence $0 \to \Omega M \to P \to M \to 0$ is the $\mho$-sequence $0 \to M^* \to P^* \to (\Omega M)^* \to 0$, and Lemma 5.4 asserts that

$$\mathbf{dim}\, M^* = \omega_a^e\, \mathbf{dim}(\Omega M)^* = \omega_a^e(z, aez - a^2 y) = (ez - aez + a^2 y, az).$$

Altogether, we see that $(y, az) = (ez - aez + a^2 y, az)$. Thus $ez - aez + a^2 y = y$ and therefore $e(1-a)z = (1-a^2)y$. Since $a \neq 1$, we see that $y = \frac{e}{a+1} z$ and therefore $|\operatorname{soc} M| = |JM| = ay = \frac{ae}{a+1} z = \frac{ae}{a+1} t(M)$. This shows that M belongs to $\mathcal{Z}(\frac{e}{a+1})$. $\square$

6.2. Proof of Theorem 1.2

Let us recall the assertion.

Let A be a short local algebra which is not self-injective. Assume that M is an indecomposable, reflexive and non-projective module with $\operatorname{Ext}^i(M, A) = 0$ for $1 = 1, 2$. Then $2 \leq a = e - 1$. If $t = t(M)$, then $\mathbf{dim}\, X = (t, at)$ for $X \in \{\Omega^2 M, \Omega M, M, \mho M\}$.

Proof. We apply Proposition 6.3 to M and to ΩM. Namely, M is reflexive and $\operatorname{Ext}^1(M,A)=0$, thus we see that M belongs to $\mathcal{Z}(\frac{e}{a+1})$, and ΩM belongs to $\mathcal{Z}(\frac{a+1}{e})$. Also, ΩM is reflexive and $\operatorname{Ext}^1(\Omega M,A)=0$. Thus we see that ΩM belongs to $\mathcal{Z}(\frac{e}{a+1})$. In this way, we see that ΩM belongs to $\mathcal{Z}(\frac{a+1}{e})\cap\mathcal{Z}(\frac{e}{a+1})$

But if $\mathcal{Z}(q)\cap\mathcal{Z}(q')$ is non-empty, then $q=q'$. It follows that $\frac{e}{a+1}=\frac{a+1}{e}$, therefore $a=e-1$. The inequality $2\le a$ is mentioned already in Theorem 1.1.

Let $\mathbf{dim}\,M=(t,s)$. Since M belongs to $\mathcal{Z}(\frac{e}{a+1})$ and $a+1=e$, it follows that $s=at$. Since the modules $\Omega M, M, \mho M$ are aligned, and $\omega_a^e(t,at)=(t,at)$, it follows that $\mathbf{dim}\,X=(t,at)$ for $X\in\{\Omega^2M,\Omega M,M,\mho M\}$. $\square$

7. The defect, defined in case $a=e-1$

Since the case $a=1$ does not provide any challenge, the interesting cases are those with $a\ge 2$. But we include the case $a=1$ in order to point out that the cases $a=e-1$ may be seen as having features which are similar to the self-injective algebras of Hilbert type $(2,1)$. The self-injective algebras of Hilbert type $(2,1)$ are exhibited in detail in Sections A.8 – A.11 in Appendix A: they are well related to the Kronecker algebra $K(2)$.

If $a=e-1$ and M is a module of Loewy length at most 2 with $\mathbf{dim}\,M=(t,s)$, let $\delta(M)=at-s$. We call $\delta(M)$ the *defect* of M.

Lemma 7.1. *Let $a=e-1\ge 1$. Let $0\to X\to P\to Z\to 0$ a bipartite sequence. Then $\mathbf{dim}\,X=\mathbf{dim}\,Z+\delta(Z)(1,1)$ and $\delta(X)=a\delta(Z)$.*

Proof. We have

$$\begin{aligned}\mathbf{dim}\,X&=(et-s,at)=((a+1)t-s,at)\\&=(t,s)+(at-s,at-s)=\mathbf{dim}\,Z+\delta(Z)(1,1),\end{aligned}$$

and $\delta(X)=a((a+1)t-s)-at=a^2t-as=a(at-s)=a\delta(Z)$. $\square$

Lemma 7.2. *Let $1\le a=e-1$. Let $0\to X\to P\to Z\to 0$ be bipartite. Then the following conditions are equivalent:*
(i) $\delta(X)=0$.
(ii) $\delta(Z)=0$.
(iii) $\mathbf{dim}\,X=\mathbf{dim}\,Z$.
(iv) $t(X)=t(Z)$.
(v) $|JX|=|JZ|$.

Proof. Since $\delta(X)=a\delta(Z)$, the conditions (i) and (ii) are equivalent. Since $\mathbf{dim}\,X=\mathbf{dim}\,Z+\delta(Z)(1,1)$, the conditions (ii) and (iii) are equivalent. Of course, (iii) implies both (iv) and (v). Now $\mathbf{dim}\,X=\mathbf{dim}\,Z+\delta(Z)(1,1)$ means that $t(X)=t(Z)+\delta(Z)$ and $|JX|=|JZ|+\delta(Z)$. Thus, if (iv) of (v) is satisfied, then $\delta(Z)=0$, thus (ii) holds. $\square$

Lemma 7.3. *Let $a=e-1\ge 1$. If $\delta(M)=0$, then either $t(\Omega M)=t(M)$ and $\delta(\Omega M)=0$, or else $t(\Omega M)>t(M)$, $\delta(\Omega M)>0$ and ΩM is not bipartite.*

If $\delta(M)>0$, then $t(\Omega M)>t(M)$ and $\delta(\Omega M)>0$.

Thus, if $\delta(M) \geq 0$ and $\delta(\Omega(M) > 0$, then

$$\cdots > \beta_{i+1}(M) > \beta_i(M) > \cdots > \beta_1(M) > \beta_0(M) = t(M).$$

Proof. The Main Lemma 5.4 asserts that $\mathbf{dim}\,\Omega M = \omega_a^e\,\mathbf{dim}\,M + (w, -w)$ for some $w \geq 0$.

First, let $\delta(M) = 0$, then $\mathbf{dim}\,M = (t, at)$ for some $t > 0$. Now, $\omega_a^e(t, at) = (t, at)$. We have $\mathbf{dim}\,\Omega M = (t, at) + (w, -w)$ for some $w \geq 0$. If $w = 0$, then trivially $t(\Omega M) = t = t(M)$ and $\delta(\Omega M) = 0)$. If $w > 0$, then $t(\Omega M) = t + w > t = t(M)$ and $\delta(\Omega M) = a(t + w) - (at - w) = (a + 1)w > 0$. Also, ΩM is not bipartite, according to Lemma 5.4.

Second, assume that $at-s = \delta(M) > 0$, thus $at > s$. Now $\mathbf{dim}\,\Omega M = (et-s+w, at-w)$ for some $w \geq 0$. Then $t(\Omega M) = et - s + w = at + t - s + w > t + w \geq t = t(M)$. Also, $a(et - s + w) = a(t + at - s + w) > a(t + w) \geq at \geq at - w$, thus $\delta(\Omega M) > 0$.

The last assertion follows by induction. □

Remark 7.4. For further considerations concerning short algebras with $a = e-1$, we refer to Sections 10, 11, 12.

8. The syzygy modules of S

Lemma 8.1. *If $e \leq a$, and $0 \to X \to P \to Z \to 0$ is a bipartite sequence, then $|\operatorname{soc} X| > |\operatorname{soc} Z|$.*

Proof. Let $\mathbf{dim}\,X = (t, s)$ and $\mathbf{dim}\,Z = (t', s')$. The Main Lemma 5.4 asserts that $(t, s) = \omega_a^e(t', s') = (et' - s', at')$. Thus $|\operatorname{soc} X| = s = at' \geq et' > s'$, since $t = et' - s' > 0$. □

If $(a_n)_n$ is a sequence of real numbers, we write (as usual) $\lim_n a_n = \infty$ provided for every integer b there is $N = N(b)$ such that $a_n > b$ for all $n \geq N$.

Proposition 8.2. *Let A be a short local algebra with $e \geq 2$. Then $\lim_n \beta_n(S) = \infty$, thus also $\lim_n |\Omega^n S| = \infty$. If, in addition, $a < e$, then the sequence of these Betti numbers $\beta_n(S)$ of S is strictly increasing: $\beta_n(S) < \beta_{n+1}(S)$ for all $n \in \mathbb{N}$.*

Proof. For any module M, we have $t(M) \leq |M| \leq (e+a+1)t(M)$, thus $\lim_n \beta_n(M) = \infty$ if and only if $\lim_n |\Omega^n(M)| = \infty$,

Let $t_n = \beta_n(S) = t(\Omega^n S)$. For $a < e$, we show that the sequence $(t_n)_n$ is strictly increasing.

First, let $a = 0$. Then $\Omega^n S = S^{e^n}$ for all $n \geq 0$. Since $e \geq 2$, we have $e^{n+1} > e^n$, thus $t_n < t_{n+1}$.

Second, let $1 \leq a \leq e - 1$. We have $t_0 = 1$, $t_1 = e$. We show by induction that $t_{n+1} > t_n$ for all $n \geq 0$. For $n = 0$, this holds true since $e \geq 2$. Thus, let $n \geq 1$. We assume that $t_{n+1} > t_n$. The Main Lemma 5.4 asserts that $t_{n+2} \geq et_{n+1} - at_n$. Thus $t_{n+2} - t_{n+1} \geq et_{n+1} - at_n - t_{n+1} = (e - 1)t_{n+1} - at_n \geq at_{n+1} - at_n = a(t_{n+1} - t_n) > 0$, where we use that $a \geq 1$. This shows that $t_{n+2} > t_{n+1}$.

Finally, let $e \leq a$. We show that $\lim_n |\Omega^n S| = \infty$. If all the modules $\Omega^n S$ are bipartite, then Lemma 8.1 asserts that $|\operatorname{soc}\Omega^{n+1}S| > |\operatorname{soc}\Omega^n S|$ for all $n \geq 0$, thus $|\Omega^n S| \geq |\operatorname{soc}\Omega^n S| > n$ for all n.

It remains to consider the case that there is some $\Omega^m S$ which is not bipartite. Let m be minimal. We claim that $\Omega^m S$ is not simple.

If $m = 1$, then ${}_AJ = \Omega S$ is of course not simple. Let $m \geq 2$. The minimality of m implies that $Z = \Omega^{m-1}S$ is bipartite. Let $p\colon P \to Z$ be a projective cover, thus $\Omega^m S$ is the kernel of p. Since Z is of Loewy length 2, we see that J^2P is contained in the kernel $\Omega^m S$ of p. We have $|J^2P| \geq |J^2| = a \geq 2$, thus $|\Omega^m S| \geq 2$. This shows that $\Omega^m S$ is not simple.

Thus, there is $m \geq 1$ such that $\Omega^m S$ is neither bipartite nor simple. We have $\Omega^m S \simeq S \oplus X$ for some $X \neq 0$. By induction, we have $\Omega^{bm}S \simeq S \oplus \bigoplus_{i=0}^{b-1}\Omega^{im}X$, for all $b \geq 0$, thus $\Omega^{bm}S$ is the direct sum of $b+1$ non-zero modules. As a consequence, $\Omega^{bm+i}S$ is the direct sum of $b+1$ non-zero modules, for all $i \geq 0$, and therefore $|\Omega^{bm+i}S| > b$ for all $i \geq 0$. Thus, let $N(b) = bm$. □

Example 8.3. *A short local algebra A with $\beta_1(S) = \beta_2(S)$.* In general, the Betti numbers are not strictly increasing, as the following example shows. Let A be the k-algebra generated by x, y with relations

$$yx,\ x^2 - y^2,\ x^3 \qquad\qquad {}_AJ \quad \begin{tikzcd} x \arrow[d, "x"'] & y \arrow[dl, "y"'] \arrow[d, "x"] \\ x^2 & xy \end{tikzcd}$$

It is a short local algebra of Hilbert type $(2,2)$. We have $\Omega S = {}_AJ$ with dimension vector $(2,2)$. As $\Omega({}_AJ)$ we can take the submodule of ${}_AA^2$ generated by $[y,x]$ and $[0,y]$, and this is a free $L(2)$-module of rank 2, thus $\mathbf{dim}\,\Omega^2 S = (2,4)$. We see that $\beta_1(S) = 2 = \beta_2(S)$.

9. Proof of Theorems 1.3 and 1.4

For the proof of Theorem 1.3, we will use the following result by Christensen and Veliche.

9.1. The Christensen-Veliche Lemma

Lemma 9.1 (Christensen-Veliche [CV]). *Let $e > 0$ and $a > 1$ be integers and let $(c_i)_{i\geq 0}$ be a sequence of positive integers with*

$$c_i = ec_{i+1} - ac_{i+2} \quad \textit{for all} \quad i \geq 0.$$

Then $a = e - 1$ and $c_i = c_0$ for all i.

Proof. See the appendix of [CV]. □

9.2. Proof of Theorem 1.3

Let A be a short local algebra which is not self-injective. Since A is not self-injective, we have $e \geq 2$. Let $P_\bullet$ be a non-zero minimal complex of projective modules which is exact. Let t_i be the rank of P_i and M_i the image of d_i. Since P_i is a projective cover of M_i, we have $t(M_i) = t_i$.

Note that we have $a \geq 1$. Namely, if $a = 0$, then the modules M_i are semisimple and $\Omega S = S^e$ shows that the sequence $\cdots, t_{i+1}, t_i, \cdots$ is strictly decreasing. Impossible.

Next, we show that M_i is bipartite for $i \ll 0$. Let $t = |M_0|$. According to Proposition 8.2, there is $N = N(b)$ such that $|\Omega^n S| > b$ for all $n \geq N$. Let $n \geq N$ and assume that S is a direct summand of M_{-n}. Then $\Omega^n S$ is a direct summand of $\Omega^n M_{-n} = M_0$, and therefore $|\Omega^n S| \leq |M_0| = b$, a contradiction. This shows that all the modules M_{-n} with $n \geq N$ are bipartite.

Using, if necessary, an index shift, we can assume that all the modules M_i with $i \leq 0$ are bipartite. Let $c_i = t_{-i} = t(M_{-i})$ for $i \geq 0$. Since all the modules M_{-i} are bipartite, Proposition 5.11 provides the recursion formula which asserts that

$$c_i = ec_{i+1} - ac_{i+2}$$

for all $i \geq 0$. Thus we can use the Christensen-Veliche Lemma 9.1 in order to conclude that $a = e-1$ and that the sequence $c_0, c_1, \ldots$ is constant, thus that the sequence $t_0, t_{-1}, t_{-2}, \ldots$ is constant.

There are two possibilities: First, all the modules M_i may be bipartite. In this case, $t_i = t_{i+1}$ for all $i \in \mathbb{Z}$.

Second, not all modules M_i are bipartite, thus there is a minimal index u such that M_{u+1} is not bipartite. As we have seen, this implies that $t_u = t_i$ for all $i \leq u$.

Since S is a direct summand of M_{u+1}, we use again Proposition 8.2 in order to see that there is some $i \geq u$ such that $t_{i+1} > t_i$. Let v be the minimal index i with this property. Thus we have

$$t_{v+1} > t_v = t_{v-1} = \cdots .$$

We apply Lemma 7.2 to the bipartite sequence $0 \to M_u \to P_{u-1} \to M_{u-1} \to 0$. Since $t(M_u) = t_u = t_{u-1} = t(M_{u-1})$, it follows that $\delta(M_u) = 0$. The first part of Lemma 7.3 yields by induction that $\delta(M_i) = 0$ for $u \leq i \leq v$ and then that $\delta(M_{u+1}) > 0$. The last part of Lemma 7.3 asserts that

$$\cdots > t(M_{i+1}) > t(M_i) > \cdots > t(M_{v+1}) > t(M_v)$$

(with $i \geq v$). This completes the proof. $\square$

9.3. Complexes of type I and of type II

We will say that a complex $P_\bullet$ is of *type* I, provided it is a non-zero minimal exact complex of projective modules, and all the modules P_i have the same rank.

We will say that a complex $P_\bullet$ is of *type* II, provided it is a non-zero minimal exact complex of projective modules P_i, and there is some integer v such that

$$\cdots > t_{v+2} > t_{v+1} > t_v = t_{v-1} = t_{v-2} = \cdots ,$$

where t_i is the rank of P_i.

Example 9.2. *An algebra A of Hilbert type $(2,1)$ with $J^2 \subset \operatorname{soc}{}_A A$ and $J^2 \subset \operatorname{soc} A_A$ with a complex of type* I.

In contrast to the commutative case, we cannot assert in Theorem 1.3 that $J^2 = \operatorname{soc}_A A$ or that $J^2 = \operatorname{soc} A_A$, as the following example shows: Let A be the k-algebra with generators x, y and relations x^2, xy, y^2.

$$x^2,\ xy,\ y^2. \qquad\qquad {}_AJ \qquad \begin{tikzcd} x \arrow[d, "y"'] & y \\ yx \end{tikzcd}$$

Note that y belongs to $\operatorname{soc}_A A$ and x belongs to $\operatorname{soc} A_A$, but neither x nor y belong to J^2. The ideal J^2 is 1-dimensional, whereas $\operatorname{soc}_A A$ and $\operatorname{soc} A_A$ are 2-dimensional.

The complex

$$\cdots \xrightarrow{\ x\ } {}_AA \xrightarrow{\ x\ } {}_AA \xrightarrow{\ x\ } \cdots$$

is non-zero, minimal and exact (here, x denotes the right multiplication by x, thus all the images are equal to $M = Ax = A/Ax$). (Note that x is a left Conca element, as defined in Section 10.3.)

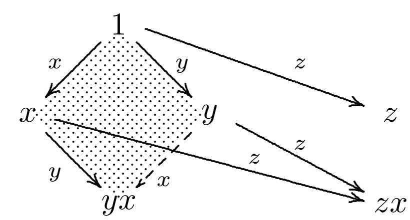
Example 9.3. *An algebra A of Hilbert type $(3,2)$ with $J^2 \neq \operatorname{soc}_A A$, with a complex of type* II.

