# Koszul modules (and the $\Omega$-growth of modules) over short local algebras. 

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#### Abstract

Following the well-established terminology in commutative algebra, any (not necessarily commutative) finite-dimensional local algebra $A$ with radical $J$ will be said to be short provided $J^{3}=0$. As in the commutative case, also in general, the asymptotic behavior of the Betti numbers of modules seems to be of interest. As we will see, there are only few possibilities for the growth of the Betti numbers of modules. And there is a class of modules, the Koszul modules, where the Betti numbers can be calculated quite easily. We generalize results which are known for commutative algebras, but some of our results seem to be new also in the commutative case.


Key words. Short local algebra, Betti number, $\Omega$-growth, Koszul module, left Koszul algebra, left Conca ideal.

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## 1. Introduction.

The algebras which we consider are finite-dimensional local $k$-algebras $A$, where $k$ is a field, and $J$ will denote the radical of $A$. Usually, we will assume that $A / J=k$. The modules to be considered are left $A$-modules of finite length.

We denote by $|M|$ the length of the module $M$ and define $t(M)=t_{0}(M)=\mid$ top $M \mid$. For $n \in \mathbb{N}$, let $t_{n}(M)=t\left(\Omega^{n} M\right)$, where $\Omega M=\Omega_{A} M$ is the first syzygy module of $M$ (as in commutative algebra $[\mathrm{BH}, \mathrm{L}]$, one may call the numbers $t_{n}(M)$ the Betti numbers of $M$ ).
1.1. The $\Omega$-growth of a module. We draw the attention to the asymptotic behavior of the Betti numbers $t_{n}(M)$ of a module $M$. If $M$ is a module, we consider the following numerical invariant

$$
\gamma(M)=\limsup _{n} \sqrt[n]{t_{n}(M)}
$$

which we call the $\Omega$-growth of $M$. If $M$ has finite projective dimension, then $\gamma(M)=0$; otherwise $\gamma(M) \geq 1$. Note that

$$
\gamma(M)=\limsup _{n} \sqrt[n]{t_{n}(M)}=\limsup _{n} \sqrt[n]{\left|\Omega^{n} M\right|}
$$

This follows from the fact

$$
t_{n}(M) \leq\left|\Omega^{n} M\right| \leq\left.\right|_{A} A \mid \cdot t_{n}(M)
$$

for all $n \geq 0\left(\right.$ since, $t(N) \leq|N| \leq\left|{ }_{A} A\right| \cdot t(N)$ for any module $\left.N\right)$.

If $A$ is a local algebra with simple module $S$, we define

$$
\gamma_{A}=\gamma(S)
$$

Theorem 1. Let $A$ be a local algebra. Then $\gamma_{A} \leq\left|{ }_{A} J\right|$ and $\gamma(M)=\gamma(\Omega M) \leq \gamma_{A}$ for any module $M$. If $M$ is a module such that $S$ is a direct summand of $\Omega^{n} M$ for some $n \geq 0$, then $\gamma(M)=\gamma_{A}$.
1.2. A local algebra $A$ with radical $J=J(A)$ is said to be short provided $J^{3}=0$. Let $e=e(A)=\left|J / J^{2}\right|$ and, for $A$ being short, let $a=a(A)=\left|J^{2}\right|$. If $A$ is a short local algebra, we call $(e(A), a(A))$ the Hilbert-type of $A$. Let us assume now that $A$ is short and that $A / J=k$, so that $S=k$.

An $A$-module has Loewy length at most 2 iff it is annihilated by $J^{2}$. If $M$ is a module with Loewy length at most 2 , we call $\operatorname{dim} M=(t(M),|J M|)$ (or its transpose, if we need to invoke matrix multiplication) the dimension vector of $M$. Let us remark that $|M|=t(M)+|J M|$. Recall from [RZ] that a module $M$ is said to be bipartite provided $\operatorname{soc} M=J M$. A module has Loewy length at most 2 if and only if it is the direct sum of a bipartite and a semisimple module.

Let $A$ be a short local algebra of Hilbert type $(e, a)$. The following matrix plays an important role

$$
\omega_{a}^{e}=\left[\begin{array}{cc}
e & -1 \\
a & 0
\end{array}\right]
$$

since it controls the change of the dimension vectors of modules of Loewy length at most 2, when we apply $\Omega=\Omega_{A}$. Namely, the vectors $\operatorname{dim} \Omega M$ and $\omega_{a}^{e} \operatorname{dim} M$ differ only slightly (see 3.1, where we recall the Main Lemma of [RZ]). A module $M$ of Loewy length at most 2 will be said to be aligned provided $\operatorname{dim} \Omega M=\omega_{a}^{e} \operatorname{dim} M$. We study the aligned modules very carefully in section 3 .
1.3. Koszul modules. The main aim of the paper is to discuss the existence and the structure of Koszul modules. For a general ring, there are different proposals of what should be seen as a "Koszul module". For dealing with short local algebras and modules of Loewy length at most 2, all these approaches coincide (see 4.8). Let $A$ be a short local algebra. A module $M$ of Loewy length at most 2 will be said to be a Koszul module provided $\operatorname{dim} \Omega^{n} M=\left(\omega_{a}^{e}\right)^{n} \operatorname{dim} M$ for all $n \geq 0$, thus provided all modules $\Omega^{n} M$ are aligned, for $n \geq 0$. A short local algebra is called a left Koszul algebra provided the simple module $S$ is a Koszul module.

If $M$ has Loewy length at most 2 and $\Omega^{n} M$ is bipartite for all $n>0$, then $M$ is Koszul (see 4.2). Thus, if $M$ is not Koszul, then $S$ is a direct summand of $\Omega^{n} M$ for some $n>0$, therefore Theorem 1 asserts that $\gamma(M)=\gamma_{A}$. The following theorem deals with the short local algebras which have a non-zero Koszul module.

Theorem 2. Let $A$ be a short local algebra of Hilbert type (e,a). If there exists a non-zero Koszul module $M$ of Loewy length at most 2, then the algebra is left Koszul, we have $a \leq \frac{1}{4} e^{2}$ and $\gamma_{A}=\frac{1}{2}\left(e+\sqrt{e^{2}-4 a}\right)$. In addition, either $\gamma(M)=\gamma_{A}$ or else $a>0$ and $\gamma(M)=\frac{1}{2}\left(e-\sqrt{e^{2}-4 a}\right)$.

We recall that the spectral radius $\rho(\omega)$ of a linear transformation $\omega: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is the maximum of the absolute values of the (complex) eigenvalues of $\omega$. Note that for $a \leq \frac{1}{4} e^{2}$, we have $\rho\left(\omega_{a}^{e}\right)=\frac{1}{2}\left(e+\sqrt{e^{2}-4 a}\right)$, see 5.4. Thus, Theorem 2 asserts that the existence of a non-zero Koszul module $M$ implies that $\gamma_{A}=\rho\left(\omega_{a}^{e}\right)$.

Theorem 2 provides a generalization of what Lescot [L] calls his key lemma: the assertion that (for a commutative short local algebra $A$ with soc $A=J^{2}$ ) the existence of a non-zero Koszul module of Loewy length at most 2 implies that $S$ is a Koszul module (see [L], 3.6).

Theorem 3. Let $A$ be a short local algebra of Hilbert type $(e, a)$. Let $M$ be a non-zero module of Loewy length at most 2 with $\gamma(M)<\gamma_{A}$. Then $M$ is a Koszul module, and the numbers $\gamma(M)$ and $\gamma_{A}$ are positive integers with

$$
e=\gamma(M)+\gamma_{A}, \quad \text { and } \quad a=\gamma(M) \cdot \gamma_{A} .
$$

In particular, we have $0<\gamma(M)<\frac{1}{2} e<\gamma_{A}<e$ and $e^{2}-4 a=\left(\gamma_{A}-\gamma(M)\right)^{2}$ (thus $e^{2}-4 a$ is the square of a positive integer; in particular, positive). Also, $\operatorname{dim} M$ is a multiple of $\left(1, \gamma_{A}\right)$ and $\operatorname{dim} \Omega^{n} M=\gamma(M)^{n} \operatorname{dim} M$ for all $n \in \mathbb{N}$.

For example, let us look at the special case $e=7$. If there is a non-zero module $M$ of Loewy length at most 2 with $\gamma(M)<\gamma_{A}$, then $\gamma(M)=1$ or 2 or 3 , thus $a=6,10,12$, respectively (and $\operatorname{dim} M$ is a multiple of $(1,6),(1,5),(1,4)$, respectively). Let us exhibit for $e=7$ the graph of $\rho\left(\omega_{a}^{e}\right)=\frac{1}{2}\left(e^{2}+\sqrt{e-4 a}\right)$ as a function of $a$ (it contains the pairs $(6,6),(10,5),(12,4)$, they are marked by bullets $\bullet$ ), as well as (marked by small circles o) the three possible pairs $(a, \gamma(M))$, where $M$ is a Koszul module with $\gamma(M)<\gamma_{A}$, namely the pairs $(6,1),(10,2),(12,3)$ :

1.4. Left Conca ideals. Let $A$ be a local algebra and $U$ an ideal of $A$. We say that $U$ is a left Conca ideal provided $U^{2}=0$ and $J^{2} \subseteq J U$. If $A$ has a left Conca ideal $U$, then $A$ is short (namely, $J^{3} \subseteq J^{2} U \subseteq J U^{2} \subseteq U^{2}=0$ ). Since $J^{2} \subseteq U$, the modules annihilated by $U$ have Loewy length at most 2 .

Theorem 4. Let $A$ be a short local algebra. If $A$ has a left Conca ideal $U$, then any module annihilated by $U$ is a Koszul module; in particular, $S$ is a Koszul module, thus $A$ is a left Koszul algebra.

This generalizes part of Theorem 1.1 of [AIS].

### 1.5. Construction of left Koszul algebras.

Theorem 5. Given a pair e, a of natural numbers, then the following assertions are equivalent.
(i) There is a short local algebra of Hilbert type (e,a) which is left Koszul.
(ii) There is a commutative short local algebra of Hilbert type (e,a) which is left Koszul.
(iii) We have $a \leq \frac{1}{4} e^{2}$.

Theorem 6. Let $c, d$ be positive integers. Let $e=c+d, a=c d$. Then there are short local algebras of Hilbert type $(e, a)$ (even commutative ones) with a Koszul module with dimension vector $(1, c)$.

Of course, if $c, d$ are positive integers and $e=c+d$ and $a=c d$, then we have $a \leq \frac{1}{4} e^{2}$. The algebras which we construct in the proof of Theorem 5 (showing that (iii) implies (ii)) and of Theorem 6 are short local algebras with a left Conca ideal.

### 1.6. A lower bound for $\gamma_{A}$.

Theorem 7. Let $A$ be a short local algebra of Hilbert type $(e, a)$. If $a \leq \frac{1}{4} e^{2}$, then $\gamma_{A} \geq \frac{1}{2}\left(e+\sqrt{e^{2}-4 a}\right)$.