The algebra A will be similar to the algebra Λ_0 considered in Section 11 (and before in [RZ1]), but with the relation $xz = 0$ instead of $xz = zx$. To be precise: A is generated by x, y, z, subject to the relations:

$$x^2,\ y^2,\ z^2,\ xy + qyx,\ xz,\ yz,\ zy - zx,$$

with $q \in k$ having infinite multiplicative order. Following [RZ1], we may visualize the algebra as follows:

$\Lambda :$



The algebra A has the basis $1,\ x,\ y,\ z,\ yx,\ zx$. We have $|\operatorname{soc}_A A| = 3$ with basis yx, zx, z, whereas, of course, $|J^2| = 2$. We get a complex of type II by taking the projective covers of the modules $A(x - \alpha y)$ where $\alpha = q^{-i}$ with $i \geq 2$, and a minimal projective resolution of $A(x - q^{-1}y)$. Note that $\Omega A(x - q^{-1}y) = A(x - y) \oplus S$.

Question 9.4. Let $P_\bullet = (P_i, d_i)$ be a complex of type II with $M_i = \operatorname{Im} d_i$ for all $i \in \mathbb{Z}$. Let $u(P_\bullet)$ be the minimum of $i \in \mathbb{Z}$ with M_{i+1} not bipartite (thus S is a direct summand of M_{u+1}, but not of M_i, for $i \leq u$) and $v(P_\bullet)$ the minimum of all $i \in \mathbb{Z}$ such that M_i is not aligned (thus M_v is not aligned, whereas M_i is aligned for all $i < v$). According to Corollary 1 in 5.5, we have $u \leq v$, and one may ask whether $u < v$ is possible. As we will see in Corollary 13.3, we have $u = v$ provided $J^2 = \operatorname{soc}_A A$.

The question may be rephrased as follows: We look at exact sequences of the form

$$0 \to M' \to P \to \cdots \to P \to M \to 0$$

with P a projective module which occurs $s \geq 1$ times, with M bipartite, $t(M') > t(P) = t(M)$ and S a direct summand of ΩM (namely, $M = M_u$, $P = P_u = \cdots = P_v$, and $M' = M_{v+1}$). The question is the following: Does there exist an exact sequence of this kind with $s \geq 2$ such that, in addition, M has a projective coresolution?

9.4. Proof of Theorem 1.4

Proof of the first part. We assume that A is a short local algebra which is not self-injective and that there exists a module which is indecomposable, non-projective and either semi-Gorenstein-projective or ∞-torsionfree. Thus, there is a reflexive module which is not projective and therefore Theorem 1.1 asserts that $a \geq 2$. Also, there exists an $\mho$-path of length 4, thus Theorem 1.2 asserts that $a = e - 1$ and $J^2 = \operatorname{soc}{}_AA = \operatorname{soc} A_A$. This is the first part of Theorem 1.4. $\square$

Proof of (1). Let M be indecomposable, non-projective, semi-Gorenstein-projective and torsionless. Let $t_i = t(\Omega^i M)$ for $i \geq 0$. According to the Dictionary 2.3, M is the start of an infinite $\mho$-path and the end of an $\mho$-path of length 1. Thus, there exists the following $\mho$-path

$$\cdots \dashleftarrow \Omega^2 M \dashleftarrow \Omega M \dashleftarrow M \dashleftarrow \mho M.$$

We see that all the modules $N = \Omega^i M$ with $i \geq 1$ are middle modules of $\mho$-paths of length 4, but this means that N is indecomposable, reflexive, non-projective, and satisfies $\operatorname{Ext}^j(N, A) = 0$ for $j = 1, 2$. According to Theorem 1.2, we have $\mathbf{dim}\, N = \mathbf{dim}\, \mho N = (t_i, at_i)$, with $t_i = t(N)$. In particular, $t_{i-1} = t(\Omega^{i-1}M) = t(\mho N) = t_i$.

Altogether, we see that all t_i with $i \geq 0$ are equal, thus equal to $t_0 = t(M)$, and that $\mathbf{dim}\, \Omega^i(M) = (t, at)$ for all $i \geq 0$. $\square$

Proof of (2). Let M be indecomposable, non-projective and ∞-torsionfree. Let $t_i = t(\mho^i M)$ for $i \geq 0$. According to the Dictionary 2.3, M is the end of an infinite $\mho$-path:

$$M \dashleftarrow \mho M \dashleftarrow \mho^2 M \dashleftarrow \mho^3 M \dashleftarrow \cdots.$$

We see that all the modules $N = \mho^i M$ with $i \geq 2$ are middle modules of $\mho$-paths of length 4, but this means that N is indecomposable, reflexive, non-projective, and satisfies $\operatorname{Ext}^j(N, A) = 0$ for $j = 1, 2$. According to Theorem 1.2, we have $\mathbf{dim}\, \Omega^2 N = \mathbf{dim}\, \Omega N = (t_i, at_i)$, where $t_i = t(N)$. Now $\Omega^2 N = \mho^{i-2} N$ and $\Omega N = \mho^{i-1} N$, thus $t(\mho^{i-2}N) = t_i = t(\mho^{i-1}N)$.

Altogether, we see that all t_i with $i \geq 0$ are equal, thus equal to $t_0 = t(M)$, and that $\mathbf{dim}\, \mho^i(M) = (t, at)$ for all $i \geq 0$. $\square$

Proof of (3). This concerns M^*. First, assume that M is ∞-torsionfree. There is an $\mho$-sequence $0 \to M \to P \to \mho M \to 0$. Since both M and $\mho M$ are reflexive, the A-dual sequence $0 \leftarrow M^* \leftarrow P^* \leftarrow (\mho M)^* \leftarrow 0$ is also an $\mho$-sequence. By (2), we have $t(\mho M) = t$, thus P has rank t, therefore P^* has rank t. This implies that $t(M^*) = t$. Since M^* is bipartite, torsionless and semi-Gorenstein-projective, it follows from the right version of (1) that $\mathbf{dim}\, M^* = (t, at)$.

Second, assume that M is semi-Gorenstein-projective and reflexive. We consider an $\mho$-sequence $0 \to \Omega M \to P \to M \to 0$. Since both M and ΩM are reflexive, also the A-dual

$0 \leftarrow (\Omega M)^* \leftarrow P^* \leftarrow M^* \leftarrow 0$ is an $\mho$-sequence. Now, the rank of P is t, thus the rank of P^* is t and therefore $|\operatorname{top}(\Omega M)^*| = t$. Now, $(\Omega M)^* = \mho M^*$. Since M^* is ∞-torsionfree, it follows from the right version of (2) that $\mathbf{dim}\, M^* = \mathbf{dim}\, \mho M^* = (t, at)$.

Proof of (4). If M is Gorenstein-projective, then M is both semi-Gorenstein projective and reflexive, as well as ∞-torsionfree. Thus (4) follows from (1), (2) and (3), $\square$

Example 9.5. *A short local algebra with an indecomposable module M which is semi-Gorenstein-projective and torsionless (but not reflexive), with* $\mathbf{dim}\, M^* \neq \mathbf{dim}\, M$. Let $A = \Lambda_0$ as discussed in Section 11 (and before in [RZ1]) and let M be the **right** module $m_1 A = (x-y)A$ (as above in 5.3). The module M is indecomposable, semi-Gorenstein-projective, and torsionless (but not reflexive). We have $(m_1A)^* = M(q)^{**} = \Omega M(1)$, see 6.7 in [RZ1]. Therefore $\mathbf{dim}\ m_1A = (1,2)$, whereas $\mathbf{dim}\ (m_1A)^* = (2,1)$.

Since m_1A is torsionless, also $\mho(m_1A)$ is semi-Gorenstein-projective. On the other hand, $\mho(m_1A)$ has Loewy length 3, see [RZ1] 7.3.

10. Some complexes of type I

10.1. Local modules

First, let us consider local modules. Note that a module M with Loewy length at most 2 is local iff $\mathbf{dim}\, M = (1, s)$ for some natural number $s \geq 0$.

Lemma 10.1. *Let A be a short local algebra with $a = e - 1$ and assume that A is not self-injective. If $0 \to X \to P \to Z \to 0$ is a bipartite sequence, with X a local module, then* $\mathbf{dim}\, X = \mathbf{dim}\, Z = (1, a)$. *In particular, also Z is local.*

Proof. First, let $e = 2$, thus $a = 1$. Since A is not self-injective, Lemma 3.2 asserts that $J^2 \subset \operatorname{soc}{}_AA$, thus ${}_AJ = I \oplus S$, where I is indecomposable and of length 2. Let B be the factor algebra of A modulo the annihilator of I, thus of ${}_AJ$. Then $a(B) = 0,\ e(B) = 1$, thus I and S are the only indecomposable B-modules. Since X is cogenerated by ${}_AJ$, it is a B-module. Since X is bipartite, we have $X = I$, thus $\mathbf{dim}\, X = (1, 1)$. Since the cokernel of the embedding $X \to P$ has Loewy length at most 2, we see that the projective module P has rank 1, thus $\mathbf{dim}\, Z = (1, 1)$.

Second, let $e \geq 3$. Since X is local and not simple, $\mathbf{dim}\, X = (1, s)$ for some s with $1 \leq s \leq e$. According to the Main Lemma 5.4, $\mathbf{dim}\, Z = (\frac{s}{a}, -1 + \frac{s}{a}(a+1))$. It follows that $\frac{s}{a}$ has to be an integer. Since $a \leq s \leq a+1$ and $a \geq 2$, it follows that $s = a$ and therefore $\mathbf{dim}\, X = (1, a) = \mathbf{dim}\, Z$. $\square$

Remark 10.2. Let A be a short local algebra with $a = e-1$ and assume that A is not self-injective. Let $0 \to X \to P \to Z \to 0$ be a bipartite sequence. *If Z is a local module, then X does not have to be local.* For an example, take an algebra of the form $A = \Lambda_0$ as discussed in Section 11 (and before in [RZ1]). Let X be the submodule of $P = {}_AA$ generated by x and y and $Z = P/X$. Then both X and Z are indecomposable of Loewy length 2. We have $\mathbf{dim}\, X = (2, 2)$, and $\mathbf{dim}\, Z = (1, 1)$, thus Z is local whereas X is not local. Note that $\delta(X) = 2$, and $\delta(Z) = 1$.

Corollary 10.3. *Let A be a short local algebra with $a = e - 1$ and assume that A is not self-injective. If X is a local reflexive module, then* $\mathbf{dim}\, X = \mathbf{dim}\, \mho X = (1, a)$.

Proof. Since X is torsionless, there is an exact sequence ϵ: $0 \to X \to P \to \mho X \to 0$. Since X is even reflexive, we know that $\mho X$ has Loewy length at most 2, thus ϵ is a bipartite sequence. $\square$

10.2. Commutative short local algebras

We consider now the case of a commutative short local algebra with $a = e - 1$. First, let A be an arbitrary commutative artinian ring.

Lemma 10.4. *Let A be a commutative artinian ring. If M, ΩM and $\Omega^2 M$ are local modules, then $\Omega^3 M \simeq \Omega M$.*

Proof. Let $p\colon A \to M$, $p'\colon A \to \Omega M$, $p''\colon A \to \Omega^2 M$ be projective covers. Let $u\colon \Omega M \to A$ be the kernel of p and $u'\colon \Omega^2 M \to A$ be the kernel of p'. Then we have $(up')(u'p'') = 0$. Now $up', u'p''$ are right multiplications by elements of A. Since A is commutative, we have $(u'p'')(up') = 0$, thus $p''u = 0$ (since p' is epi and u' is mono). The sequence $0 \to \Omega M \xrightarrow{u} A \xrightarrow{p''} \Omega^2 M \to 0$ is a short exact sequence, since u is mono, p'' epi, $p''u = 0$, and $|\Omega M| + |\Omega^2 M| = |{}_A A|$. Thus $\Omega^3 M = \Omega M$. $\square$

Corollary 10.5. *Let A be a commutative short local algebra. Then any complex of type* I *involving a projective module of rank* 1 *is periodic of period* 2*, and there is no complex of type* II *involving a projective module of rank* 1*.*

If A is a non-commutative short local algebra, then there may exist non-periodic complexes of type I involving a projective module of rank 1, as well as complexes of type II involving a projective module of rank 1. A typical example is the algebra $A = \Lambda_0$ discussed in Section 11 (and before in [RZ1]).

Proposition 10.6. *Let A be a commutative short local algebra with $a = e - 1$ and assume that A is not self-injective. If X is a local module and an $\mho$-path of length* 4 *ends in X, then X is Gorenstein-projective with Ω-period* 2 *and $\mathbf{dim}\,\Omega X = \mathbf{dim}\,X$.*

Proof. The $\mho$-path shows that the modules X, $\mho X, \mho^2 X$ are reflexive. Corollary 10.1 shows successively that the modules $\mho X$, then $\mho^2 X$, finally $\mho^3 X$ are local. We apply Lemma 10.2 to $M = \mho^3 X$ (with $\Omega M = \mho^2 X$, $\Omega^2 M = \mho X$, $\Omega^3 M = X$) and see that $X \simeq \mho^2 X$. This shows that X is Gorenstein-projective with Ω-period 2. Also we see that $\mathbf{dim}\,\Omega X = \mathbf{dim}\,X$. $\square$

10.3. Conca elements

Here is a simple way for obtaining complexes of type I. Following [AIS] (but dealing also with non-commutative local algebras), a non-zero element x will be called a *left Conca element* provided $x^2 = 0$ and $J^2 = Jx$. And x is called a *Conca element*, provided $x^2 = 0$ and $J^2 = Jx = xJ$. If x is a left Conca element, Ax is bipartite with $\mathbf{dim}\,Ax = (1, a)$. Let $r(x)\colon {}_A A \to {}_A A$ be the right multiplication by x, defined by $r(x)(a) = ax$ for $a \in A$. Obviously, the existence of a left Conca element implies that $1 \le a \le e-1$ (namely, r_x maps J onto J^2 and has $Ax = J^2 + Ax$ in its kernel, thus we get a surjective map $J/Ax \to J^2$, and $|J/Ax| = e - 1$, whereas $|J^2| = a$). In 15.1, we will see that if $1 \le a \le e - 1$, there are algebras of Hilbert type (e, a) with a Conca element x such that Ax is reflexive.

If $a = e - 1$, then for any Conca element x, the module Ax has to be reflexive, even Gorenstein-projective, as the following proposition shows.

Proposition 10.7. *Let A be a short local algebra of Hilbert type $(e, e-1)$ with $e \geq 2$. Let x be a left Conca element in A, and $P_\bullet = (P_i, d_i)$ with $P_i = {}_AA$ and $d_i = r(x)\colon {}_AA \to {}_AA$ for all $i \in \mathbb{Z}$. Then $P_\bullet = (P_i, d_i)$ is a minimal exact complex of projective complexes with all images being equal to Ax. In particular, $P_\bullet$ is a complex of type I. If x is a Conca element, then also $P_\bullet^*$ is exact, thus Ax is Gorenstein projective.*

Proof. Since $x^2 = 0$, we have $\operatorname{Im} r_x \subseteq \operatorname{Ker} r_x$, thus $P_\bullet$ is a complex. We have $Ax = \operatorname{Im} r_x \subseteq \operatorname{Ker} r_x = \Omega(Ax)$, and $\dim Ax = a+1$, whereas $\dim \Omega(Ax) = (1+e+a)-(a+1) = e$. Our assumption $a = e-1$ implies that $Ax = \Omega(Ax)$, thus $P_\bullet$ is exact. Of course, $P_\bullet$ is minimal, since $x \in J$. Altogether we see that $P_\bullet$ is a complex of type I with all images being equal to Ax.

The A-dual complex $P_\bullet^*$ is (P_i^*, d_i^*) with $P_i^* = A_A$ and $d_i^* : A_A \to A_A$ the left multiplication defined by x (defined by $l(x)(a) = xa$ for $a \in A$). If we assume that x is a Conca element, then x is a left Conca element of A^{op}, therefore $P_\bullet^*$ is exact. Altogether we see: If x is a Conca element, then both $P_\bullet$ and $P_\bullet^*$ are exact complexes, thus Ax is Gorenstein-projective. □

Remark 10.8. Of course, a left Conca element is not necessarily a Conca element. A typical example is the element x in $A = k\langle x, y\rangle/\langle x^2, xy, y^2\rangle$: Here, $Jx = kyx = J^2$, whereas $xJ = 0$.

10.4. Exact complexes $P_\bullet^*$ with $H_i(P_\bullet^*) \neq 0$ for all $i \in \mathbb{Z}$

Next, let us draw the attention to minimal exact complexes $P_\bullet$ such that $H_i(P_\bullet^*) \neq 0$ for all $i \in \mathbb{Z}$. Answering questions in [CV], Hughes-Jorgensen-Şega [HJS] provided examples of such complexes over a commutative ring A, namely over a short local algebra of Hilbert type $(5, 4)$. In the non-commutative setting, there are such examples already over short local algebras of Hilbert type $(2, 1)$ and $(3, 2)$.

Examples 10.9. *Short local algebras with minimal exact complexes $P_\bullet$ such that $H_i(P_\bullet^*) \neq 0$ for all $i \in \mathbb{Z}$.*

As first example, take the algebra A of Hilbert type $(2, 1)$ exhibited in 9.3 and the complex $P_\bullet$ mentioned there, where $d_i\colon {}_AA \to {}_AA$ is the multiplication by y for all $i \in \mathbb{Z}$. All images are equal to Ay, thus 2-dimensional, and therefore $P_\bullet$ is exact. In the A-dual complex $P_\bullet^*$, all images are yA, thus 1-dimensional. Thus $H_i(P_\bullet^*) \neq 0$ for all $i \in \mathbb{Z}$.

An example A with Hilbert type $(3, 2)$ is the algebra $A = \Lambda_0$ as discussed in Section 11 (but also in [RZ1]; actually, one may take any algebra of the form $\Lambda(q)$ as considered in [RZ1], with arbitrary q). Let $M = Ay$. Then $\Omega M \simeq M$. If $P_\bullet$ is the complex with $P_i = {}_AA$ and with all maps $d_i\colon P_i \to P_{i-1}$ being the right multiplication by y, then $P_\bullet$ is exact and minimal, all images in $P_\bullet$ are Ay (thus bipartite), whereas all images in $P_\bullet^*$ are isomorphic to the 2-dimensional right module yA and therefore $\dim H_i(P_\bullet^*) = 2$ for all $i \in \mathbb{Z}$.

10.5. Complexes of type I and of type II

Any ∞-torsionfree module M has a projective coresolution which is the concatenation of $\mho$-sequences, we may call it its *$\mho$-coresolution.* We may concatenate the $\mho$-coresolution of M with a minimal projective resolution of M and obtain in this way a minimal exact complex $P_\bullet(M)$ of projective modules. Given an ∞-torsionfree module M, one may ask whether $P_\bullet(M)$ is a complex of type I or of type II.

Let us stress that both cases are possible, as the algebra $A = \Lambda_0$ considered in Section 11 (and before in [RZ1]) shows. The Λ_0-module $M(1)$ is ∞-torsionfree, and $\Omega M(1)$ has a simple direct summand, thus the minimal projective resolution of $M(1)$ consists of projective modules whose rank is not bounded (see Proposition 8.2), thus $P_\bullet(M(1))$ is a complex of type II.

On the other hand, if M is Gorenstein-projective, then both $P_\bullet(M)$ and $P_\bullet(M)^*$ are exact complexes of projective modules, thus $P_\bullet(M)$ has to be a complex of type I.

But there are also ∞-torsionfree modules which are not Gorenstein-projective, such that $P_\bullet(M)$ is a complex of type I. For example, the Λ_0^{op}-module $m_{q^2}\Lambda_0$ is ∞-torsionfree. Here, for $\alpha \in k$, we define $m_\alpha = x - \alpha y \in \Lambda_0$. The syzygies of $m_{q^2}\Lambda_0$ are the modules $m_{q^i}\Lambda_0$ with $q \le 1$, thus of rank 1. We see that $P_\bullet(m_{q^2}\Lambda_0)$ is a complex of type I.

In addition, let us remark that there are complexes $P_\bullet = (P_i, d_i)$ of type I such that the image M of some d_i is semi-Gorenstein-projective, but not Gorenstein-projective. An example is the Λ_0^{op}-module $M = m_1\Lambda_0$ in [RZ1].

11. Some short local algebras of Hilbert type $(e, e-1)$

In this section, we are going to construct a short local algebra of Hilbert type $(e, e-1)$, where $e \ge 3$, with semi-Gorenstein-projective modules which are not Gorenstein-projective. The algebra which we construct will be denoted by Λ_c, with $c = e - 3$. The algebras Λ_0 have been exhibited already in [RZ1] and [RZ2] (and the general case is a straightforward generalization).

We need to assume that the base field k contains an element $q \in k$ with infinite multiplicative order. Thus, let $c \ge 0$. We define $\Lambda = \Lambda_c$ by generator and relations. The algebra $\Lambda = \Lambda_c$ is generated by $x, y, z, u_1, \dots, u_c$, subject to the relations:

$$x^2,\ y^2,\ z^2,\ yz,\ xy + qyx,\ xz - zx,\ zy - zx,$$
$$xu_i - u_ix,\ yu_i,\ u_iy,\ zu_i,\ u_iz,\ u_iu_j,$$

for all $1 \le i, j \le c$. We obtain a short local algebra of Hilbert type $(3 + c, 2 + c)$ say with radical J, such that $yx, zx, u_1x, \dots, u_cx$ is a basis of $J^2 = \operatorname{soc} \Lambda J = \operatorname{soc} J_\Lambda$.

We may visualize (the coefficient quiver of) ${}_\Lambda J$ as follows:

$$\begin{tikzcd}
y \arrow[d, dashed, "x"'] \arrow[dr, "y"] & z \arrow[d, "x"'] & u_1 \arrow[d, "x"'] & \cdots & u_c \arrow[d, "x"'] \\
yx & zx & u_1x & \cdots & u_cx \\
 & x \arrow[ul, "y"] \arrow[u, "z"'] \arrow[ur, "u_1"] \arrow[urrr, "u_c"] & & &
\end{tikzcd}$$

using the usual convention that a solid arrow $v \to v'$ labeled say by x means that $xv = v'$, a dashed arrow $v \dashrightarrow v'$ labeled by x means that xv is a non-zero multiple of v' (in our case, $xy = -qyx$). Here, the middle layer with the vertices $yx, zx, u_1x, \dots, u_cx$ is the basis of J^2, as mentioned already.

We are interested in the modules $M(\alpha)$ with $\alpha \in k$ with basis $v, v', v'', v_1, \dots, v_c$, such that $xv = \alpha v',\ yv = v',\ zv = v'', u_iv = v_i$, for all $1 \le i \le c$ and such that $v', v'', v_1, \dots, v_c$ are annihilated by all generators.

$$\begin{tikzcd} v \arrow[d, shift right, dashed, "x"'] \arrow[d, shift left, dotted, "y"] \arrow[dr, "z"] \arrow[drr, "u_1"] \arrow[drrrr, "u_c"] & & & & \\ v' & v'' & v_1 & \cdots & v_c \end{tikzcd}$$

The modules $M(\alpha)$ with $\alpha \in k$ are pairwise non-isomorphic indecomposable Λ-modules of length $3 + c$. As in [RZ1] one shows:

(1) *The module $M(0)$ is Gorenstein-projective and Ω-periodic with period* 1. In particular, there are non-zero minimal exact complexes of projective modules of type I.

(2) *The module $M(q)$ is semi-Gorenstein-projective and not torsionless.*

(3) *The module $M(1)$ is ∞-torsionfree and $\Omega M(1)$ has a simple direct summand.* Therefore $P_\bullet(M(1))$ is a non-zero minimal exact complex of projective modules of type II.

Of course, also further properties of Λ_0 shown in [RZ1] carry over to the algebras Λ_c with arbitrary $c \ge 0$. Here, we only want to stress that for *any $a \ge 2$, there does exist a short local algebra A, namely $A = \Lambda_{a-2}$, of Hilbert type $(a+1, a)$ which has modules M, M', M'' of length $a+1$ such that M is Gorenstein-projective, M' is semi-Gorenstein-projective and not torsionless, and M'' is ∞-torsionfree, with $\Omega M''$ having a simple direct summand.*

12. The Auslander-Reiten conjecture (proof of Theorem 1.5)

12.1. Preliminary considerations

We need some preliminary considerations (they are well-known, see for example Iyama [I], Section 2.1, and also [M2]). If M, N are modules, $\underline{\mathrm{Hom}}(\Omega M, N) = \mathrm{Hom}(M, N)/\mathcal{P}(M, N)$, where $\mathcal{P}(M, N)$ denotes the set of homomorphisms $M \to N$ which factor through a projective module.

Lemma 12.1. *Let* $\mathrm{Ext}^1(Z, A) = 0$. *Then, for any module N, we have*

(a) $$\mathrm{Ext}^1(Z, N) \simeq \underline{\mathrm{Hom}}(\Omega Z, N),$$

(b) $$\underline{\mathrm{Hom}}(Z, N) \simeq \underline{\mathrm{Hom}}(\Omega Z, \Omega N).$$

Proof. Let $0 \to X \xrightarrow{u} PZ \to Z \to 0$ be exact, where PZ is a projective cover of Z. Thus $X = \Omega Z$.

(a) We get the exact sequence

$$\mathrm{Hom}(PZ, N) \xrightarrow{\mathrm{Hom}(u,N)} \mathrm{Hom}(X, N) \xrightarrow{\delta} \mathrm{Ext}^1(Z, N) \to 0$$

Of course, the image of $\operatorname{Hom}(u,N)$ always lies in $\operatorname{Hom}(X,N)_{\operatorname{add} A}$ (the set of homomorphisms $X \to N$ which factor through $\operatorname{add} A$). Since $\operatorname{Ext}^1(Z,A) = 0$, the map u is a left $\operatorname{add}(A)$-approximation, thus any homomorphism $X \to N$ which factors through $\operatorname{add}(A)$ factors through $u\colon X \to PZ$. This shows that the image of $\operatorname{Hom}(u,N)$ is equal to $\operatorname{Hom}(X,N)_{\operatorname{add} A}$. By definition, $\underline{\operatorname{Hom}}(X,N) = \operatorname{Hom}(X,N)/\operatorname{Hom}(X,N)_{\operatorname{add} A}$, thus δ yields an isomorphism $\underline{\operatorname{Hom}}(X,N) \simeq \operatorname{Ext}^1(Z,N)$.

(b) Let $0 \to \Omega N \to PN \to N \to 0$ be exact. Any map $f\colon Z \to N$ lifts to a map $f'\colon PZ \to PN$ and thus yields by restriction a map $f''\colon X \to \Omega N$. If f factors though $\operatorname{add} A$, then f'' factors also through $\operatorname{add} A$. In this way, we obtain an additive map $\eta\colon \underline{\operatorname{Hom}}(Z,N) \to \underline{\operatorname{Hom}}(X,\Omega N)$. Since u is a left $\operatorname{add}(A)$-approximation, the map η is bijective. $\square$

Lemma 12.2. *If* $\operatorname{Ext}^i(Z,A) = 0$ *for* $i = 1,2$*, then, for any module* N

$$\operatorname{Ext}^1(Z,N) \simeq \operatorname{Ext}^1(\Omega Z,\Omega N).$$

Proof. Since $\operatorname{Ext}^1(Z,A) = 0$, we have $\operatorname{Ext}^1(Z,N) \simeq \underline{\operatorname{Hom}}(\Omega Z,N)$. Since $\operatorname{Ext}^1(\Omega Z,A) = 0$, we have $\underline{\operatorname{Hom}}(\Omega Z,N) \simeq \underline{\operatorname{Hom}}(\Omega^2 Z,\Omega N)$ and $\underline{\operatorname{Hom}}(\Omega^2 Z,\Omega N) \simeq \operatorname{Ext}^1(\Omega Z,\Omega N)$. $\square$

Corollary 12.3. *If M is semi-Gorenstein-projective, and N is an arbitrary module, we have* $\operatorname{Ext}^i(M,N) \simeq \operatorname{Ext}^i(\Omega M,\Omega N)$, *for all* $i \geq 1$.

Proof. We apply Lemma 12.2 to $\Omega^{i-1}M$ and see:

$$\operatorname{Ext}^i(M,N) \simeq \operatorname{Ext}^1(\Omega^{i-1}M,N) \simeq \operatorname{Ext}^1(\Omega^i M,\Omega N) \simeq \operatorname{Ext}^i(\Omega M,\Omega N).$$

$\square$

12.2. The non-vanishing of $\operatorname{Ext}^i(M,M)$ for all $i \geq 1$

Proposition 12.4. *Let A be a short local algebra which is not self-injective, and let M be a non-projective semi-Gorenstein-projective module. Then* $\operatorname{Ext}^i(M,M) \neq 0$ *for* **all** $i \geq 1$.

Proof. We can assume that M is indecomposable, then also all the modules $\Omega^i M$ are indecomposable with $i \geq 0$. Let (e,a) be the Hilbert-type of A. Let $t = t(\Omega M)$. According to Theorem 1.4, we have $a = e-1 \geq 2$ and $\mathbf{dim}\,\Omega^i M = (t,at)$ for all $i \geq 1$. We have for $i \geq 1$

$$\operatorname{Ext}^i(M,M) = \operatorname{Ext}^i(\Omega M,\Omega M) = \operatorname{Ext}^1(\Omega^i M,\Omega M)$$

where we use Corollary 12.3. Now $\mathbf{dim}\,\Omega^i M = (t,at) = \mathbf{dim}\,\Omega M$, thus both modules $\Omega^i M$ and ΩM are regular modules, see Section A.2 in Appendix A. Since $e \geq 3$, it follows that $\operatorname{Ext}^1_{L(e)}(\Omega^i M,\Omega M) \neq 0$. But then also $\operatorname{Ext}^1_A(\Omega^i M,\Omega M) \neq 0$, since $L(e)$ is a factor algebra of A. (In general, of B is a factor algebra of A, and M', M'' are B-modules, then $\operatorname{Ext}^1_B(M',M'')$ can be considered as a subset of $\operatorname{Ext}^1_A(M',M'')$.) $\square$

12.3. Proof of Theorem 1.5

Let A be a short local algebra and let M be a non-projective semi-Gorenstein-projective module. First, we consider the case that A is not injective. According to Proposition 12.4

we have $\mathrm{Ext}^t(M,M)\neq 0$ for all $t\geq 1$. In particular, $\mathrm{Ext}^1(M,M)\neq 0$. Second, let A be self-injective. Then we also have $\mathrm{Ext}^1(M,M)\neq 0$, now according to Hoshino, see the first part of Proposition A.5 in Appendix A. $\square$

13. The Main Lemma, revisited

13.1. Main Lemma in the case $J^2=\mathrm{soc}\,_AA$

Lemma 13.1. *Let A be a short local algebra with $J^2=\mathrm{soc}\,_AA$. Let M be a module of Loewy length at most 2. Let $\Omega M = X\oplus S^w$ with X bipartite and $w\in\mathbb{N}$. Then*

$$\mathbf{dim}\,\Omega M = \omega_a^e\,\mathbf{dim}\,M + (w,-w).$$

Proof. Let $M'=\Omega M$ and take an exact sequence $0\to M'\to P\to M\to 0$ with P projective and with an inclusion map $M'\to P$. Let $U=J^2P$. As in the proof of 5.4, we see that $JM'\subseteq U\subseteq \mathrm{soc}\,M'$ and that

$$\mathbf{dim}\,M' = \omega_a^e\,\mathbf{dim}\,M + (w,-w).$$

where $w=|U/JM'|$.

Now $J^2=\mathrm{soc}\,_AA=\mathrm{soc}\,_AJ$ means that $_AJ^2$ is bipartite, thus also JP is bipartite. Therefore $M'\subseteq JP$ implies that $\mathrm{soc}\,M'\subseteq \mathrm{soc}\,JP=J^2P=U$, and therefore $U=\mathrm{soc}\,M'$.

Write $M'=X\oplus W$ with X bipartite and W semisimple. Then $U=\mathrm{soc}\,M'=JX\oplus W$, and $JM'=JX\oplus JW=JX$. Altogether, we get $U=JM'\oplus W$. It follows that $w=|U/JM'|=|W|$. Thus, W is isomorphic to S^w and therefore $M'=X\oplus W=X\oplus S^w$ with X bipartite. $\square$

13.2. Consequences

Recall that a module M of Loewy length at most 2 is said to be aligned (see Section 5.5), provided $\mathbf{dim}\,\Omega M=\omega_a^e\,\mathbf{dim}\,M$.

Corollary 13.2. *Let A be a short local algebra with $J^2=\mathrm{soc}\,_AA$. Then a module M of Loewy length at most 2 is aligned if and only if ΩM is bipartite.*

Proof. Let M be a module of Loewy length at most 2. We have seen in Corollary 5.6 that if ΩM is bipartite, then M is aligned. For the converse, we need the assumption that $J^2=\mathrm{soc}\,_AA$. By Lemma 13.1, we know that $\Omega M=X\oplus S^w$ with X bipartite and $\mathbf{dim}\,\Omega M=\omega_a^e\,\mathbf{dim}\,M+(w,-w)$. If M is aligned, then $\mathbf{dim}\,\Omega M=\omega_a^e\,\mathbf{dim}\,M$, thus $w=0$, and therefore ΩM is bipartite. $\square$

Using Lemma 13.1, we are able to improve Theorem 1.3 in the case $J^2=\mathrm{soc}\,_AA$.

Corollary 13.3. *Let A be a short local algebra of Hilbert type $(e,e-1)$ which is not self-injective and assume that $J^2=\mathrm{soc}\,_AA$.*

Let $P_\bullet=(P_i,d_i)_i$ be a non-zero minimal exact complex of projective modules of type II*, let M_i be the image of d_i and $t_i=t(P_i)=t(M_i)$. As we know, there is $v\in\mathbb{Z}$ with $t_{v+1}>t_v=t_{v-1}$. Let $t=t_v$. Then all the modules M_i with $i\leq v$ are bipartite, whereas M_{v+1} is not bipartite.*

Proof. By Theorem 1.3, we know that M_{v+1} is not bipartite and that $\mathbf{dim}\, M_i = (t, at)$ for all $i \le v$. Suppose that M_i is not bipartite, say $M_i = U \oplus S^w$ with U bipartite and $w \ge 1$. Let $M = M_{i-1}$. According to 13.1, we have $\mathbf{dim}\, M_i = \mathbf{dim}\, \Omega M = \omega_a^e\, \mathbf{dim}\, M + (w, -w)$. Thus $t(M_i) = t + w > t$ and therefore $i > v$. $\square$

Remark 13.4. Let us return to the Main Lemma 5.4 itself. Let M be a module of Loewy length at most 2. If we use covering theory, the number w provided by the Main Lemma 5.4 can be understood well. Thus, let $\widetilde{A}$ be a $\mathbb{Z}$-cover of A (we assume that the set of vertices of the quiver of $\widetilde{A}$ is $\mathbb{Z}$, and that the arrows go from i to $i+1$, for all i). Let π be the push-down functor. Let $\widetilde{M}$ be a module with $\pi(\widetilde{M}) = M$, such that top $\widetilde{M}$ is a direct sum of copies of $S(0)$ (we recall the definition of $\widetilde{M}$ in Section A.2 in Appendix A). Then $\Omega\widetilde{M} = U \oplus S(2)^w \oplus S(1)^{w'}$, with U being bipartite (and having support equal to $\{1, 2\}$ provided $U \neq 0$). It follows that $\Omega M = \pi(\Omega\widetilde{M}) = \pi(U) \oplus S^{w+w'}$. Here we see the number w which is mentioned in the Main Lemma 5.4. If we consider $\Omega\widetilde{M}$ as a representation of the e-Kronecker quiver with vertices $1, 2$, then $S(2)^w$ is a maximal direct summand of $\Omega\widetilde{M}$ which is semisimple and projective, whereas $S(1)^{w'}$ is a maximal direct summand of $\Omega\widetilde{M}$ which is semisimple and injective.

14. Algebras without non-projective reflexive modules and without non-zero minimal exact complexes of projective modules

Proposition 14.1. *Let $e \ge 2$. For any $0 \le a \le e^2$, there exists a short local algebra of Hilbert type (e, a) such that any reflexive module is projective and such that the only minimal exact complex of projective modules is the zero complex.*

Proof. Let E be a vector space of dimension e say with basis $x_1, \dots, x_e$ and let T be the truncated tensor algebra $T = k \oplus E \oplus (E \otimes E)$. Of course, T is a short local algebra with $J(T) = E \oplus (E \otimes E)$ and $J(T)^2 = E \otimes E$, thus $e(T) = e,\ a(T) = e^2$.

Let $0 \le a \le e^2$. We will choose a suitable subspace $U \subseteq E \otimes E$ with $\dim U = e^2 - a$ and define $A = T/U$. Then $J(A) = J(T)/U$. Always, $J(A) = J(T)/U$ will be decomposable, thus Theorem 1.1 asserts that A has no non-projective reflexive modules.

If $a = 0$, then we have to take $U = E \otimes E$ and obtain $A = L(e)$. Since $e \ge 2$, $J(A) = E$ is a semisimple A-module of length e, thus decomposable.