In view of Theorems 5 and 2, the assertion of Theorem 7 can be strengthened as follows. Let $\mathcal{A}(a, e)$ be the class of all short local algebras of Hilbert type $(e, a)$. Then: For $a \leq \frac{1}{4} e^{2}$, the subset $\left\{\gamma_{A} \mid A \in \mathcal{A}(e, a)\right\}$ of $\mathbb{R}$ has a minimal element, namely $\frac{1}{2}\left(e+\sqrt{e^{2}-4 a}\right)$. (On the one hand, Theorem 7 shows that $\frac{1}{2}\left(e+\sqrt{e^{2}-4 a}\right)$ is a lower bound; on the other hand, according to Theorem 5, there is a Koszul algebra $A$ in $\mathcal{A}(e, a)$ and Theorem 2 asserts that $\left.\gamma_{A}=\frac{1}{2}\left(e+\sqrt{e^{2}-4 a}\right).\right)$

We have seen in [RZ] that there is a trichotomy for short local algebras: There are the two special cases, first $a=1$, second $a=e-1$, and then there are the remaining algebras with $a \notin\{1, e-1\}$ (for example, Gorenstein projective modules or non-zero minimal acyclic complexes of projective modules do not exist if $a \notin\{1, e-1\})$. Theorem 5 yields a further separation: namely between $a \leq \frac{1}{4} e^{2}$ and $a>\frac{1}{4} e^{2}$ : The class $\mathcal{A}(e, a)$ contains a Koszul algebra iff $a \leq \frac{1}{4} e^{2}$. The disparity between $a \leq \frac{1}{4} e^{2}$ and $a>\frac{1}{4} e^{2}$ can be seen well if one looks at the spectral radius $\rho\left(\omega_{a}^{e}\right)$ as a function of $a$ (fixing $e$ ): we have $\rho\left(\omega_{a}^{e}\right)=\frac{1}{2}\left(e+\sqrt{e^{2}-4 a}\right)$ for $a \leq \frac{1}{4} e^{2}$, and $\rho\left(\omega_{a}^{e}\right)=\sqrt{a}$ for $a \geq \frac{1}{4} e^{2}$.


Note that if $a \leq \frac{1}{4} e^{2}$, thus Theorem 7 asserts that $\rho\left(\omega_{a}^{e}\right)$ is a lower bound for $\gamma_{A}$, and it seems that this is also true for $a>\frac{1}{4} e^{2}$.
1.7. Outline of the paper. Sections 3 and 4 provide characterizations of the aligned modules and the Koszul modules, respectively. The $\Omega$-growth of modules is discussed in sections 2 and 5 ; in section 2, there is the proof of Theorem 1, in section 5 the proof of Theorems 2 and 3. Section 6 deals with left Conca ideals and presents the proof of Theorem 4. In section 7 we construct suitable algebras with left Conca ideals in order to establish Theorems 5 and 6 . The final section 8 provides a lower bound for $\gamma_{A}$, provided $a \leq \frac{1}{4} e^{2}$.
1.8. Remark. As in the previous paper [RZ], we are dealing most of the time with a short local algebra $A$ and with modules of Loewy length at most 2 , thus with $A / J^{2}$ modules, where $J$ is the radical of $A$. But we take into account the syzygy functor $\Omega_{A}$ (note that $\Omega_{A}$ sends any module to a module of length at most 2 ). It is well-known that an $A / J^{2}$-module $M$ can be understood quite well by looking at the corresponding $K(e)$ module $\widetilde{M}=(M / J M, J M)$, where $K(e)$ is the Kronecker algebra with $e=e(A)$ arrows (see for example the appendices A. 1 and A. 2 of [RZ]). The dimension vector $\operatorname{dim} M=$ $(t(M),|J M|)$ as introduced in 1.2 is just the class of $\widetilde{M}$ in the Grothendieck group of $K(e)$ (note that it carries much more information than the class of $M$ in the Grothendieck group of $A$ ). A main tool which we use is the linear transformation $\omega_{a}^{e}$ on the Grothendieck group of $K(e)$ : the vectors $\operatorname{dim} \Omega M$ and $\omega_{a}^{e} \operatorname{dim} M$ often are equal, and always differ only slightly. Thus $\omega_{a}^{e}$ provides important information on the asymptotic behaviour of the Betti numbers of $M$. One may be tempted to write $\omega_{A}$ instead of $\omega_{a}^{e}$, but we refrain from doing so in order to stress that this transformation depends only on the parameters $e, a$, and not on the further structure of $A$.

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## 2. The $\Omega$-growth of a module.

First, let us consider an arbitrary finite-dimensional algebra $A$.
2.1. Lemma. If $M^{\prime}$ is a direct summand of $M$, then $\gamma\left(M^{\prime}\right) \leq \gamma(M)$.

Proof. If $M^{\prime}$ is a direct summand of $M$, then $\Omega^{n} M^{\prime}$ is a direct summand of $\Omega^{n} M$, thus $\left|\Omega^{n} M^{\prime}\right| \leq\left|\Omega^{n} M\right|$ for all $n \geq 0$.
2.2. Lemma. If $0 \rightarrow M^{\prime} \rightarrow M \rightarrow M^{\prime \prime} \rightarrow 0$ is an exact sequence, then

$$
\gamma(M) \leq \max \left\{\gamma\left(M^{\prime}\right), \gamma\left(M^{\prime \prime}\right)\right\} .
$$

Proof. We start with minimal projective resolutions of $M^{\prime}$ and $M^{\prime \prime}$. The horseshoe lemma provides a (not necessarily minimal) projective resolution of $M$. This shows that $t_{n}(M) \leq t_{n}\left(M^{\prime}\right)+t_{n}\left(M^{\prime \prime}\right)$ for all $n \geq 0$. Therefore

$$
\lim \sup \sqrt[n]{t_{n}(M)} \leq \max \left\{\lim \sup \sqrt[n]{t_{n}\left(M^{\prime}\right)}, \lim \sup \sqrt[n]{t_{n}\left(M^{\prime \prime}\right)}\right\}
$$

2.3. Lemma. Let $A$ be a finite-dimensional local algebra and $M$ a module. Then $\gamma(M)=\gamma(\Omega M) \leq\left|{ }_{A} J\right|$.