Let E' be the subspace of E with basis $x = x_1$, and E'' the subspace generated by $x_2, \dots, x_e$. Thus $E = E' \oplus E''$.

If $e \le a$, then $E \otimes E'$ has dimension $e(e-1) \ge e^2 - a$, thus there is a subspace $U \subseteq E \otimes E''$ of dimension $e^2 - a$. Then, for $A = T/U$, we have $J(A) = J' \oplus J''$, where $J' = E' \oplus (E \otimes E')$ and $J'' = E'' \oplus (E \otimes E'')/U)$ are non-zero submodules of ${}_AJ(A)$, thus ${}_AJ(A)$ is decomposable. Note that $\dim J(A)^2 = \dim(E \otimes E') + \dim(E \otimes E'')/U = e + (e(e-1) - (e^2 - a)) = a$.

Finally, let $1 \le a < e$. Let U' be the subspace of $E \otimes E$ with basis $x_{a+1} \otimes x, \dots, x_e \otimes x$, and let $U'' = E \otimes E''$. Let $U = U' \oplus U''$. By abuse of notation, we will denote the residue class of $z \in T$ modulo U just by z again. We note that ${}_AJ(A)$ is the direct sum of the local module N generated by $x = x_1$ (with basis $x,\ x_1 \otimes x, \dots, x_a \otimes x$, thus $\mathbf{dim}\, N = (1, a)$) and a semisimple module with basis $x_2, \dots, x_e$, thus $J(A) \simeq N \oplus S^{e-1}$. In particular, ${}_AJ(A)$ is again decomposable.

We claim that the only minimal exact complex of projective A-modules is the zero complex. According to Theorem 1.3, we only have to look at the case $a = e-1$. Note that $J(A)$ has the basis $x_1, \dots, x_e;\ x_1 \otimes x, \dots, x_a \otimes x$.

The only indecomposable modules cogenerated by ${}_AJ(A)$ are N and S (namely, the annihilator C of ${}_AJ(A)$ is the ideal generated by J^2 and the element x_e, thus $A'' = A/C$ is of the form $L(a)$, and ${}_{A''}N$ is the indecomposable projective $L(a)$-module).

We have $\Omega S = {}_AJ(A) = N \oplus S^{e-1}$. And we have $\Omega N = S^e$ (namely, the map $f\colon {}_AA \to N$ with $f(1) = x$ maps x_i to $x_i \otimes x$, thus its kernel has basis $x_1 \otimes x, \dots, x_a \otimes x$ and x_e, thus ΩN is of the form S^e.)

Assume now that $P_\bullet$ is a minimal exact complex of projective modules and that M is one of the images. Then M is torsionless of Loewy length at most 2, thus of the form $M = N^u \oplus S^v$ for some natural numbers $u,\ v \geq 0$. We have $t(M) = u + v$. Since

$$\Omega M = \Omega(N^u \oplus S^v) = S^{eu} \oplus N^v \oplus S^{(e-1)v},$$

we have $t(\Omega M) = eu + v + (e-1)v = e(u+v)$. It follows that $t(P_{i+1}) = et(P_i)$ for all $i \in \mathbb{Z}$. Since $e \geq 2$, this is only possible if $t(P_i) = 0$ for all $i \in \mathbb{Z}$, thus $P_\bullet$ is the zero complex. $\square$

Remark. The assumption $e \geq 2$ is necessary, since all short local algebras with $e = 1$ are self-injective and not semisimple (thus, the simple module is non-projective and reflexive and occurs as an image in a minimal exact complex of projective modules).

15. Algebras with $\mho$-paths of length 2 and 3

The existence of an $\mho$-path of length 2 means the existence of a non-projective reflexive module; the existence of an $\mho$-path of length 3 means the existence of a non-projective 3-torsionfree module, thus of a non-projective module M such that both M and $\mho M$ are reflexive modules.

15.1. Algebras with $\mho$-paths of length 2

Proposition 15.1. *Let $1 \leq a \leq e-1$. There exists an (even commutative) short local algebra A of Hilbert type (e, a) with a reflexive module of Loewy length 2 with dimension vector $(1, a)$.*

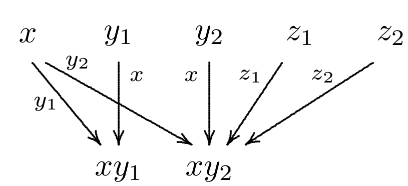
Proof. Let $c = e - a - 1$. Let A be the commutative algebra with generators

$$x,\ y_1, \dots, y_a,\ z_1, \dots, z_c,$$

and relations

$$x^2,\ xz_j,\ y_iy_{i'},\ y_iz_j,\ z_j^2 - xy_a,\ z_jz_{j'},$$

for all $i, i' \in \{1, \dots, a\}$ and all $j, j' \in \{1, \dots, c\}$ with $j' \neq j$. The elements $xy_1, \dots, xy_a$ form a basis of the vector space $J^2 = \operatorname{soc}{}_AA = \operatorname{soc}A_A$. For $a = c = 2$, the module ${}_AJ$ looks as follows



Let $M = Ax$. Then M is a module with Loewy length 2 and $\mathbf{dim}\, M = (1, a)$. Let us show that the embedding $\iota\colon Ax \to {}_AA$ is a left $\operatorname{add}({}_AA)$-approximation.

First, consider an element $m = \alpha x + \sum \beta_i y_i + \sum \gamma_j z_j$ with coefficients $\alpha,\ \beta_i,\ \gamma_j \in k$ and assume that there is a surjective map $Ax \to Am$. We have $xm = \sum \beta_i x y_i$. Since the element x annihilates Ax, we must have $xm = 0$, thus $\beta_i = 0$ for all i. We have $z_j m = \gamma_j x y_a$. Since the element z_j annihilates x, we must have $\gamma_j = 0$. It follows that $m = \alpha x$. This shows that for any homomorphism $f\colon Ax \to {}_AJ$, there is a scalar $\alpha \in k$ such that $f - \alpha\iota$ maps into J^2.

Second, we show that all the maps $g\colon Ax \to {}_AJ^2$ factor through ι. Let $g(m) = \sum \delta_i x y_i$ with $\delta_i \in k$. Let g' be the right multiplication on ${}_AA$ with $\sum \delta_i y_i$ Since

$$g'\iota(m) = g'(x) = x \sum \delta_i y_i = \sum \delta_i x y_i = g(m),$$

it follows that $g'\iota = g$. Altogether, we see that ι is a left $\operatorname{add}({}_AA)$-approximation.

It remains to show that the factor module $\mho M = {}_AA/Ax$ is cogenerated by ${}_AJ$. Now ${}_AA/Ax$ maps onto Ax as well as onto all the modules Az_j with $1 \le j \le c$ and the intersection of the kernels of these maps is zero. This shows that ${}_AA/Ax$ can be embedded into $Ax \oplus \bigoplus_j Az_j$. $\square$

Note that the element x constructed in the proof is a Conca element of A, as defined in [AIS] (see Section 10.3, and also [RZ3]).

15.2. An algebra with an $\mho$-path of length 3

Proposition 15.2. *There exists an (even commutative) short local algebra A of Hilbert type $(6, 2)$ with a non-projective 3-torsionfree module M having dimension vector $(2, 2)$.*

Proof. Let A be the commutative local algebra with generators $x_1, y_1, z_1, x_2, y_2, z_2$, and with the following relations: all squares of the generators (these are 6 relations), all products of pairs of generators with different indices (these are 9 relations), as well as the four additional relations

$$y_1 z_1,\ y_2 z_2,\ x_1 y_1 - x_2 y_2,\ x_1 z_1 - x_2 z_2.$$

Altogether, we have 19 relations. The ideal J^2 has the basis x_1y_1, x_1z_1, thus the Hilbert type of A is $(6, 2)$.

We visualize J as follows:

$$\begin{tikzcd}
x_1 \arrow[dr, "y_1"'] \arrow[drr, "z_1"] & y_1 \arrow[d, "x_1"] & z_1 \arrow[d, "x_1"] \\
 & x_1y_1 & x_1z_2 \\
x_2 \arrow[ur, "y_2"] \arrow[urr, "z_2"'] & y_2 \arrow[u, "x_2"'] & z_2 \arrow[u, "x_2"']
\end{tikzcd}$$

and we may mention that x_1 and x_2 are Conca elements.

Let $M = Ax_1 + Ax_2 \subset A$. Thus $\mathbf{dim}\, M = (2, 2)$.

Claim: *The module M is reflexive with $\mho M = M^*$.*

Proof. Let $f\colon M \to A$ be a homomorphism.

An easy calculation shows that there is $\lambda \in k$ such that $f(c) - \lambda c \in \operatorname{soc} A$ for all $c \in M$. [Namely, let $f(x_1) \equiv \alpha_1 x_1 + \beta_1 y_1 + \gamma_1 z_1 + \alpha_2 x_2 + \beta_2 y_2 + \gamma_2 z_2$ modulo $\operatorname{soc} A$, for some scalars $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2)$. Then $0 = f(x_1 x_1) = x_1 f(x_1) = \beta_1 x_1 y_1 + \gamma_1 x_1 z_1$ shows that $\beta_1 = 0 = \gamma_1$. Second, $0 = f(x_2 x_1) = x_2 f(x_1) = \beta_2 x_1 y_1 + \gamma_2 x_1 z_1$ shows that $\beta_2 = 0 = \gamma_2$. Third, $0 = f(y_2 x_1) = y_2 f(x_1)$ shows that $\alpha_2 = 0$. Altogether, we see that $f(x_1) \equiv \lambda x_1$ with $\lambda = \alpha_1$. Similarly, there is $\lambda' \in k$ with $f(x_2) \equiv \lambda' x_2$ But we also have $0 = f(x_1 y_1 - x_2 y_2) = y_1 f(x_1) - y_2 f(x_2) = (\lambda - \lambda') x_1 y_1$ thus $\lambda = \lambda'$ and therefore $f(c) \equiv \lambda c$ for all $c \in M$.]

If we use in addition that $\operatorname{soc} A \subset M$, we see that f is the restriction to M of an endomorphism of A.

In this way, we see that the inclusion map $M \to A$ is a (minimal) left $(\operatorname{add} A)$-approximation. As a consequence, we have $\mho M = A/M$. Now A/M is the algebra $C = L(4)$ with radical generators y_1, z_1, y_2, z_2. The monomorphism $u\colon A/M \to A^2$ defined by $u(1) = (x_1, x_2)$ shows that $\mho M = A/M$ is torsionless, therefore M is reflexive.

Since M is reflexive, and $\mathbf{dim}\, M = (2, 2)$, Proposition 6.1 asserts that $\mathbf{dim}\, M^* = (1, 4)$, thus M^* is a local module. Since M is annihilated by x_1, x_2, also M^* is annihilated by x_1, x_2, thus M^* is the free module of rank 1 over the algebra C (actually, the calculations presented above yield a direct way to see that $M^* = C$). It follows that $M^* = \mho M$.

If M is reflexive, also M^* is reflexive. Thus we see that in our case $\mho M = M^*$ is reflexive. Since both modules M and $\mho M$ are reflexive, M is 3-torsionfree. $\square$

Here is the $\mho$-path with the dimension vectors $\mathbf{dim}\, M$, $\mathbf{dim}\, \mho M$, $\mathbf{dim}\, \mho^2 M$ mentioned below (note that the module $\mho^3 M$ has Loewy length 3).

$$\begin{array}{ccccccccc} & M & \dashleftarrow & \mho M & \dashleftarrow & \mho^2 M & \dashleftarrow & \mho^3 M, \\ \mathbf{dim} & (2,2) & & (1,4) & & (2,11) & & \end{array}$$

Since $M \in \mathcal{Z}(\frac{1}{2})$, we have $\mho M \in \mathcal{Z}(2)$, and $\mho^2 M \in \mathcal{Z}(\frac{11}{4})$, as asserted in Proposition 6.1.

16. Final remarks

16.1. The torsionless modules for a short local algebras

The modules we have been interested in are mainly torsionless modules, namely syzygy modules; therefore we often have restricted the attention to the A-modules of Loewy length at most 2, thus to $L(e)$-modules, or, better, to the factor category $\operatorname{mod} L(e)/\operatorname{add}{}_A A$ (here, we factor out the ideal of $\operatorname{mod} L(e)$ given by all maps which factor through a projective A-module). Of course, the syzygy functor Ω_A has also to be taken into account; it is an endo-functor of the category $\operatorname{mod} L(e)/\operatorname{add}{}_A A$.

Note that the syzygy modules in $\operatorname{mod} A$ are the modules cogenerated by $W = {}_A J$. This means: We start with an $L(e)$-module W (namely the radical $W = {}_A J$ of A) and look at the category $\operatorname{sub} W$ of all $L(e)$-modules cogenerated by W, as well as at the endo-functor Ω_A of $\operatorname{sub} W/\operatorname{add}{}_A A$.

In dealing with $L(e)$-modules M, the main invariant is the dimension vector $\mathbf{dim}\, M$; it is a pair of natural numbers, thus an element of $\mathbb{Z}^2$. Here, $\mathbb{Z}^2$ is the Grothendieck group of the $L(e)$-modules with respect to the exact sequences of the form $0 \to JM \to M \to M/JM \to 0$, where M is any $L(e)$-module (equivalently, given an $L(e)$-module, we may consider the corresponding $K(e)$-module $\widetilde{M}$, see Section A.2 in Appendix A, and take as $\mathbf{dim}\, M$ the usual dimension vector of $\widetilde{M}$). As we have mentioned, the main tool in this paper has been the transformation ω_a^e on $\mathbb{Z}^2$, since it describes for the modules M in $\operatorname{sub} W$ the dimension vector $\mathbf{dim}\, \Omega_A M$ in terms of $\mathbf{dim}\, M$, at least roughly. The transformation ω_a^e plays a role quite similar to the usual use of ω_1^e (or better of $(\omega_1^e)^2$) in the representation theory of the e-Kronecker quiver (where $(\omega_1^e)^2$ describes the change of the dimension vectors of indecomposable non-projective modules when we apply the Auslander-Reiten translate τ). A decisive difference if of course the fact that ω_1^e is invertible, whereas, for $a \ge 2$, ω_a^e is not invertible over $\mathbb{Z}$.

16.2. Auslander-Reiten theory and homological behavior

We want to stress that the Auslander-Reiten-quiver of an algebra A does not determine the homological behavior of $\operatorname{mod} A$, see for example the short local self-injective algebras with $e = 2$ as discussed in Sections A.8 – A.11 in Appendix A: For all self-injective short local k-algebras with $e = 2$, the isomorphism classes of the indecomposable modules are indexed by the same set: namely, there are the indecomposable $L(e)$-modules and there is one additional module, the projective-injective indecomposable module P. The Auslander-Reiten quivers coincide: always P is inserted at the same place. But the homological behavior may be completely different, as the structure of the $\mho$-components shows. The operator Ω_A yields an arbitrary Möbius transformation on the projective line $\mathbb{P}^1(k)$ and this transformation is not displayed by the Auslander-Reiten quiver of A.

16.3. Projective coresolutions

Part of the paper has been devoted to the study of acyclic minimal complexes of projective modules, thus to the study of minimal projective coresolutions (of a module without non-zero projective direct summands): Note that a minimal projective coresolution determines uniquely an acyclic minimal complex of projective modules and any acyclic minimal complex of projective modules is obtained in this way. As we have seen, a minimal projective coresolution of a module seldom does exist. Also, if it exists, then it may not be unique (see for example the module $M(0,0,1)$ mentioned in [RZ2], 1.7). However, if it exists, then its structure may be very restricted: If A is a short local algebra, and $P_0 \to P_{-1} \to P_{-2} \to \cdots$ is a non-zero minimal projective coresolution of some module, let $t_i = t(P_i)$. Then either $t_i = t_{i-1}$ for $i \ll 0$ (and $a = e - 1$) or else $t_{i+1} + t_{i-1} = et_i$ for all $i \ll 0$ (and A is self-injective with $a = 1$), see Theorem 1.3 and Proposition A.7 in Appendix A.

Appendix A. Radical-square-zero algebras and self-injective algebras

This Appendix aims to describe the categories $\operatorname{mod} A$ where A is a short local algebra which is self-injective (equivalently, Gorenstein, see Remark 3.3) or has radical-square zero. We start in A.2 with the radical-square zero k-algebra $A = L(e)$ (with radical J of dimension e and $A/J = k$). In order to do so, we look in A.1 at a related algebra, the path algebra $K(e)$ of the e-Kronecker quiver.

A.1. The structure of $\operatorname{mod} K(e)$

We denote by $K(e)$ the *e-Kronecker quiver* with e arrows (or its path algebra):

$$\underset{0}{\circ} \xrightarrow{\langle e\rangle} \underset{1}{\circ}$$

(here and also later, we will depict a set of e arrows with same source and same sink by a single arrow endowed with the symbol $\langle e\rangle$). A representation (or module) V of $K(e)$ will be written in the form $V = (V_0, V_1;\ \phi\colon k^e \otimes V_0 \to V_1)$. There are two simple representations, namely $S(0) = (k, 0; 0)$ and $S(1) = (0, k; 0)$.

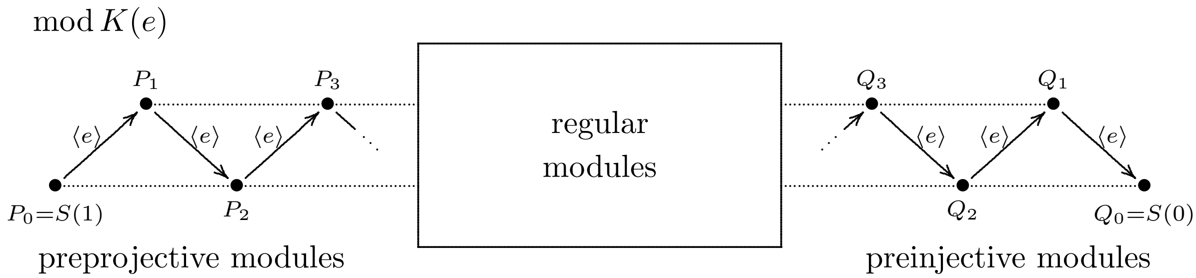
The Grothendieck group of $\operatorname{mod} K(e)$ (with respect to exact sequences) is $\mathbb{Z}^2$. Given a representation V of $K(e)$, the corresponding element in the Grothendieck group is the dimension vector $\mathbf{dim}\, V = (\dim V_0, \dim V_1)$ of V. On $\mathbb{Z}^2$, we consider the quadratic form $q(x, y) = x^2 + y^2 - exy$. This form q is positive definite, if $e = 1$, it is positive semidefinite, if $e = 2$ and indefinite, if $e \geq 3$. If $e \neq 2$, there is no non-zero pair (x, y) with $q(x, y) = 0$.