Proof. Let $J$ be the radical of $A$ and $d=\left.\right|_{A} J \mid$. Let $m$ be the Loewy length of ${ }_{A} J$ (thus $J^{m} \neq 0$, and $J^{m+1}=0$ ). Let $c=\left|{ }_{A} J^{m}\right|$.

For $n \geq 1$, the module $\Omega^{n} M$ is a submodule of $J P\left(\Omega^{n-1} M\right)$, thus it has Loewy length at most $m$. But if $N$ is a module of Loewy length at most $m$, then $J^{m} P(N) \subseteq \Omega N$. This shows that for $n \geq 1$, we have

$$
J^{m} P\left(\Omega^{n} M\right) \subseteq \Omega^{n+1} M \subseteq J P\left(\Omega^{n} M\right)
$$

Since $\left|J^{m} P\left(\Omega^{n} M\right)\right|=c t_{n}(M)$ and $\left|J P\left(\Omega^{n} M\right)\right|=d t_{n}(M)$, we get

$$
c t_{n}(M) \leq\left|\Omega^{n+1} M\right| \leq d t_{n}(M)
$$

thus

$$
\gamma(M)=\limsup \sqrt[n]{t_{n}(M)}=\limsup \sqrt[n]{\left|\Omega^{n+1} M\right|}=\gamma(\Omega M)
$$

On the other hand, we have by induction $t_{n+1}(M) \leq d^{n} t_{1}(M)$, therefore

$$
\gamma(\Omega M)=\lim \sup \sqrt[n]{t_{n+1}(M)} \leq \lim \sup \sqrt[n]{d^{n} t_{1}(M)}=d=\left|{ }_{A} J\right|
$$

2.4. Proof of Theorem 1. We have seen in 2.3 that $\gamma_{A}=\gamma(S) \leq\left|{ }_{A} J\right|$ and that $\gamma(M)=\gamma(\Omega M)$.

It follows from 2.2 that $\gamma(M) \leq \gamma(S)$, using induction on the length of $M$. Thus $\gamma(M) \leq \gamma_{A}$.

Now assume that $S$ is a direct summand of $\Omega^{n} M$ for some $n \geq 0$. Using 2.1 and 2.3, we get $\gamma(M) \leq \gamma(S) \leq \gamma\left(\Omega^{n} M\right)=\gamma(M)$.
2.5. Remark. We should stress that the $\Omega$-growth $\gamma(M)$ of a module $M$ measures the exponential growth of the Betti numbers. A similar, but deviating measure, the complexity, was introduced by Alperin and Evens [AE] in 1981 dealing with representations of a finite group $G$ : The complexity of a $k G$-module is the least integer $c$ such that there is a constant $\kappa>0$ with $t_{n}(M) \leq \kappa \cdot n^{c-1}$ for all $n \geq 1$. In contrast to the $\Omega$-growth, the complexity measures the polynomial growth of the Betti numbers. There is the following obvious observation: If a $k G$-module $M$ has finite complexity and is not projective, then $\gamma(M)=1$. This follows from the fact that $\lim _{n} \sqrt[n]{n}=1$.

Dealing with an arbitrary finite-dimensional algebra, it may be advisable to look at various measures for the growth of the Betti numbers. However, the present investigation seems to indicate that for short local algebras, it is the $\Omega$-growth as defined in the introduction which is the decisive invariant.

## 3. Aligned modules.

From now on, $A$ will be a short local $k$-algebra with radical $J$ such that $S=A / J=k$.
3.1. We recall from [RZ] the Main Lemma. Let A be a short local algebra of Hilbert type $(e, a)$. If $M$ is a module of Loewy length at most 2, then there is a unique natural number $w$ such that

$$
\operatorname{dim} \Omega M=\omega_{a}^{e} \operatorname{dim} M+(w,-w)
$$

The module $\Omega M$ has a direct summand of the form $S^{w^{\prime}}$ with $w^{\prime} \geq w$.
According to the Main Lemma, we have $\operatorname{dim} \Omega M=\omega_{a}^{e} \operatorname{dim} M$ provided $\Omega M$ is bipartite. But this formula is valid for a larger class of modules, namely the aligned modules. We are going to provide several equivalent conditions for a module to be aligned.
3.2. If $M$ is a module of Loewy length at most 2 , let $p: P(M) \rightarrow M$ be a projective cover. We consider $\Omega M$ as a submodule of $J P(M)$ with inclusion map $u: \Omega M \rightarrow J P(M)$ and obtain in this way the exact sequence

$$
\eta_{M}=(0 \rightarrow \Omega M \xrightarrow{u} J P(M) \xrightarrow{p} J M \rightarrow 0) .
$$

(In order to see that this sequence is exact, we apply the Snake Lemma to the following commutative diagram with exact rows:


The kernel of $p: P(M) \rightarrow M$ is $\Omega M$. Since $p^{\prime \prime}$ is an isomorphism, and $p$ is surjective, we see that the cokernel of $J P(M) \rightarrow J M$ is zero.)