We have $q(\mathbf{dim}\, V) = \dim \operatorname{End}(V) - \dim \operatorname{Ext}^1(V, V)$ for every module V (see [R1]); more generally, given modules V, V' with $\mathbf{dim}\, V = \mathbf{dim}\, V'$, we have $q(\mathbf{dim}\, V) = \dim \operatorname{Hom}(V, V') - \dim \operatorname{Ext}^1(V, V')$. We can use q in order to distinguish between the regular indecomposable and the non-regular indecomposable modules: An indecomposable module V is *regular,* provided $\operatorname{Ext}^1(V, V) \neq 0$, and this happens if and only if $q(\mathbf{dim}\, V) \leq 0$. The remaining indecomposable modules are the indecomposable modules with $q(\mathbf{dim}\, V) = 1$ and then $\dim \operatorname{End}(V) = 1$. An element $(x, y) \in \mathbb{Z}^2$ is said to be a *real root of* q provided $q(x, y) = 1$ and an *imaginary root* provided $q(x, y) \leq 0$. If V is a regular indecomposable module, then there exists an indecomposable module V' with $\mathbf{dim}\, V' = \mathbf{dim}\, V$ such that V and V' are not isomorphic. If V is indecomposable with $q(\mathbf{dim}\, V) = 1$, then any indecomposable module V' with $\mathbf{dim}\, V' = \mathbf{dim}\, V$ is isomorphic to V. A non-regular indecomposable module V with $\mathbf{dim}\, V = (x, y)$ is said to be *preprojective* provided $x < y$, otherwise it is said to be *preinjective* (and then $x > y$).

For $e = 1$, there are just 3 indecomposable representations, namely $S(1), P(0), S(0)$, with $\mathbf{dim}\, S(1) = (0, 1), \mathbf{dim}\, P(1) = (1, 1)$ and $\mathbf{dim}\, S(0) = (1, 0)$.

We assume now that $e \geq 2$. The indecomposable preprojective modules can be labeled $P_0, P_1, P_2, \dots,$ with $P_0 = S(1)$, P_1 the indecomposable projective representation corresponding to the vertex 0 (thus $\mathbf{dim}\, P_1 = (1, e)$) and $\mathbf{dim}\, P_{i+1} = e\, \mathbf{dim}\, P_i - \mathbf{dim}\, P_{i-1}$ for $i \geq 1$. Similarly, the indecomposable preinjective modules can be labeled $Q_0 = S(0), Q_1, Q_2, \dots;$ with $Q_0 = S(0)$, Q_1 the indecomposable injective representation corresponding to the vertex 1 (thus $\mathbf{dim}\, Q_1 = (e, 1)$) and $\mathbf{dim}\, Q_{i+1} = e\, \mathbf{dim}\, Q_i - \mathbf{dim}\, Q_{i-1}$ for $i \geq 1$. If we define b_n for $n \geq -1$ recursively by $b_{-1} = 0,\ b_0 = 1$ and $b_{n+1} = eb_n - b_{n-1}$ for $n \geq 0$, then $\mathbf{dim}\, P_n = (b_{n-1}, b_n)$ and $\mathbf{dim}\, Q_n = (b_n, b_{n-1})$ (for example, for $e = 3$, the sequence $b_{-1}, b_0, b_1 \dots$ is just the sequence of the even-index Fibonacci numbers 0, 1, 3, 8, 21, 55, 144, ...). An explicit formula for the numbers b_n due to Avramov, Iyengar and Şega will be exhibited in Appendix B.

The global structure of $\operatorname{mod} K(e)$ can be seen by looking at the Auslander-Reiten quiver of $K(e)$. It has the following shape:



There are two Auslander-Reiten components of non-regular modules: the preprojective component (seen on the left) and the preinjective component (seen on the right). The Auslander-Reiten components of regular modules are homogeneous tubes for $e = 2$, and are of the form $\mathbb{Z}\mathbb{A}_\infty$ for $e \geq 3$.

Non-zero maps between preprojective modules (and between preinjective modules, respectively) go from left to right. Also, there are no non-zero maps from a regular module to a preprojective module, and no non-zero maps from a preinjective module to a preprojective or a regular module.

History. Here are at least some hints. The representations of $K(2)$ are called Kronecker modules, since they have been classified by Kronecker in 1890. We will give a brief survey on related investigations at the end of Section A.9.

The fact that there are just 3 indecomposable representations of $K(1)$ is a basic statement of elementary linear algebra.

The representation theory of $K(e)$ with $e \geq 3$ has attracted a lot of interest in the last 40 years, but is still very mysterious.

The algebras $K(e)$ with $e = 1$, $e = 2$, and $e \geq 3$ are typical representation-finite, tame, and wild algebras, respectively. One expects that any one-parameter family of indecomposable modules of a tame algebra is related to the regular modules of $K(2)$, and that any wild algebra has a full subcategory which is related to the regular representations of $K(3)$.

A.2. The push-down functor $\pi\colon \operatorname{mod} K(e) \to \operatorname{mod} L(e)$

We recall that $L(e)$ is the local k-algebra with radical J such that $J^2 = 0$, $\dim J = e$ and $L(e)/J = k$. We assume here that $|J| = e \geq 2$ and identify $J = k^e$.

We denote by $\pi\colon \operatorname{mod} K(e) \to \operatorname{mod} L(e)$ the push-down functor: It sends $V = (V_0, V_1;\ \phi\colon k^e \otimes V_0 \to V_1)$ to the representation

$$\pi V = \pi(V_0, V_1;\ \phi\colon k^e \otimes V_0 \to V_1) = \left(V_0 \oplus V_1;\ \begin{bmatrix} 0 & 0 \\ \phi & 0 \end{bmatrix}\right).$$

Under the functor π, the two simple representations of $K(e)$ are sent to the unique simple $L(e)$-module S. The indecomposable $K(e)$-modules of length at least 2 correspond under π bijectively to the indecomposable $L(e)$-modules of length at least 2, thus to the indecomposable bipartite $L(e)$-modules. We have $\mathbf{dim}\, \pi V = \mathbf{dim}\, V$ for any $K(e)$-module V without a simple projective direct summand.

Conversely, given an $L(e)$-module M, we denote by $\widetilde{M}$ the $K(e)$-module

$$\widetilde{M} = (\operatorname{top} M, \operatorname{rad} M;\ \overline{\mu}\colon J \otimes \operatorname{top} M \to \operatorname{rad} M),$$

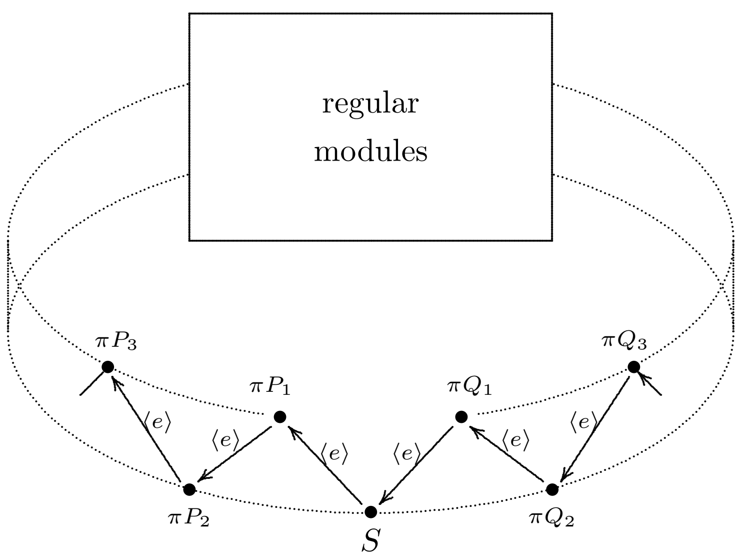
where $\overline{\mu}$ is induced by the multiplication map $\mu\colon J \otimes M \to M$ (note that $J \otimes \operatorname{rad} M$ is contained in the kernel of μ and that the image of μ is $\operatorname{rad} M$.). We have $\mathbf{dim}\,\widetilde{M} = \mathbf{dim}\, M$ for any $L(e)$-module M.

We have $\pi\widetilde{M} \simeq M$ for any $L(e)$-module M, and conversely, we have $\widetilde{\pi V} \simeq V$ for any $K(e)$-module V without a simple projective direct summand. Altogether we see: *π and $\widetilde{\ }$ provide inverse bijections between isomorphism classes as follows*:

$$\left\{\begin{matrix}\text{indecomposable}\\ K(e)\text{-modules } V\\ \text{different from } S(1)\end{matrix}\right\} \underset{\sim}{\overset{\pi}{\rightleftarrows}} \left\{\begin{matrix}\text{indecomposable}\\ L(e)\text{-modules}\end{matrix}\right\}$$

An indecomposable $L(e)$-module M will be said to be *regular* provided $\widetilde{M}$ is a regular $K(e)$-module. The Auslander-Reiten quiver for $L(e)$ is obtained from the Auslander-Reiten quiver of $K(e)$ by identifying the vertices $S(1)$ and $S(0)$ in order to obtain the vertex S.

mod $L(e)$



Proposition A.1 (Homomorphisms). *If M, M' are $L(e)$-modules, then π yields an injective map*

$$\operatorname{Hom}_{K(e)}(\widetilde{M}, \widetilde{M'}) \xrightarrow{\pi} \operatorname{Hom}_{L(e)}(M, M')$$

and

$$\operatorname{Hom}_{L(e)}(M, M') = \pi \operatorname{Hom}_{K(e)}(\widetilde{M}, \widetilde{M'}) \oplus \operatorname{Hom}_k(\operatorname{top} M, \operatorname{rad} M').$$

Proof. It is easy to show this directly. But one also may invoke the general covering theory as developed by Gabriel and his students. We use the $\mathbb{Z}$-cover Q of $L(e)$ with vertex set $\mathbb{Z}$, with e arrows $z \to z{+}1$ for all $z \in \mathbb{Z}$ and with all paths of length 2 as relations. We identify the full subquiver of Q with vertices $0, 1$ with $K(e)$.

If V is a representation of Q and $j \in \mathbb{Z}$, let $V[j]$ be the shifted representation with $V[j]_i = V_{i+j}$. The push-down functor π can be extended to a functor $\pi\colon \operatorname{mod} Q \to \operatorname{mod} L(e)$ and covering theory asserts that π yields a bijection between $\bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}_Q(V, V[j])$ and $\operatorname{Hom}_{L(e)}(\pi V, \pi V')$.

It remains to consider the indecomposable representations V, V' of Q which are either bipartite with support $\{0, 1\}$, or equal to $S(0)$. For example, if both V, V' are bipartite with support in $\{0, 1\}$, then $\operatorname{Hom}_Q(V, V'[1]) = \operatorname{Hom}_k(V_0, V_1') = \operatorname{Hom}_k(\operatorname{top} V, \operatorname{rad} V')$. $\square$

Proposition A.2 (Extensions). *Let M be an indecomposable regular $L(e)$-module. Then*

$$\operatorname{Ext}^1_{L(e)}(M, M) \neq 0.$$

If $e \geq 3$ and M, M' are indecomposable regular $L(e)$ modules with $\mathbf{dim}\, M = \mathbf{dim}\, M'$, then

$$\operatorname{Ext}^1_{L(e)}(M, M') \neq 0.$$

Proof. We have

$$\dim \operatorname{End}(\widetilde{M}) - \dim \operatorname{Ext}^1_{K(e)}(\widetilde{M}, \widetilde{M}) = q(\mathbf{dim}\, M) \leq 0.$$

Thus $\operatorname{End}(\widetilde{M}) \neq 0$ implies that $\operatorname{Ext}^1_{K(e)}(\widetilde{M}, \widetilde{M}) \neq 0$. Of course, a non-split self-extension of $\widetilde{M}$ yields under π a non-split self-extension of the $L(e)$-module M.

The second assertion is shown in the same way, now using that for $e \geq 3$ we have $q(\mathbf{dim}\, M) < 0$. $\square$

Proposition A.3 (Solid modules). *Let M be an $L(e)$-module. The following conditions are equivalent:*

(i) *M is solid.*
(ii) *$M \neq 0$ and $\operatorname{End} M = k \cdot 1_M + \{\phi \in \operatorname{End} M \mid \operatorname{Im} \phi \subseteq \operatorname{rad} M \subseteq \operatorname{Ker} \phi\}$.*
(iii) *$\dim \operatorname{End} M = 1 + |\operatorname{top} M| \cdot |\operatorname{rad} M|$.*
(iv) *$\operatorname{End}(\widetilde{M}) = k$.*

If these conditions are satisfied, then M is indecomposable.

Proof. (i) $\implies$ (ii). Assume that M is solid. An endomorphism of M which does not vanish on $\operatorname{soc} M$ has to be invertible. In particular, M has to be indecomposable: Namely, if $M = M' \oplus M''$ is a direct sum decomposition, then the projection onto M' maps $\operatorname{soc} M'$ onto itself and vanishes on $\operatorname{soc} M''$. Thus, either $M = S$ or else M is bipartite. If ϕ is an endomorphism of M and its restriction to $\operatorname{soc} M$ is the scalar multiplication by $\lambda \in k$, then $\phi - \lambda 1_M$ maps M into $\operatorname{rad} M$. This shows that $\operatorname{End}(M) = k \cdot 1_M \oplus \operatorname{Hom}(\operatorname{top} M, \operatorname{rad} M)$, thus (ii) is satisfied.

(ii) $\implies$ (iii) is trivial. The implication (iii) $\implies$ (iv) is a direct consequence of the Proposition A.1.

(iv) $\implies$ (i). Since $\widetilde{M}$ is indecomposable, also M is indecomposable. If $M = S$, then clearly M is solid. Thus, we can assume that M is bipartite. Proposition A.1 shows that any endomorphism ϕ is of the from $\phi = \lambda \cdot 1_M + \phi'$, where $\operatorname{soc} M = \operatorname{rad} M \subseteq \operatorname{Ker}(\phi')$. This shows that the restriction of ϕ to $\operatorname{soc} M$ is the scalar multiplication by λ. $\square$

Proposition A.4 (Modules without self-extensions). *Let $e \geq 2$. Let M be an indecomposable $L(e)$-module. The following conditions are equivalent.*
(i) *M is isomorphic to πP_i or πQ_i for some $i \geq 1$,*
(ii) *M is not simple and $q(\mathbf{dim}\, M) = 1$.*
(iii) $\operatorname{Ext}^1_{L(e)}(M, M) = 0$.

Proof. An indecomposable $K(e)$-module V satisfies $q(\mathbf{dim}\, V) = 1$ if and only if V is preprojective or preinjective. This yields the equivalence of (i) and (ii).

(iii) $\implies$ (i): If M is regular, then Proposition A.1 asserts that $\operatorname{Ext}^1_{L(e)}(M, M) \neq 0$. If $M = S$, then, of course, $\dim \operatorname{Ext}^1_{L(e)}(M, M) = e > 0$. This shows that an indecomposable module M with $\operatorname{Ext}^1_{L(e)}(M, M) = 0$ is isomorphic to πP_i or πQ_i for some $i \geq 1$,

(i) $\implies$ (iii). Let M be a bipartite module with $\mathbf{dim}\, M = (x, y)$. We define $g(M) = \dim \operatorname{End}(M) - 1 - xy$. Since $xy = |\operatorname{top} M| \cdot |\operatorname{rad} M|$, we see that $g(M) \geq 0$ and that $g(M) = 0$ if and only if M is solid.

The projective cover of M is isomorphic to $L(e)^x$, and ΩM is semi-simple, namely isomorphic to S^z with $z = ex - y$. We apply $\operatorname{Hom}(-, M)$ to the exact sequence $0 \to S^z \to L(e)^x \to M \to 0$ and obtain the exact sequence

$$0 \to \operatorname{Hom}(M, M) \to \operatorname{Hom}(L(e)^x, M) \to \operatorname{Hom}(S^z, M) \to \operatorname{Ext}^1_{L(e)}(M, M) \to 0.$$

We have $\dim \operatorname{Hom}(M, M) = xy + 1 + g(M)$, $\dim \operatorname{Hom}(L(e)^x, M) = x(x + y)$ and finally $\dim \operatorname{Hom}(S^z, M) = zy = (ex - y)y$. Thus

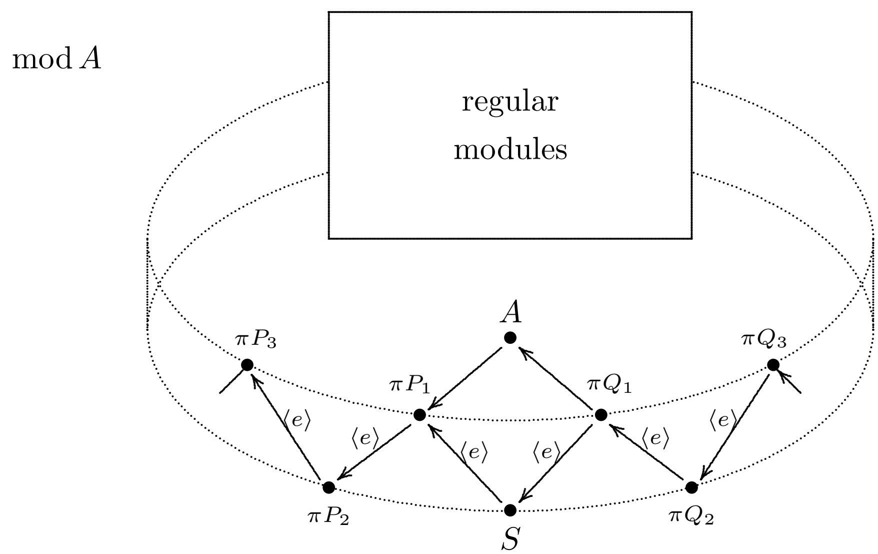
$$\begin{aligned} \dim \operatorname{Ext}^1_{L(e)}(M, M) &= xy + 1 + g(M) - x(x + y) + (ex - y)y \\ &= 1 + g(M) - x^2 + exy - y^2 = 1 - q(x, y) + g(M). \end{aligned}$$

If M is isomorphic to πP_i or πQ_i for some $i \geq 1$, then $q(x, y) = 1$ and M is solid, thus $g(M) = 0$, and therefore $\operatorname{Ext}^1_{L(e)}(M, M) = 0$. $\square$

Historical remark. The algebra $K(e)$ is obtained from $L(e)$ by a process which has been called "separation of a node" by Martinez [MV1] (a *node* is a simple module S which never occurs as a composition factor of $\operatorname{rad} M/(\operatorname{rad} M \cap \operatorname{soc} M)$, for any module M; if the algebra is given by a quiver with relations, then a vertex v is a *node* iff the composition of any arrow ending in v with any arrow starting in v is a relation). It seems that the first systematic separation of nodes was used in Gabriel's paper [Gb]: He showed that using separation of the nodes, the representations of a radical-square-zero algebra over an algebraically closed field can be obtained from the representations of a corresponding hereditary algebra (note that for a radical-square-zero algebra, all simple modules are nodes). The separation of nodes yields algebras which are stably equivalent, as later described in Auslander-Reiten-Smalø [ARS, Chapter X].

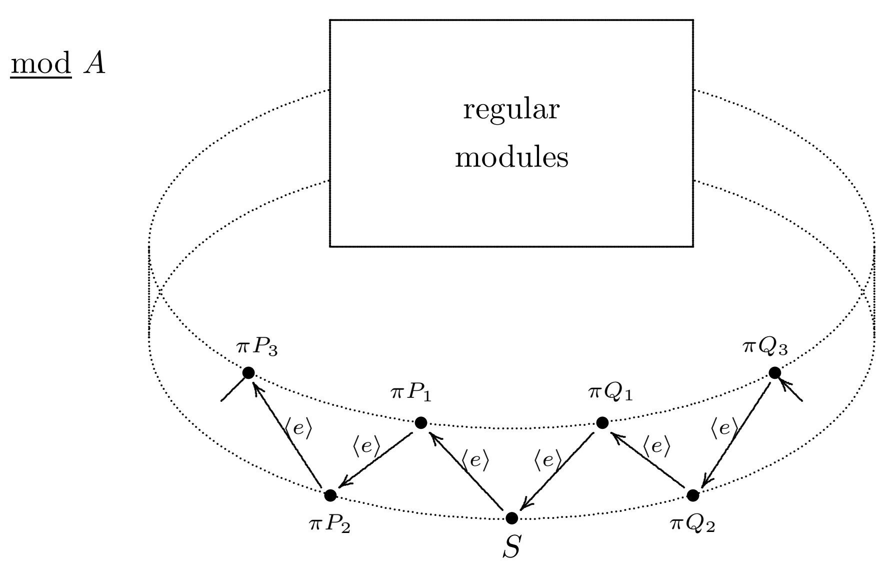
A.3. The self-injective short local algebras A with $e \geq 2$

Let A be a self-injective short algebra with $e \geq 2$. We obtain the Auslander-Reiten quiver for A from the Auslander-Reiten quiver of A/J^2 by inserting the vertex A.



The modules πP_i with $i \geq 1$ are the indecomposable A-modules which are different from ${}_AA$ and preprojective in the sense of Auslander-Smalø [AS]. The modules πQ_i with $i \geq 1$ are the indecomposable A-modules which are different from ${}_AA$ and preinjective in the sense of Auslander-Smalø.

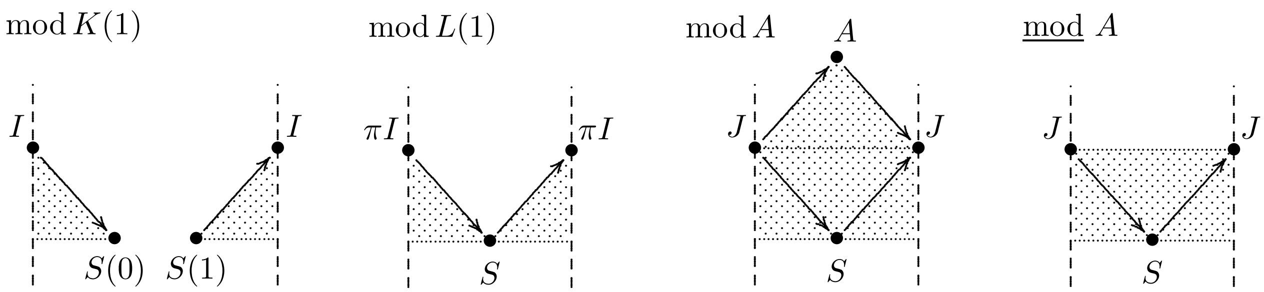
Finally, let us present the Auslander-Reiten quiver of the triangulated category $\underline{\mathrm{mod}}\, A$.



A.4. The cases $e = 1$

If A is a self-injective short algebra with $e = 1$, then either $a = 0$ or $a = 1$. In both cases, A is uniserial, thus its module category is well understood.

It may be of interest to draw the four relevant pictures in the case $e = a = 1$, so that one may compare them with the pictures for $e \geq 2$ exhibited above. Note that the last three categories shown below (the categories $\operatorname{mod} L(1)$, $\operatorname{mod} A$, and $\underline{\operatorname{mod}}\, A$) live (again) on a cylinder. For a unified presentation, we also show $\operatorname{mod} K(1)$ as embedded into a cylinder — a rather unusual display of a single triangle. Always, the dashed vertical lines are lines which have to be identified. The indecomposable representation of $K(1)$ of length 2 is denoted by I. Of course, if A is a short local algebra of Hilbert type $(1,1)$, then $J = \operatorname{rad} A = \pi I$.



A.5. Extensions of modules over self-injective algebras

In the following proposition, the first assertion is due to Hoshino [Ho1, Theorem 3.4].

Proposition A.5. *Let A be a self-injective short local algebra.*

(a) (Hoshino [Ho1]) *If M is a non-projective module, then* $\operatorname{Ext}^1(M, M) \neq 0$.

(b) *If $e \neq 2$ and M, M' are non-projective indecomposable modules with* $\mathbf{dim}\, M = \mathbf{dim}\, M'$. *Then* $\operatorname{Ext}^1(M, M') \neq 0$.

Proof. Since M, M' are non-projective indecomposable modules, they have Loewy length at most 2. Since there are non-projective modules, we must have $e \geq 1$ and thus $\operatorname{Ext}^1(S, S) \neq 0$, where S is the simple A-module.

If $e = 1$, see Section A.4: Either M is simple, thus $M' \simeq M$ and $\operatorname{Ext}^1(M, M') \neq 0$, or else $a = 1$ and M is of length 2. Then again $M' \simeq M$ and there is an exact sequence $0 \to M \to {}_AA \oplus S \to M \to 0$, which shows that $\operatorname{Ext}^1(M, M) \neq 0$, thus $\operatorname{Ext}^1(M, M') \neq 0$.

Thus, we can assume that $e \geq 2$. If M and M' are regular, then $\operatorname{Ext}^1_{L(e)}(M, M') \neq 0$, see Proposition A.2. Since there is a non-split exact sequence $0 \to M' \to M'' \to M \to 0$ in $\operatorname{mod} L(e)$, this sequence is also a non-split exact sequence in $\operatorname{mod} A$, therefore $\operatorname{Ext}^1_A(M, M') \neq 0$. If M is not regular, then $\mathbf{dim}\, M = \mathbf{dim}\, M'$ implies that $M \simeq M'$ and M belongs to the orbit of S under Ω and Ω^{-1}. The Corollary 12.3 asserts that $\operatorname{Ext}^1(M, M) \simeq \operatorname{Ext}^1(S, S) \neq 0$ (since A is self-injective, all modules are semi-Gorenstein-projective). $\square$

A.6. The BGP-functors

We want to show that for a self-injective short local algebra A of Hilbert-type $(e,1)$, the syzygy functor $\Omega = \Omega_A$ corresponds to a BGP-reflection functor for the $K(e)$-modules, as considered in [DR].

A BGP-functor σ_μ for the representations of $K(e)$ starts with two k-k-bimodules ${}_0W_1$, ${}_1W_0$ of dimension e and a non-degenerate bilinear form $\mu\colon {}_0W_1 \otimes {}_1W_0 \to k$. By definition,

$$\sigma_\mu(V_0, V_1;\ \phi\colon {}_1W_0 \otimes V_0 \to V_1) = (\operatorname{Ker}\phi,\ \phi'\colon {}_0W_1 \otimes \operatorname{Ker}\phi \to V_0),$$

where ϕ' is the composition

$${}_0W_1 \otimes \operatorname{Ker}\phi \xrightarrow{1\otimes u} {}_0W_1 \otimes_1 W_0 \otimes V_0 \xrightarrow{\mu\otimes 1} k \otimes V_0 = V_0,$$

with $u\colon \operatorname{Ker}\phi \to {}_1W_0 \otimes V_0$ the canonical inclusion map. We have $\sigma_\mu(S(1)) = 0$. Let $\operatorname{mod}_0 K(e)$ (and $\operatorname{mod}_1 K(e)$) be the full subcategory of all $K(e)$-modules without simple projective (and injective, respectively) direct summands. The restriction of σ_μ to $\operatorname{mod}_0 K(e)$ is an equivalence $\operatorname{mod}_0 K(e) \to \operatorname{mod}_1 K(e)$. If we denote the matrix ω_1^e just by σ, then $\mathbf{dim}\,\sigma_\mu M = \sigma\,\mathbf{dim}\,M$, for any indecomposable $K(e)$-module M which is not simple projective.

If M is indecomposable and not isomorphic to $S(1)$, then $\mathbf{dim}\,\sigma_\mu M = \sigma\,\mathbf{dim}\,M$. It follows that for $e \geq 2$, we have

$$\begin{aligned} \sigma_\mu P_i &= \begin{cases} P_{i-1} & \text{if} \quad i \geq 1, \\ 0 & \text{if} \quad i = 0, \end{cases} \\ \sigma_\mu Q_i &= Q_{i+1} \quad \text{for all } i \geq 0. \end{aligned}$$

Now we fix a self-injective algebra A of Hilbert-type $(e,1)$ and an embedding of k^e as a complement of J^2 in J, thus we identify J/J^2 with $W = k^e$. Let ${}_1W_2 = {}_2W_1 = W$ and take as bilinear form $\mu\colon W \otimes W \to k$ the multiplication map $J/J^2 \otimes J/J^2 \to J^2 = k$. Since A is self-injective, μ is non-degenerate and we write $\sigma_A = \sigma_\mu$. For any A-module M, let $\Omega_A M$ be its first syzygy module.

Proposition A.6. *Let A be a self-injective short local algebra with $e(A) = e$. Let M be in $\operatorname{mod}_0 K(e)$. Then the A-module $\pi(\sigma_A M)$ is isomorphic to $\Omega_A \pi(M)$.*

We have to exclude $S(1)$, since $\pi(\sigma_A S(1))) = 0$, whereas $\Omega_A \pi S(1) = \Omega_A S = {}_AJ$.

Proof. Let us start with the A-module $M = \pi(T,\ \phi\colon W \otimes T \to JM)$, where $T = \operatorname{top} M = M/JM$ (thus, we identify M with $T \oplus JM$, this is the right column in the following diagram). Its projective cover is $PM = A \otimes T = (k \oplus W \oplus J^2) \otimes T$ (this is the middle column) with canonical map $p = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \phi & 0 \end{smallmatrix}\right]\colon PM \to M$. This yields $\Omega_A M$ (namely the left column) as the kernel of p. Altogether, we deal with five exact sequences of vector spaces (displayed in the upper five rows), organized in two commutative diagrams. In this way, we obtain the exact sequence of representations of $K(e)$ exhibited as the lowest row:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & k\otimes T & \xrightarrow{1} & T & \longrightarrow & 0\\
0 & \longrightarrow & W\otimes 0 & \longrightarrow & W\otimes T & \xrightarrow{1} & W\otimes T & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & \mathrm{Ker}(\phi) & \xrightarrow{u} & W\otimes T & \xrightarrow{\phi} & JM & \longrightarrow & 0\\
0 & \longrightarrow & W\otimes\mathrm{Ker}(\phi) & \xrightarrow{1\otimes u} & W\otimes W\otimes T & \xrightarrow{1\otimes\phi} & W\otimes JM & \longrightarrow & 0\\
 & & {\scriptstyle (\mu\otimes 1)(1\otimes u)}\downarrow & & \downarrow{\scriptstyle \mu\otimes 1} & & \downarrow & & \\
0 & \longrightarrow & J^2\otimes T & \xrightarrow{1} & J^2\otimes T & \longrightarrow & 0 & \longrightarrow & 0\\
 & & & & & & & & \\
0 & \longrightarrow & \Omega_A M & \longrightarrow & PM & \xrightarrow{p} & M & \longrightarrow & 0
\end{array}$$

There is the following commutative diagram of functors:

$$\begin{array}{ccc}
\mathrm{mod}_0\, K(e) & \xrightarrow{\sigma_A} & \mathrm{mod}\, K(e)\\
\downarrow{\scriptstyle \iota\pi} & & \downarrow{\scriptstyle \iota\pi}\\
\mathrm{mod}\, A/\,\mathrm{add}(A) & \xrightarrow{\Omega_A} & \mathrm{mod}\, A/\,\mathrm{add}(A)
\end{array}$$

where $\iota\colon \mathrm{mod}\, L(e) \to \mathrm{mod}\, A$ is the canonical embedding.

A detailed study of the operation of Ω_A on the set of indecomposable modules of length 2 in case $e(A) = 2$ will be given at the end of Section A.8.

Historical remark. Reflection functors for quivers were introduced by Bernstein-Gelfand-Ponomarev [BGP] and play an important role in the representation theory of quivers. (The printer has asked us to define BGP as used in the head line of A.6: these are the initionals of the authors of the paper [BGP].) They have been generalized to species in [DR]. As we have seen above, this generalization is actually also of interest for quivers, for example for the e-Kronecker quiver $K(e)$, since one avoids in this way the use of a fixed basis of the arrow space. But we should stress that the account given here deviates from the usual convention (say used in [BGP] and [DR]) which is based on changing the orientation of arrows. Indeed, the BGP-reflection functors considered in [BGP] and [DR] send a representation of the e-Kronecker quiver $\circ \xrightarrow{\langle e\rangle} \circ$ to a representation of the quiver $\circ \xleftarrow{\langle e\rangle} \circ$ (with opposite orientation). In contrast, we relabel the vertices in order to obtain σ_μ as an endo-functor of $\mathrm{mod}\, K(e)$. As a consequence, the change of the dimension vector under σ_μ is described by the product σ of the usual BGP-reflection matrix $\left[\begin{smallmatrix}1 & 0\\ e & -1\end{smallmatrix}\right]$ and the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ (corresponding to the exchange of the coordinate axes):

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ e & -1 \end{bmatrix} = \begin{bmatrix} e & -1 \\ 1 & 0 \end{bmatrix} = \omega_1^e.$$

A.7. The $\mho$-quiver

We are going to analyze the $\mho$-quivers of the short local algebras A which have radical square zero or which are self-injective.

A.7.1. The radical-square-zero algebras $L(e)$ with $e \geq 2$. First, let us look at those algebras which are not self-injective. These are the algebras $A = L(e)$ with $e \geq 2$. Here, the simple module S is the only non-projective indecomposable module which is torsionless, and ΩS is not isomorphic to S. Thus, all but one $\mho$-components are of type $\mathbb{A}_1$, the remaining one is the $\mho$-component containing S and this component is of the form $\mathbb{A}_2$:

$$\begin{tikzcd} {[S]} & {[\mho S]} \arrow[l, dashed] \end{tikzcd}$$

The corresponding $\mho$-sequence is $0 \to S \to {}_AA^e \to \mho S \to 0$, and $\mathbf{dim}\, \mho S = (e, e^2 - 1)$.

Now, we look at the self-injective algebras.

A.7.2. The $\mho$-quiver of a self-injective algebra. If A is any finite-dimensional algebra, then A is self-injective algebra, iff all modules are torsionless, iff any module M satisfies $\operatorname{Ext}^1(M, A) = 0$; thus iff any vertex of the $\mho$-quiver of A is the end of an arrow, iff any vertex of the $\mho$-quiver of A is the start of an arrow; thus iff any $\mho$-component of A is either of the form $\widetilde{\mathbb{A}}_n$ with $n \geq 0$ or of the form $\mathbb{Z}$. For a self-injective algebra, the operator $\mho$ coincides with Σ, where ΣM is the cokernel of the canonical map $M \to IM$, where IM is the injective envelope of M. It is usual also to write in this case $\Omega^{-1}M = \Sigma M = \mho M$, since for M indecomposable and not projective, we have $\Sigma\Omega M = M = \Omega\Sigma M$.

A.7.3. The radical-square-zero algebra $A = L(1)$. In this case, S is the only non-projective indecomposable module, thus the $\mho$-quiver has just one component, namely the loop

$$\begin{tikzcd} {[S]} \arrow[loop right, dashed] \end{tikzcd}$$

The remaining self-injective short local algebras A have $a(A) \neq 0$, thus $a(A) = 1$.

A.7.4. The case $e = 1$ and $a = 1$. Let $S[2]$ be the indecomposable module of length 2, thus $S,\ S[2]$ are the only non-projective indecomposable modules and $\Omega S = \Sigma S = S[2]$.

$$\begin{tikzcd} {[S]} \arrow[r, dashed, bend right] & {[S[2]]} \arrow[l, dashed, bend right] \end{tikzcd}$$

For the cases $e \geq 2$, we will use A.6, in order to describe the $\mho$-quiver of A.

A.7.5. The regular modules. By definition, these are the indecomposable A-modules of the form $M = \pi X$, where X is a regular $K(e)$-module. An $\mho$-component which contains a regular module M contains only regular modules and looks as follows:

$$\begin{array}{ccccccccccc} \cdots & \dashleftarrow & \Omega_A^2 M & \dashleftarrow & \Omega_A M & \dashleftarrow & M & \dashleftarrow & \Omega_A^{-1} M & \dashleftarrow & \Omega_A^{-2} M & \dashleftarrow & \cdots \\ & & \sigma^2 \,\mathbf{dim}\, M & & \sigma\, \mathbf{dim}\, M & & \mathbf{dim}\, M & & \sigma^{-1}\, \mathbf{dim}\, M & & \sigma^{-2}\, \mathbf{dim}\, M & & \end{array}$$

(below any module, we show the corresponding dimension vector). In general, such an $\mho$-component is of type $\mathbb{Z}$. *Only for* $e = 2$, M *may be* Ω_A*-periodic,* and then, of course, we deal with an $\mho$-component of type $\widetilde{A}_n$ for some $n \geq 0$. See Sections A.8 – A.11 for further discussion of the case $e = 2$. To repeat: *If* $e \geq 3$, *then all* $\mho$*-components containing regular modules are of type* $\mathbb{Z}$ (and, as we will see next, also the only additional component is of type $\mathbb{Z}$).