Considering the top of the modules, the exact sequence $\eta_{M}$ yields the exact sequence

$$
\bar{\eta}_{M}=(\operatorname{top} \Omega M \xrightarrow{\bar{u}} \operatorname{top} J P(M) \xrightarrow{\bar{p}} J M \rightarrow 0),
$$

here we use that $J M$ is semisimple, since the Loewy length of $M$ is at most 2 .
3.3. Proposition. Let $M$ be a module of Loewy length at most 2. The following conditions are equivalent.
(i) $M$ is aligned (by definition, this means that $\operatorname{dim} \Omega M=\omega_{a}^{e} \operatorname{dim} M$ ).
(ii) $t(\Omega M)=e t(M)-|J M|$.
(iii) $|J \Omega M|=a t(M)$.
(iv) $J \Omega M=J^{2} P(M)$.
(v) $J \Omega M=J^{2} P(M) \cap \Omega M$.
(vi) $J^{2} P(M) \subseteq J \Omega M$.
(vii) The inclusion map $u$ yields an injective map $\bar{u}: \operatorname{top} \Omega M \rightarrow \operatorname{top} J P(M)$.
(viii) The sequence $\eta_{M}$ induces an exact sequence $0 \rightarrow \operatorname{top} \Omega M \rightarrow \operatorname{top} J P(M) \rightarrow J M \rightarrow 0$.
(ix) A minimal projective presentation $P_{1} \rightarrow P_{0} \rightarrow M \rightarrow 0$ induces an exact sequence $0 \rightarrow \operatorname{top} P_{1} \rightarrow \operatorname{top} J P_{0} \rightarrow J M \rightarrow 0$.

Proof. Let us start with the equivalence of (ii), (iii). The Main Lemma (see 3.1) asserts that

$$
(t(\Omega M),|J \Omega M|)=(e t(M)-|J M|+w, a t(M)-w)
$$

for some $w$. Thus, if $t(\Omega M)=e t(M)-|J M|$ (the condition (ii)), then $w=0$ and therefore $|J \Omega M|=a t(M)$ (the condition (iii)). And conversely, if the condition (iii) is satisfied, then again we have $w=0$, thus condition (ii) is satisfied.

Assertion (i) is the conjunction of (ii) and (iii), thus it is of course equivalent to (i) and to (iii). The inclusion map $J \Omega M \subseteq J^{2} P(M)$ shows that (iii) and (iv) are equivalent.

Since we assume that $M$ has Loewy length at most 2 , we have $J^{2} M=0$, thus $M=$ $P(M) / \Omega M$ implies that $J^{2} M \subseteq \Omega M$. This shows the equivalence of (iv) and (v).

Since $\Omega M \subseteq J P(M)$, we always have $J \Omega M \subseteq J^{2} P(M)$. Thus (iv) and (vi) are equivalent.

For the equivalence of (iv) and (vii), we apply the Snake Lemma to the following commutative diagram with exact rows


We have $\operatorname{Ker}(u)=0$ and $\operatorname{Cok}(u)=J M$. Also, the vertical map $\bar{u}$ on the right is part of the exact sequence $\bar{\eta}_{M}$, thus its cokernel is also $J M$. Altogether, the Snake Lemma yields the exact sequence $0 \rightarrow \operatorname{Ker}(\bar{u}) \rightarrow \operatorname{Cok} u^{\prime} \rightarrow J M \rightarrow J M \rightarrow 0$. The surjective map $J M \rightarrow J M$ has to be an isomorphism, thus the nap $\operatorname{Ker}(\bar{u}) \rightarrow \operatorname{Cok} u^{\prime}$ has to be an isomorphism. This mean that $u^{\prime}$ is surjective (the condition (iv)) if and only if $\bar{u}$ is injective (the condition (vii)).

The conditions (vii) and (viii) are of course equivalent, since $\eta_{M}$ induces the exact sequence $\bar{\eta}_{M}=(\operatorname{top} \Omega M \xrightarrow{\bar{u}} \operatorname{top} J P(M) \xrightarrow{\bar{p}} \operatorname{top} J M \rightarrow 0)$, and this is a short exact sequence if and only if $\bar{u}$ is injective.

The assertions (viii) and (ix) are equivalent, since $P_{1}=P(\Omega M)$ and in this way, top $P_{1}$ is identified with top $\Omega M$.
3.4. Remark. Let $V$ be a proper left ideal of $A$. Then ${ }_{A} A / V$ is aligned if and only if $J^{2} \subseteq J V$. Namely, $\Omega\left({ }_{A} A / V\right)=V$, thus we deal with condition (iv).

In particular: The simple module $S$ is always aligned, since here we have $V=J$. (Of course, we also may look at $\operatorname{dim} S=(1,0)$; we have $\Omega S={ }_{A} J$ and $\operatorname{dim}_{A} J=(e, a)=$ $\omega_{a}^{e}(1,0)$, this is condition (i).)
3.5. We recall from [RZ], 13.2: Let $A$ be a short local algebra and $M$ a module of Loewy length at most 2. If $\Omega M$ is bipartite, then $M$ is aligned. Conversely, if $J^{2}=\operatorname{soc}_{A} A$, and $M$ is aligned, then $M$ is bipartite.

The condition $J^{2}=\operatorname{soc}_{A} A$ has been discussed quite carefully in section 13 of [RZ].

### 3.6. Examples of aligned modules $M$ such that $\Omega M$ is not bipartite.

(1) Note that 3.4 provides such an example, namely $S$ is always aligned, whereas $\Omega S={ }_{A} J$ is bipartite iff $J^{2}=\operatorname{soc}_{A} A$. Note that for all short local algebras with $a=0$ and $e \geq 1$, but also for many other short local algebras, we have $J^{2} \neq \operatorname{soc}_{A} A$.

Here are two additional examples of indecomposable modules $M$ of Loewy length 2 which are aligned, but $\Omega M$ is not bipartite.
(2) Here is a typical example of a short local algebra with $J^{2} \neq \operatorname{soc}_{A} A$ : the algebra $A$ generated by $x, y, z$, with relations $x^{2}, y^{2}, z^{2}, x y-y x, x z, z x, y z, z y$.

Let $M=A y \simeq A /(A y+A z)$, thus $\Omega M=A y \oplus A z$. Then $\operatorname{dim} M=(1,1)$ and $\operatorname{dim} \Omega M=(2,1)$.

(3) Consider now the algebra $A A$ generated by $x, y, z$ with the relations $y x-x y, z y-$ $x^{2}, z x, y^{2}, x z, y z, z^{2}$ (thus $J^{2}$ has the basis $x^{2}, y x$ ). We define $M$ by taking a suitable submodule $U$ of a projective module $P$ and define $M=P / U$ so that $\Omega M=U$. Namely, let $P=A^{3}$ and let $\Omega M$ be the submodule of $P$ generated by $(x, y, 0),(0, x, y),(0, z, 0)$.


It turns out that $M$ is indecomposable and of Loewy length 2 . Since $A$ has Hilbert type $(3,2)$ and $\operatorname{dim} \Omega M=\operatorname{dim} M=(3,6)$, we see that $M$ is aligned (the condition (i) is satisfied). But $U=\Omega M$ is not bipartite: the submodule $J \Omega M$ is generated by

$$
\begin{aligned}
& x(x, y, 0)=\left(x^{2}, x y, 0\right), y(x, y, 0)=(x y, 0,0), z(x, y, 0)=\left(0, x^{2}, 0\right), \\
& x(0, x, y)=\left(0, x^{2}, x y\right), y(0, x, y)=(0, x y, 0), z(0, x, y)=\left(0,0, x^{2}\right),
\end{aligned}
$$

thus equal to $J^{2} P$, and therefore $\Omega M$ is isomorphic to the direct sum of two copies of $A / J^{2}$ and one copy of the simple module $S$. Altogether, we see that $M$ is aligned, but $\Omega M$ is not bipartite.

Of course, in all the examples of 3.6, we have $J^{2} \neq \operatorname{soc}_{A} A$, see 3.5.

## 4. Koszul modules.

4.1. Koszul modules. Following Herzog-Iyengar [HI] (see also [AIS]) a module $M$ will be said to be a Koszul module provided a minimal projective resolution

$$
\cdots \rightarrow P_{n+1} \quad \rightarrow \quad P_{n} \quad \rightarrow \quad P_{n-1} \quad \rightarrow \cdots \rightarrow \quad P_{0} \quad \rightarrow \quad M \quad \rightarrow \quad 0
$$

induces for any $n \geq 0$ an exact sequence

$$
\epsilon_{n}^{M}: \quad 0 \rightarrow P_{n} / J P_{n} \rightarrow J P_{n-1} / J^{2} P_{n-1} \rightarrow \cdots \rightarrow J^{n} P_{0} / J^{n+1} P_{0} \rightarrow J^{n} M / J^{n+1} M \rightarrow 0
$$

(note that the image of $d_{i+1}: P_{i+1} \rightarrow P_{i}$ is contained in $J P_{i}$, thus $d_{i+1}\left(J^{n} P_{i+1}\right) \subseteq J^{n+1} P_{i}$ for all $n \geq 0$ ).

A local algebra is called a left Koszul algebra provided the simple module $S$ is a Koszul module.

Of course, the projective modules are always Koszul modules. If $M$ is a Koszul module, then also $\Omega M$ is a Koszul module and has Loewy length at most 2 (since we assume, as always, that $A$ is a short local algebra). In the following, we usually will restrict the attention to Koszul modules of Loewy length at most 2.

Proposition. Let A be a short local algebra and $M$ a module of Loewy length at most 2. The following conditions are equivalent:
(i) $M$ is a Koszul module.
(ii) For every $n \geq 1$, the exact sequence $0 \rightarrow \Omega^{n} M \rightarrow P\left(\Omega^{n-1} M\right) \rightarrow \Omega^{n-1} M \rightarrow 0$ induces an exact sequence $0 \rightarrow \Omega^{n} M / J \Omega^{n} M \rightarrow J P\left(\Omega^{n-1} M\right) / J^{2} P\left(\Omega^{n-1} M\right) \rightarrow J \Omega^{n-1} M \rightarrow 0$.
(iii) $\operatorname{dim} \Omega^{n} M=\left(\omega_{a}^{e}\right)^{n} \operatorname{dim} M$ for all $n \geq 0$.
(iv) The modules $\Omega^{n} M$ with $n \geq 0$ are aligned.

Proof of the equivalence of (i) and (ii). We use the isomorphisms top $P\left(\Omega^{i} M\right) \rightarrow$ top $\Omega^{i} M$. Also note that $J^{2} \Omega^{i} M=0$ for all $i \geq 0$.

We can rewrite $\epsilon_{n}^{M}$ as:

$$
\epsilon_{n}^{M}: \quad 0 \rightarrow \operatorname{top} P\left(\Omega^{n} M\right) \rightarrow \operatorname{top} J P\left(\Omega^{n-1} M\right) \rightarrow \cdots \rightarrow \operatorname{top} J^{n} P(M) \rightarrow \operatorname{top} J^{n} M \rightarrow 0
$$

For $n=0$, this sequence $\epsilon_{0}^{M}: 0 \rightarrow \operatorname{top} P(M) \rightarrow \operatorname{top} M \rightarrow 0$ is always exact.
For $n \geq 1$, we can use the isomorphism top $P\left(\Omega^{n} M\right) \rightarrow \operatorname{top} \Omega^{n} M$ in order to rewrite $\epsilon_{n}^{M}$ as

$$
0 \rightarrow \operatorname{top} \Omega^{n} M \rightarrow \operatorname{top} J P\left(\Omega^{n-1} M\right) \rightarrow J \Omega^{n-1} M \rightarrow 0 .
$$

Note that this is just the exact sequence $\bar{\eta}_{\Omega^{n} M}$ as considered in 3.2.
The equivalence of (ii) and (iii) for $n \geq 1$ is given by Proposition 3.3, namely we use the equivalence of (i) and (vii) for $M$ replaced by $\Omega^{n-1} M$. By the definition of an aligned module, the conditions (iii) and (iv) are the same.
4.2. Proposition. Assume that $M$ has Loewy length at most 2 and all the modules $\Omega^{n} M$ with $n \geq 1$ are bipartite. Then $M$ is a Koszul module.