A.7.6. The non-regular modules. For all self-injective short local algebras A with $e \geq 2$, there is in addition the $\mho$-component containing the simple module S. It is always of type $\mathbb{Z}$ and consists of S and the modules πP_i and πQ_i with $i \geq 1$. We have $\pi Q_i = \Omega_A^i S$ and $\pi P_i = \Omega_A^{-i} S$; in particular, we have $\pi Q_1 = {}_AJ$, and $\pi P_1 = {}_AA/J^2$.

$$\begin{array}{ccccccccccccc} \cdots & \dashleftarrow & \pi Q_2 & \dashleftarrow & \pi Q_1 & \dashleftarrow & S & \dashleftarrow & \pi P_1 & \dashleftarrow & \pi P_2 & \dashleftarrow & \cdots \\ \cdots & & \begin{bmatrix} b_2 \\ b_1 \end{bmatrix} & & \begin{bmatrix} b_1 \\ b_0 \end{bmatrix} & & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \vdots & \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} & & \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} & & \cdots \end{array}$$

(again, we show below any module the corresponding dimension vector). Since for $i \geq 0$, we have $\Omega^i S = \pi Q_i$ and $\mathbf{dim}\, \pi Q_i = \mathbf{dim}\, Q_i = (b_i, b_{i-1})$, we see that

$$\beta_i(S) = b_i$$

for all $i \geq 0$. This means that *the numbers* b_i *for* $i \geq 0$ *are just the Betti numbers of* S.

In the display of the $\mho$-component of S we have inserted a dotted vertical line between the dimension vectors of S and of πP_1. This separation line should stress that $\Omega(\pi P_1) = S$, whereas $\sigma(\mathbf{dim}\, \pi P_1) = \sigma \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{dim}\, S$. There is just one $\mho$-sequence which is not bipartite, namely the sequence starting in S (as mentioned already in 2.4(a)):

$$0 \to S \to {}_AA \to \pi P_1 \to 0.$$

It is this sequence which is marked by the dotted separation line.

A.7.7. The $\mho$-paths of length 2. All but two $\mho$-paths of length 2 are controlled by a single formula which relates the tops of the modules involved:

Proposition A.7. *Let A be a self-injective short local algebra. Let $M_1 \leftarrow M_0 \leftarrow M_{-1}$ be an $\mho$-path. If M_0 is not isomorphic to S nor to $\pi P_1 = {}_AA/J^2$, then*

$$t(M_1) + t(M_{-1}) = e t(M_0).$$

Proof. The only $\mho$-sequence which is not bipartite is the sequence $0 \to S \to A \to A/S \to 0$. Thus, if If M_0 is not isomorphic to S nor to A/J^2, then both sequences $0 \to M_1 \to P(M_0) \to M_0 \to 0$ and $0 \to M_0 \to P(M_{-1}) \to M_{-1} \to 0$ are bipartite. Let $\mathbf{dim}\, M_{-1} = (t, s)$. Then $\mathbf{dim}\, M_0 = (et - s, t)$ and $\mathbf{dim}\, M_1 = (e(et-s) - t, et - s)$. Since $t(M_1) = e(et-s) - t,\ t(M_0) = et - s,\ t(M_{-1}) = t$, we see that $t(M_1) + t(M_{-1}) = eM_0$. $\square$

There are the two remaining $\mho$-paths $M_1 \leftarrow M_0 \leftarrow M_{-1}$ with $M_0 = S$ and $M_0 = {}_AA/J^2$. Both are part of the $\mho$-component which contains S. This $\mho$-component has been displayed above. Let us show again the relevant part:

$$\begin{array}{ccccccc}
\pi Q_1 & \dashleftarrow & S & \dashleftarrow & \pi P_1 & \dashleftarrow & \pi P_2 \\
= {}_AJ & & & & = {}_AA/J^2 & & = \mho({}_AA/J^2)
\end{array}$$

and recall that $t({}_AJ) = b_1 = e,\ t(S) = b_0 = 1,\ t({}_AA/J^2) = b_0 = 1,\ t(\mho({}_AA/J^2)) = b_1 = e$.

Corollary A.8. *Let $e \geq 2$. Let $P_\bullet$ be a minimal exact complex of projective modules and let $t_i = t(P_i)$. If all images of $P_\bullet$ are bipartite, then*

$$t_{i-1} + t_{i+1} = et_i \qquad (*)$$

for all $i \in \mathbb{Z}$. If S is the image of $P_0 \to P_{-1}$, then $(*)$ holds for all $i \notin \{0, -1\}$ and $t_{-1} = t_0 = 1,\ t_{-2} = t_1 = e$. $\square$

Historical Remark. For a self-injective algebra A, the $\mho$-quiver just depicts the graph of the operation Ω on the set of isomorphism classes of indecomposable non-projective modules, thus it visualizes a basic concept which has been used since the early days of homological algebra.

A.8. Self-injective algebras with $e = 2$: The modules of length 2

Let A be a self-injective short local algebra with $e = e(A) = 2$. In Sections A.8 and A.9, we are going to survey some properties of the regular modules. We start with the indecomposable modules of length 2; they always are regular.

Lemma A.9. *Let A be a self-injective short local algebra with $e = 2$. If M is an indecomposable module of length 2, also ΩM and $\Sigma M (= \mho M)$ are indecomposable modules of length 2.*

Proof. An indecomposable module M of length 2 is local, thus its projective cover is a free module of rank 1 and therefore ΩM has length 2, again. Since ΩM is a submodule of $PM = {}_AA$ and ${}_AA$ has simple socle, also ΩM has simple socle, thus ΩM is indecomposable. This shows that ΩM is an indecomposable module of length 2. Similarly, one shows that ΣM is an indecomposable module of length 2. $\square$

Corollary A.10. *Let A be a self-injective short local algebra with $e = 2$. If M is an indecomposable module of length 2, then all modules in the $\mho$-component containing M are indecomposable modules of length 2.*

If M is indecomposable and of length 2, then $\operatorname{Ext}^1_{L(2)}(M,M) \neq 0$, according to Proposition A.1 in Appendix A. Therefore also $\operatorname{Ext}^1_A(M,M) \neq 0$. The (uniquely defined) exact sequence $0 \to M \to M' \to M \to 0$ with $J^2(M') = 0$ will be called the *Kronecker extension* of M (and we write $M' = M[2]$). Also $\operatorname{Ext}^1(M, \Omega M) \neq 0$, since there is the exact sequence $0 \to \Omega M \to PM \to M \to 0$, and this is an $\mho$-sequence. These two kinds of extensions, the Kronecker extension and the $\mho$-extension, are the basic data for dealing with indecomposable modules of length 2.

Proposition A.11. *Let A be a self-injective short local algebra with $e = 2$. Let M, M' be indecomposable modules of length 2. Then $\operatorname{Ext}^1(M, M') = 0$ iff $M' \not\simeq M$ and $M' \not\simeq \Omega M$.*

Proof. We assume that M' is not isomorphic to M or ΩM, and have to show that $\operatorname{Ext}^1(M, M') = 0$. Let $\epsilon\colon 0 \to M' \to Y \to M \to 0$ be a non-split exact sequence. If $J^2 Y = 0$, then this is an exact sequence of $L(e)$-modules, thus the Kronecker extension of M. Assume now that $J^2 Y \neq 0$. Then Y has an indecomposable direct summand isomorphic to ${}_AA$. Since both ${}_AA$ and Y have length 4, we see that $Y = {}_AA$. Thus $Y \to M$ is a projective cover of M and and ϵ is the $\mho$-extension of M. In particular, $M' = \Omega M$. $\square$

Corollary A.12. *Let A be a self-injective short local algebra with $e = 2$. Let M be an indecomposable module of length 2 and $t \geq 0$. Then $\operatorname{Ext}^{t+1}(M,M) = 0$ iff $\Sigma^t M$ is not isomorphic to M or ΩM.*

Proof. We have $\operatorname{Ext}^{t+1}(M,M) = \operatorname{Ext}^1(M, \Sigma^t M)$. $\square$

Corollary A.13. *Let A be a self-injective short local algebra with $e = 2$. Let M be an indecomposable module of length 2 and $t \geq 0$. Then $\operatorname{Ext}^t(M,M) = 0$ for all $t \geq 2$ iff the $\mho$-component containing M is of type $\mathbb{Z}$.* $\square$

Corollary A.14. *Let A be a self-injective short local algebra with $e = 2$. Let M be an indecomposable module of length 2. The following conditions are equivalent.*

(i) $\operatorname{Ext}^2(M,M) = 0$.
(ii) *There is $i \geq 1$ with* $\operatorname{Ext}^i(M,M) = 0$.
(iii) *The $\mho$-component containing M has cardinality at least* 3.
(iv) $\Omega^2 M \not\simeq M$.

Proof. (i) implies (ii) is trivial.

(ii) implies (iii): Assume that the $\mho$-component containing M has cardinality at most 2. Then the modules belonging to the component are M and ΩM. Thus, for any $i \geq 1$, the module $\Sigma^{i-1} M$ is isomorphic to M or to ΩM. Thus, for any $i \geq 1$, the group $\operatorname{Ext}^i(M,M) = \operatorname{Ext}^1(M, \Sigma^{i-1} M)$ is equal to $\operatorname{Ext}^1(M,M)$ or to $\operatorname{Ext}^1(M, \Omega M)$ and both groups are non-zero. This contradicts (ii).

(iii) implies (iv) is trivial.

(iy) implies (i). We assume that $\Omega^2 M \not\simeq M$. Then clearly also $\Omega M \not\simeq M$. Now $\Omega^2 M \not\simeq M$ implies that $\Omega M \not\simeq \Sigma M$ and $\Omega M \not\simeq M$ implies that $M \not\simeq \Sigma M$. According to Corollary 1 (with $t = 1$), we have $\operatorname{Ext}^2(M,M) = 0$. $\square$

If M is an indecomposable non-projective module with $\Omega^2 M \simeq M$, then the $\mho$-component containing M has cardinality at most 2: it consists of the two modules M, ΩM which may or may not be isomorphic. Here are the two cases: on left the case that $\Omega M \simeq M$; on the right, the case that $\Omega M \not\simeq M \simeq \Omega^2 M$.

$$\begin{tikzcd}[M] \arrow[loop, dashed] \end{tikzcd} \qquad\qquad \begin{tikzcd}[M] \arrow[r, bend right, dashed] & {[\Omega M]} \arrow[l, bend right, dashed]\end{tikzcd}$$

In the left case $\Omega M \simeq M$, the vector space $\operatorname{Ext}^1(M, M)$ is 2-dimensional, a basis of $\operatorname{Ext}^1(M, M)$ is given by the Kronecker extension and the $\mho$-extension. In the right case $\Omega M \not\simeq M \simeq \Omega^2 M$, the vector space $\operatorname{Ext}^1(M, M)$ is 1-dimensional with basis the Kronecker extension. Also the vector space $\operatorname{Ext}^1(M, \Omega M)$ is 1-dimensional, it has the $\mho$-extension as basis.

Of special interest are the self-injective short local algebras with $e = 2$ which have $\mho$-components of simple regular modules of cardinality at least 3. Such $\mho$-components don't exist for commutative algebras, as Huneke-Şega-Vraciu [HSV] have shown: *If A is a commutative self-injective short local algebra, then any indecomposable non-projective module M satisfies* $\operatorname{Ext}^i(M, M) \neq 0$ *for all* $i \geq 1$. (Note that we have seen in Lemma 10.4 that for any commutative, self-injective, short local algebra A, any $\mho$-component consisting of local modules has cardinality at most 2.)

Let us look at the operation of Ω_A on the set of indecomposable modules of length 2. Let V be a vector space. Non-zero elements x, y of V are called *equivalent* provided there is $\lambda \in k$ (necessarily non-zero) with $y = \lambda x$. We denote by $\mathbb{P}(V)$ the set of equivalence classes of non-zero elements of V. If $\dim V = n+1$, then $\mathbb{P}(V)$ is called the *n-dimensional projective space*. We write $\mathbb{P}^1(k)$ instead of $\mathbb{P}(k^2)$ and call this the *projective line* over k.

If A is a self-injective short local algebra with $e(A) = 2$, we may identify $\mathbb{P}(J/J^2) = \mathbb{P}^1(k)$ with the set of indecomposable modules of length 2. Namely, let us fix a generating set x_0, x_1 of ${}_AJ$, as well as a non-zero element $z \in J^2$. If $(\alpha_0, \alpha_1) \in k^2$ is a non-zero pair, then $M(\alpha_0, \alpha_1) = A(\alpha_0 x_0 + \alpha_1 x_1)$ is an indecomposable module of length 2, all are obtained in this way, and $M(\alpha_0, \alpha_1) \simeq M(\alpha'_0, \alpha'_1)$ iff (α_0, α_1) and (α'_0, α'_1) are equivalent.

Note that $\Omega = \Omega_A$ provides a permutation of the set of (isomorphism classes of) indecomposable modules of length 2, thus of $\mathbb{P}(J/J^2) = \mathbb{P}^1(k)$. We are going to describe this permutation. The multiplication in A yields a (non-degenerate) bilinear form $\mu\colon J/J^2 \otimes J/J^2 \to J^2$, say given by the (2×2)-matrix B:

$$(\alpha'_0 x_0 + \alpha'_1 x_1) \cdot (\alpha_0 x_0 + \alpha_1 x_1) = \mu((\alpha'_0, \alpha'_1), (\alpha_0, \alpha_1)) \cdot z = (\alpha'_0, \alpha'_1) B (\alpha_0, \alpha_1)^t \cdot z$$

We define a linear map $\eta_B\colon k^2 \to k^2$ by

$$\eta_B(\alpha_0, \alpha_1) = (\alpha_0, \alpha_1) \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) B^{-1},$$

of course, η_B is invertible.

Lemma A.15. *For any pair non-zero (α_0, α_1) in k^2, we have*

$$\Omega_A M(\alpha_0, \alpha_1) = M(\eta_B(\alpha_0, \alpha_1)).$$

Proof. Let $x = \alpha_0 x_0 + \alpha_1 x_1$ and $y = \alpha'_0 x_0 + \alpha'_1 x_1$, where $(\alpha'_0, \alpha'_1) = \eta_B(\alpha_0, \alpha_1)$. We have

$$\begin{aligned} yx &= \mu((\alpha'_0, \alpha'_1), (\alpha_0, \alpha_1)) = \mu(\eta_B(\alpha_0, \alpha_1), (\alpha_0, \alpha_1)) \\ &= (\alpha_0, \alpha_1) \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) B^{-1} B \, (\alpha_0, \alpha_1)^t = 0. \end{aligned}$$

But if x, y are elements in $J \setminus J^2$ with $yx = 0$, then there is the following exact sequence

$$ {}_AA \xrightarrow{\rho(y)} {}_AA \xrightarrow{\rho(x)} {}_AA, $$

where $\rho(y)$ and $\rho(x)$ denote the right multiplication by y, and by x, respectively. The image of $\rho(y)$ is Ay, the image of $\rho(x)$ is Ax, thus $\Omega Ax = Ay$. $\square$

It is clear that any invertible linear transformation $\eta\colon k^2 \to k^2$ induces a permutation of $\mathbb{P}^1(k)$, sending the equivalence class of (α_0, α_1) to the equivalence class of $\eta(\alpha_0, \alpha_1)$. Such a permutation is called a *Möbius transformation.* Thus, we have shown the first part of the following assertion:

Proposition A.16. *The operation of Ω_A on the set $\mathbb{P}^1(k)$ of indecomposable modules of length 2 is given by a Möbius transformation, and any Möbius transformation of $\mathbb{P}^1(k)$ occurs in this way.*

Proof. It remains to show that any Möbius transformation is of the form η_B for some invertible matrix B. Let μ be the non-degenerate bilinear form on k^2 given by the matrix B. We just need a self-injective short local algebra such that the multiplication map $J/J^2 \otimes J/J^2 \to J^2$ is given by μ. Then Ω_A will operate on the set of indecomposable modules of length 2 by η_B.

But for any non-degenerate bilinear form μ on an e-dimensional vector space W, we can define the algebra A_μ with underlying vector space $k \oplus W \oplus k$ and with multiplication

$$ (c, w, d)(c', w', d') = (cc', cw' + c'w, cd' + c'd + \mu(w, w')), $$

where $c, c', d, d' \in k$ and $w, w' \in W$. Of course, A_μ is a self-injective short local algebra with $e(A_\mu) = e$. And by definition, the multiplication map $J/J^2 \otimes J/J^2 \to J^2$ is just $\mu\colon W \otimes W \to k$. $\square$

A.9. Self-injective algebras with $e = 2$: The regular modules

As we have mentioned, the aim of Sections A.8 and A.9 is to survey properties of the regular modules for a self-injective short local algebra A with $e = e(A) = 2$. Until now, we were dealing just with the indecomposable modules of length 2 (they always are regular).

In our case $e(A) = 2$, an indecomposable A-module M is regular provided provided it has Loewy length at most 2 and $\mathbf{dim}\, M = (m, m)$ for some m. A regular module M is said to be *simple regular* provided the only proper submodule of M which is regular, is the zero module. Of course, the indecomposable modules of length 2 are simple regular. We should stress that in case the base field k is algebraically closed, the indecomposable modules of length 2 are the only simple regular modules! However, if k is not algebraically closed, then there are additional simple regular modules.

We denote by $\mathcal{R}$ the full subcategory of all modules which are direct sums of regular indecomposable modules. Also, we denote by $\widetilde{\mathcal{R}}$ the the full subcategory of $\operatorname{mod} K(2)$ consisting of all $K(2)$-modules which are direct sums of indecomposable regular $K(2)$-modules. The push-down functor $\pi\colon \operatorname{mod} K(2) \to \operatorname{mod} L(2)$ provides a bijection between the indecomposable objects in $\widetilde{\mathcal{R}}$ and the indecomposable objects in $\mathcal{R}$. Actually, there is an ideal $\mathcal{I}$ of the category $\operatorname{mod} L(2)$, namely the class of all maps with semisimple image, such that $\widetilde{\mathcal{R}}/(\mathcal{I}\cap\widetilde{\mathcal{R}}) = \mathcal{R}$. To phrase this differently: if X, Y are regular $L(2)$-modules, then $\operatorname{Hom}_{L(2)}(X, Y)$ is just the direct sum $\operatorname{Hom}_{K(2)}(\widetilde{X}, \widetilde{Y}) \oplus \mathcal{I}(X, Y)$, and $\mathcal{I}(X, Y)$ corresponds to the set of linear maps $\operatorname{top} X \to \operatorname{soc} Y$.

It is important to know that the subcategory $\widetilde{\mathcal{R}}$ of $\operatorname{mod} K(2)$ is an abelian subcategory and the embedding functor is exact. In particular, $\widetilde{\mathcal{R}}$ is a hereditary length category. The simple objects of the abelian category $\widetilde{\mathcal{R}}$ are called the *simple regular* $K(e)$-modules. It is clear that an $L(2)$-module R is simple regular iff $R = \pi\widetilde{R}$ for some simple regular $K(2)$-module.

Any indecomposable regular $K(2)$-module M has a unique Jordan-Hölder sequence in $\widetilde{\mathcal{R}}$, and all the factors are isomorphic. We write $\widetilde{R}[t]$ for the indecomposable regular $K(2)$-module with a filtration with t factors of the form $\widetilde{R}$. In this way, we obtain a bijection between the isomorphism classes of the indecomposable objects of $\widetilde{\mathcal{R}}$ and the pairs $\widetilde{R}, t$, where $\widetilde{R}$ is simple regular and $t \in \mathbb{N}_1$. Using the pushdown functor π, we obtain a bijection between the isomorphism classes of the indecomposable objects of $\mathcal{R}$ and the pairs R, t, where R is simple regular and $t \in \mathbb{N}_1$ (the number t is called the *regular length* of $R[t]$).

Proposition A.17. *Let A be a self-injective short local algebra with $e = 2$. Let R be a simple regular A-module and $t \geq 1$.*
(a) *ΩR is simple regular.*
(b) *$\Omega(R[t]) = (\Omega R)[t]$.*
(c) *The type of the $\mho$-component of $R[t]$ is the same as the type of the $\mho$-component of R.*

Proof of (a). Since R is regular, its dimension vector is of the form (m, m), thus also $\mathbf{dim}\, \Omega R = (m, m)$ and therefore ΩR is regular. Let U be a proper regular submodule of ΩR. Let $V = R/U$, this is also a regular module. If $\mathbf{dim}\, U = (u, u)$ and $\mathbf{dim}\, V = (v, v)$, then we have $u+v = m$. Starting with injective envelopes of U and V, the horseshoe lemma asserts that there is an injective module I such that $\Sigma\Omega R \oplus I$ has a submodule of the form ΣU with factor module ΣV. Since $\mathbf{dim}\, U + \mathbf{dim}\, V = \mathbf{dim}\, R$, we see that $I = 0$, thus ΣU is a submodule of R with factor module ΣV. By assumption, $V \neq 0$, thus $\Sigma V \neq 0$. It follows that ΣU is a proper regular submodule of R. Since R is simple regular, we have $\Sigma U = 0$, thus $U = 0$. This shows that R is simple regular.

Proof of (b). We use induction on t. Let $t \geq 2$. There is an exact sequence $0 \to R[t-1] \to R[t] \to R \to 0$. The horseshoe lemma asserts that there is a projective module P such that $\Omega(R[t]) \oplus P$ is an extension of $\Omega(R[t-1])$ by ΩR, and by induction we have $\Omega(R[t-1]) = (\Omega R)[t-1]$. Now $\Omega(R[t])$ is indecomposable, thus $\Omega(R[t]) = (\Omega R)[s]$ for some $1 \leq s \leq t$. Assume that $s < t$. We apply Σ. On the one hand, we have $\Sigma\Omega(R[t]) = R[t]$, on the other hand, we have $\Sigma((\Omega R)[s]) = \Sigma\Omega(R[s]) = R[s]$, where we use again induction. But for $s < t$, the module $R[t]$ is not isomorphic to $R[s]$. It follows that $s = t$, thus $\Omega(R[t]) = (\Omega R)[t]$.

Proof of (c). The type of the $\mho$-component of an indecomposable module M is $\widetilde{\mathbb{A}}_n$ iff $n \geq 0$ is minimal with $\Omega^n M \simeq M$ (and $\mathbb{Z}$, if there is no n of this kind). According to (b) we have $\Omega^n(R[t]) \simeq R[t]$ iff $\Omega^n R \simeq R$. $\square$

Historical Remark for A.8 and A.9. The representations of $K(2)$ have been classified by Kronecker in 1890, completing earlier partial results by Jordan and Weierstrass, as mentioned for example in [ARS]. This classification plays an important role in many parts of mathematics. A standard reference for the matrix approach (in the language of matrix pencils) is Gantmacher's book on matrix theory [Gm]. There is the equivalent theory of coherent sheaves over the projective line, where the usual reference is the splitting theorem of Grothendieck (but one should be aware that this result can be traced back to Hilbert (1905), Plemelj (1908), and G. D. Birkhoff (1913), see [OSS]).

The category $\operatorname{mod} L(2)$ plays also a prominent role in modular representation theory, since the group algebra kG of the Klein four group G in characteristic 2 is a self-injective short local algebra with $e = 2$ (namely $kG = A_1$). The usual reference are papers by Bashev (1961) and Heller-Reiner (1961), see Benson [B].

A.10. Self-injective algebras with $e = 2$: Normal forms

Let us now assume that k is algebraically closed. In this case, it is easy to determine normal forms for the self-injective short local algebras A with $e(A) = 2$.

- There are the algebras $A_q = k\langle x, y\rangle/\langle x^2, y^2, xy + qyx\rangle$ with $q \in k^*$; note that A_q is isomorphic to $A_{q^{-1}}$.
- In addition, there is the algebra $A_0 = k\langle x, y\rangle/\langle x^2, y^2 + xy, y^2 - yx\rangle$.

Sketch of proof. Let A be a self-injective short local k-algebra with $e(A) = 2$. First one shows that there is always an element $x \in J \setminus J^2$ with $x^2 = 0$. Then one takes $z \in J$ such that x, z is a generating set for ${}_AJ$. Since A is self-injective, the elements xz and zx have to be non-zero, thus there is $q \in k^*$ with $xz + qzx = 0$. If we have $z^2 = 0$, then $A \simeq A_q$. Thus, let $z^2 \neq 0$. Then there is $\alpha \in k^*$ with $z^2 = \alpha zx$. If $q \neq 1$, then $y = z + (q-1)^{-1}\alpha x$ satisfies $y^2 = 0$ and we deal again with the previous case (with z replaced by y). Thus, the case $z^2 \neq 0$ and $q = 1$ remains: here, the elements x and $y = \alpha^{-1}z$ satisfy the defining relations for A_0. □

If $A = A_q$, with $q \in k^*$, then the modules Ax and Ay are Ω-periodic of period 1. The Ω-orbit of a module $A(x + \alpha y)$ with $\alpha \neq 0$ is the set $\{A(x + q^t\alpha y) \mid t \in \mathbb{Z}\}$; its cardinality is equal to the multiplicative order $o(q)$ of q. The algebra A_1 is the exterior algebra in 2 generators; in this case all regular indecomposable modules are Ω-periodic of period 1, thus the corresponding Möbius transformation is the identity. The remaining algebras A_q (with $q \in k^*$ and $q \neq 1$) have precisely two Ω-orbits of cardinality 1 which consist of indecomposable modules of length 2, namely the orbits of Ax and Ay. This means that the corresponding Möbius transformation has precisely two fixed points. All other orbits of indecomposable modules of length 2 have cardinality $o(q)$. (If $o(q) = \infty$, then A_q is just a quantum exterior algebra in 2 generators as discussed in A.11.)

For $A = A_0$, the module Ax is Ω-periodic of period 1. The Ω-orbit of the module $A(y + \alpha x)$ with $\alpha \in k$ is the set $\{A(y + (\alpha + t)x) \mid t \in \mathbb{Z}\}$; thus its cardinality is equal to the characteristic $\operatorname{char} k$ of k. The corresponding Möbius transformation has precisely one fixed points (these Möbius transformations are often called *parabolic*).

If A is commutative, then $A = A_{-1}$ or else the characteristic of k is 2 and $A = A_0$. In both cases, the corresponding Möbius transformation has order 2.

Of course, all the algebras A_q with $q \in k^*$ are special biserial. If the characteristic of k is 2, also A_0 is special biserial, namely isomorphic to $A\langle x, y\rangle/\langle xy, yx, x^2 - y^2\rangle$. If the characteristic of k is different from 2, then A_0 is (biserial, but) not special biserial. +

A.11. Example: The quantum exterior algebra in 2 generators

The quantum exterior algebra A in two variables x, y is the k-algebra generated by x, y with the relations $x^2,\ y^2,\ xy+qyx$, where $q \in k^* = k\setminus\{0\}$ has infinite (multiplicative) order. Note that the elements $1,\ x,\ y$, and yx form a basis for A.

We consider the left ideals $M_\alpha = A(x+\alpha y)$ with $\alpha \in k$; these are indecomposable modules of length 2.

Lemma A.18. *Let A be the quantum exterior algebra A in two variables. If $\alpha \in k$,, then*

$$\Omega M_\alpha = M_{q\alpha}.$$

Proof. If $a \in A$, let $\rho_a\colon {}_AA \to {}_AA$ be the right multiplication by a. Of course, the image of ρ_a is the left ideal Aa. The relations show that $(x+q\alpha y)(x+\alpha y) = 0$. This implies that the composition

$${}_AA \xrightarrow{\rho_{x+q\alpha y}} {}_AA \xrightarrow{\rho_{x+\alpha y}} {}_AA$$

is zero. The image of the left map is $M_{q\alpha}$, the image of the right map is M_α. It follows that $\Omega M_\alpha = M_{q\alpha}$. □

Let A be the quantum exterior algebra in two variables x, y. The $\mho$-component containing M_1 looks as follows:

$$\cdots \dashleftarrow M_{q^2} \dashleftarrow M_q \dashleftarrow M_1 \dashleftarrow M_{q^{-1}} \dashleftarrow M_{q^{-2}} \dashleftarrow \cdots$$

Since the modules M_{q^i} with $i \in \mathbb{Z}$ are pairwise non-isomorphic, we see that this $\mho$-component is of type $\mathbb{Z}$. Thus we see:

Proposition A.19. *If A is the quantum exterior algebra in 2 generators, then there exists a two-dimensional indecomposable module M* (namely $M = M_1$) *with $\mho$-component of type $\mathbb{Z}$. Thus*

$$\operatorname{Ext}^i(M,M) = 0 \quad \textit{for all } i \geq 2, \textit{ whereas} \quad \operatorname{Ext}^1(M,M) \neq 0.$$

□

Corollary A.20 (Smalø [Sm]). *If A is the quantum exterior algebra in 2 generators, there are indecomposable modules M and N_i with $i \in \mathbb{N}_1$ such that $\operatorname{Ext}^i(M,N_i) \neq 0$ and $\operatorname{Ext}^j(M,N_i) = 0$ for all $j > i$.*

Proof. Let M be a 2-dimensional indecomposable module with $\mho$-component of type $\mathbb{Z}$. Let $N_i = \Omega^{i-1}M$. Then

$$\operatorname{Ext}^i(M,N_i) = \operatorname{Ext}^i(M,\Omega^{i-1}M) = \operatorname{Ext}^1(M,\Sigma^{i-1}\Omega^{i-1}M) = \operatorname{Ext}^1(M,M) \neq 0.$$

Also, for $j > i$, we have

$$\operatorname{Ext}^j(M,N_i) = \operatorname{Ext}^{j-i+1}(M,\Sigma^{i-1}\Omega^{i-1}M) = \operatorname{Ext}^{j-i+1}(M,M) = 0,$$

since $j-i+1 \geq 2$. $\square$

Recall that Auslander had conjectured that for every module M there exists a bound $b(M)$ with the following property: if N is a module with $\operatorname{Ext}^j(M,N)=0$ for $j \gg 0$, then $\operatorname{Ext}^j(M,N)=0$ for $j > b(M)$. Corollary A.20 shows:

Corollary A.21 (Smalø [Sm]). *The quantum exterior algebra in two variables is a counter-example to the Auslander conjecture.* $\square$

The first counter-example for the Auslander conjecture was given by Jorgensen-Şega [JS1].

A.12. Koszul modules

The paper [RZ3] will draw the attention to Koszul modules as defined by Herzog and Iyengar [HI], see also [AIS]. If A is a short local algebra, then *an A-module M of Loewy length at most 2 is a Koszul module if and only if all the modules $\Omega^n M$ with $n \geq 0$ are aligned*, see [RZ3].

Since for a self-injective algebra A, any A-module is Gorenstein-projective, the minimal projective resolutions of all indecomposable non-projective modules are displayed by the $\mho$-quiver. It follows:

Proposition A.22 ([Sj, MV2, AIS]). *Let A be a self-injective short local algebra with $e \geq 2$. If M is indecomposable, then M is Koszul if and only if M is not preprojective in the sense of Auslander-Smalø (thus not of the form $\pi P_1, \pi P_2, \dots$).* $\square$

Let us add:

Proposition A.23. *Let A be a self-injective short local algebra. If $e \geq 2$, then the simple module S is a Koszul module, and for any module M, there exists $m \geq 0$ such that $\Omega^m M$ is Koszul. If $e=1$, and $a=1$, then the only Koszul modules are the projective modules.*

Proof. We can assume that M is an indecomposable module. First, let $e \geq 2$ and assume that M is not Koszul, then $M = \pi P_m$ for some $m \geq 1$ and therefore $\Omega^m(\pi P_m) = S$ is Koszul. If $e=1$, and $a=1$, then A is uniserial, thus M is isomorphic to k, ${}_AJ$ or ${}_AA$, and, of course, the modules k and ${}_AJ$ are not Koszul. $\square$

Historical Remark. The Koszul modules over a self-injective short local algebra have been determined by Sjödin [Sj], Martínez-Villa [MV2] and Avramov-Iyengar-Şega [AIS]. We hope that our outline of the general setting explains what is considered as a surprising behavior in [AIS].

Already in 1979, Sjödin [Sj] has looked for indecomposable non-projective modules M at the power series $P_M^A = \sum_{n \geq 0} \beta_n(M) T^n$ (called the *Poincaré series* of M). He showed that for a self-injective short local algebra A, the series P_M^A is rational (this follows from the fact that $\Omega^m M$ is Koszul for some $m \geq 0$).

Appendix B. A formula of Avramov-Iyengar-Şega

B.1. The sequences $b(e,a)_n$

Let e, a be real numbers. We define recursively the sequence $b_n = b(e,a)_n$ with $n \geq -1$ as follows: $b_{-1} = 0$, $b_0 = 1$ and

$$(*) \qquad b_{n+1} = eb_n - ab_{n-1},$$

for $n \geq 0$. In this paper, we are interested it is the case that e, a are natural numbers and $a \leq e^2$. Namely, if A is a short local algebra with Hilbert type (e,a), then e, a are natural numbers with $a \leq e^2$, and the recursion rule $(*)$ has popped up in Section 5.6, when dealing with a module M such that both M and ΩM are aligned.

As a consequence, we see the relevance of the numbers $b_n = b(e,a)_n$: *We have $\beta_n(S) = b_n$ for all $0 \leq n \leq N$ if and only if the modules $\Omega^n S$ with $0 \leq n < N$ are aligned.* As we have mentioned in A.12 (with reference to [RZ3]), the module S is a Koszul module in the sense of [HI] iff all the modules $\Omega^n M$ with $n \geq 0$ are aligned. Thus *S is a Koszul module iff $\beta_n(S) = b_n$ for all $n \geq 0$* (and then $\mathbf{dim}\,\Omega^n S = (b_n, b_{n-1})$).

The paper [AIS] aimed to provide a concise formula for the numbers $b(e,1)_n$ with $e \geq 3$, but the formula presented there was slightly distorted and usually did not even give integers. We are indebted to Avramov, Iyengar and Şega for communicating to us a proper revision and to allow us to include it here.

B.2. The formula of Avramov, Iyengar, Şega

Theorem B.1 (Avramov, Iyengar, Şega). *If $a < \frac{1}{4}e^2$, then for all $n \geq 0$*

$$b(e,a)_n = \frac{1}{2^n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2j+1} (e^2 - 4a)^j e^{n-2j}.$$

Proof (Avramov, Iyengar, Şega): Since we assume that $a < \frac{1}{4}e^2$, the roots of the polynomial $1 - eT + aT^2$ are real numbers, and do not coincide. The roots are

$$\lambda = \frac{e-q}{2}, \quad \text{and} \quad \rho = \frac{e+q}{2}, \quad \text{where} \quad q = \sqrt{e^2 - 4a} \ > \ 0.$$

Starting with the factorization

$$1 - eT + aT^2 = (1 - \rho T)(1 - \lambda T),$$

we may look at the power series expansion of the rational function $(1 - eT + aT^2)^{-1}$:

$$\frac{1}{1 - eT + aT^2} \ = \ \frac{1}{(\rho - \lambda)} \left(\frac{\rho}{1 - \rho T} - \frac{\lambda}{1 - \lambda T} \right) \ = \ \frac{1}{q} \sum_{n \geq 0} (\rho^{n+1} - \lambda^{n+1})\, T^n$$

Of course, we have

$$\frac{1}{1 - eT + aT^2} = \sum_{n \geq 0} b(e,a)_n T^n,$$

therefore

$$b(e,a)_n = \tfrac{1}{q}(\rho^{n+1} - \lambda^{n+1}).$$

The binomial expansions of ρ^{n+1} and λ^{n+1} yield

$$\begin{aligned}
\rho^{n+1} - \lambda^{n+1} &= \sum_{i=0}^{n+1} \binom{n+1}{i} \frac{1}{2^{n+1}} \left(e^{n+1-i}q^i - (-1)^i e^{n+1-i} q^i\right) \\
&= \frac{1}{2^n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2j+1} q^{2j+1} e^{n-2j}
\end{aligned}$$

Altogether, one gets that

$$\begin{aligned}
b(e,a)_n = \tfrac{1}{q}(\rho^{n+1} - \lambda^{n+1}) &= \frac{1}{2^n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2j+1} q^{2j} e^{n-2j}, \\
&= \frac{1}{2^n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2j+1} (e^2 - 4a)^j e^{n-2j}.
\end{aligned}$$

□

Note that the formula exhibited above is already of interest in the case $e = 3$ and $a = 1$. In this case the numbers $b_n = b(3,1)_n$ are just the even-index Fibonacci numbers (see Section A.1 in Appendix A).

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