If $\operatorname{soc}_{A} A=J^{2}$, and $M$ is a Koszul module, then all the modules $\Omega^{n} M$ with $n \geq 1$ are bipartite.

Proof. If $m \geq 0$ and $\Omega^{m+1} M$ is bipartite, then $\Omega^{m} M$ is aligned, see 3.4 (2). Thus, the assumption implies that all the modules $\Omega^{m} M$ with $m \geq 0$ are aligned. It follows from Proposition 4.1 that $M$ is Koszul.

Assume now that $\operatorname{soc}_{A} A=J^{2}$. Then [RZ] 13.2 asserts that any aligned module is bipartite. Thus, if $M$ is a Koszul module, then all the modules $\Omega^{n} M$ with $n \geq 1$ are aligned, thus bipartite.

Remarks. (1) If $A$ is a short local algebra with $a=0$, then $\Omega S=S^{e}$ shows that $S$ is a Koszul module, thus $A$ is a Koszul algebra.
(2) If $A$ is a Koszul algebra, then usually not all modules are Koszul. A typical example is the $k$-algebra $A$ with generators $x, y$ and relations $x^{2}, x y, y^{2}$. Let $I=A x \simeq A / A x$. As
we see, $I$ is $\Omega$-periodic with period 1. It follows that $I$ is a Koszul module. But it is easy to see that the remaining indecomposable modules of length 2 are not Koszul.

Also, all self-injective short local algebras $A$ with $e \geq 2$ are Koszul algebras, but Proposition A. 8 in the Appendix of [RZ] asserts that there are countably many indecomposable modules which are not Koszul.
(3) Here is an example of a Koszul module $M$ such that none of the modules $\Omega^{n} M$ with $n \geq 1$ is bipartite. Let $A$ be generated by $x, y$ with relations $x^{2}, x y, y^{2}$ (this algebras has been considered already in [RZ], 9.3). Let $I=A x$. Then $\Omega S={ }_{A} J=I \oplus S$, and $\Omega I=I$. Thus, by induction, we see that $\Omega^{n} S=I^{n} \oplus S$ for all $n \geq 0$. It follows that $M=S$ is a Koszul module. On the other hand, $S$ is a proper direct summand of $\Omega^{n} M=\Omega^{n} S$, for any $n \geq 1$.
4.3. We say that a short exact sequence $0 \rightarrow M^{\prime} \rightarrow M \rightarrow M^{\prime \prime} \rightarrow 0$ is $t$-exact, provided $t(M)=t\left(M^{\prime}\right)+t\left(M^{\prime \prime}\right)$. A submodule $M^{\prime}$ of $M$ will be called a $t$-submodule provided the canonical exact sequence $0 \rightarrow M^{\prime} \rightarrow M \rightarrow M / M^{\prime} \rightarrow 0$ is $t$-exact, thus provided $t(M)=$ $t\left(M^{\prime}\right)+t\left(M / M^{\prime}\right)$. Of course, if $M^{\prime}$ is a submodule of $M$, then $t(M)=t\left(M^{\prime}\right)+t\left(M / M^{\prime}\right)$ if and only if $P(M)$ is isomorphic to $P\left(M^{\prime}\right) \oplus P\left(M / M^{\prime}\right)$.

Similarly, a filtration $0=M_{0} \subseteq M_{1} \subseteq \cdots \subseteq M_{m}=M$ will be called a $t$-filtration provided $t(M)=\sum_{j=1}^{m} t\left(M_{j} / M_{j-1}\right)$, or, equivalently, provided $P(M)$ is isomorphic to $\bigoplus_{j} P\left(M_{j} / M_{j-1}\right)$. Note that if $M$ has Loewy length at most 2 and $0=M_{0} \subseteq M_{1} \subseteq \cdots \subseteq$ $M_{m}=M$ is a $t$-filtration then $\operatorname{dim} M=\sum_{j} \operatorname{dim} M_{j} / M_{j-1}$.

Lemma. Let $A$ be a short local algebra. Let $0 \rightarrow M^{\prime} \rightarrow M \rightarrow M^{\prime} \rightarrow 0$ be $t$ exact. If $M^{\prime}, M^{\prime \prime}$ are aligned, then also $M$ is aligned and there is a t-exact sequence $0 \rightarrow \Omega M^{\prime} \rightarrow \Omega M \rightarrow \Omega M^{\prime \prime} \rightarrow 0$.

Proof. We can assume that the map $M^{\prime} \rightarrow M$ is the inclusion map of a submodule. Since $P(M) \simeq P\left(M^{\prime}\right) \oplus P\left(M^{\prime \prime}\right)$, the horseshoe lemma yields an exact sequence $0 \rightarrow \Omega M^{\prime} \rightarrow$ $\Omega M \rightarrow \Omega M^{\prime \prime} \rightarrow 0$. Multiplying with $J^{2}$, there is the exact sequence $0 \rightarrow J^{2} \Omega M^{\prime} \rightarrow$ $J^{2} \Omega M \rightarrow J^{2} \Omega M^{\prime \prime} \rightarrow 0$. Now we have the inclusion maps $u^{\prime}: J \Omega M^{\prime} \subseteq J^{2} \Omega M^{\prime}, u: J \Omega M \subseteq$ $J^{2} \Omega M$, and $u^{\prime \prime}: J \Omega M^{\prime \prime} \subseteq J^{2} \Omega M^{\prime \prime}$. If $M^{\prime}, M^{\prime \prime}$ are aligned, the maps $u^{\prime}, u^{\prime \prime}$ are bijective, thus also $u$ has to be bijective. This shows that $M$ is aligned and therefore $\operatorname{dim} \Omega M=$ $\omega(\operatorname{dim} M)$. But this implies that

$$
\begin{aligned}
\operatorname{dim} \Omega M & =\omega(\operatorname{dim} M)=\omega\left(\operatorname{dim} M^{\prime}+\omega\left(\operatorname{dim} M^{\prime \prime}\right)\right. \\
& =\omega\left(\operatorname{dim} M^{\prime}\right)+\omega\left(\operatorname{dim} M^{\prime \prime}\right)=\operatorname{dim} \Omega M^{\prime}+\operatorname{dim} \Omega M^{\prime \prime}
\end{aligned}
$$

In particular, we have $t(\Omega M)=t\left(\Omega M^{\prime}\right)+t\left(\Omega M^{\prime \prime}\right)$. This shows that $\Omega M^{\prime}$ can be identified with a $t$-submodule of $\Omega M$ with factor module $\Omega M^{\prime \prime}$.
4.4. Corollary. Let $A$ be a short local algebra. Let $M$ be a module of Loewy length at most 2 and $0=M_{0} \subseteq M_{1} \subseteq \cdots \subseteq M_{m}=M$ a t-filtration. Let $n \geq 0$.
(a) If all the modules $\Omega^{i}\left(M_{j} / M_{j-1}\right)$ with $0 \leq i \leq n$ and $1 \leq j \leq m$ are aligned, then $\Omega^{n} M$ is aligned and $\Omega^{n+1} M$ has a $t$-filtration with factors $\Omega^{n+1}\left(M_{j} / M_{j-1}\right)$, where $1 \leq j \leq m$.
(b) If all the modules $M_{j} / M_{j-1}$ with $1 \leq j \leq m$ are Koszul modules, then also $M$ is a Koszul module.

Proof. Let $\omega=\omega_{a}^{e}$. It is sufficient to show the assertions (a) and (b) for $m=2$, the general case follows easily by induction on $m$. Thus, let $M^{\prime}$ be a submodule of $M$ and let $M^{\prime \prime}=M / M^{\prime}$. We assume that $M^{\prime}$ is a $t$-submodule of $M$, thus $t(M)=t\left(M^{\prime}\right)+t\left(M^{\prime \prime}\right)$ and $P(M) \simeq P\left(M^{\prime}\right) \oplus P\left(M^{\prime \prime}\right)$.
(a) We show: If $\Omega^{i} M^{\prime}, \Omega^{i} M^{\prime \prime}$ are aligned for $0 \leq i \leq n$, then $\Omega^{n} M$ is aligned and $\Omega^{n+1} M^{\prime}$ can be identified with a t-submodule of $\Omega^{n+1} M$ with factor module $\Omega^{n+1} M^{\prime \prime}$. Proof by induction on $n$. The case $n=0$ has been shown in 4.3. Thus assume the assertion is true for some $n \geq 0$, and assume now that the modules $\Omega^{i} M^{\prime}, \Omega^{i} M^{\prime \prime}$ are aligned for $0 \leq i \leq n+1$. By induction, we know that we may consider $\Omega^{n+1} M^{\prime}$ as a $t$-submodule of $\Omega^{n+1} M$ with factor module $\Omega^{n+1} M^{\prime \prime}$. Since $\Omega^{n+1} M^{\prime}$ and $\Omega^{n+1} M^{\prime \prime}$ are aligned, we apply (1) in order to conclude that $\Omega^{n+1} M$ is aligned and that $\Omega^{n+2} M^{\prime}$ can be identified with a $t$-submodule of $\Omega^{n+2} M$ with factor module $\Omega^{n+2} M^{\prime \prime}$. This completes the proof of (a).
(b) If $M^{\prime}, M^{\prime \prime}$ are Koszul modules, then $M$ is Koszul. Proof. We use the equivalence of (i) and (iv) in 4.1: If $M^{\prime}, M^{\prime \prime}$ are Koszul, then all the modules $\Omega^{n} M^{\prime}, \Omega^{n} M^{\prime \prime}$ are aligned. According to (a), all the modules $\Omega^{n} M$ are aligned. Thus $M$ is Koszul.
4.5. Corollary. Let $A$ be a short local algebra and $U$ an ideal of $A$. Let $n \geq 0$.
(a) Assume that for any local module $N$ annihilated by $U$, the modules $\Omega^{i} N$ with $0 \leq i \leq n$ are aligned. Then for any module $M$ annihilated by $U$, the module $\Omega^{n} M$ is aligned.
(b) Assume that any local module $N$ annihilated by $U$ is a Koszul module, then any module $M$ annihilated by $U$ is a Koszul module.

Proof. Any module $M$ annihilated by $U$ has a $t$-filtration whose factors are local modules (of course annihilated by $U$ ). Namely, any composition series of top $M$ lifts to a $t$-filtration of $M$. Thus, we can apply 4.4.
4.6. Proposition. Let $A$ be a short local algebra. If there exists a non-projective Koszul module, then $A$ is a left Koszul algebra.

Proof. If $N$ is a non-projective Koszul module, then $\Omega N$ is a non-zero module of Loewy length at most 2 and is a Koszul module. Thus we can assume that there is given a module $M \neq 0$ of Loewy length at most 2 which is a Koszul module. According to 3.2, we have the exact sequence

$$
\eta_{M}=(0 \rightarrow \Omega M \xrightarrow{u} J P(M) \xrightarrow{p} J M \rightarrow 0) .
$$

Since $M$ is a Koszul module, $M$ is aligned, thus 3.3 (vii) asserts that $\eta_{M}$ is $t$-exact.
Consider the sequences

$$
\eta_{M}(n)=\left(0 \rightarrow \Omega^{n+1} M \xrightarrow{\Omega^{n} u} \Omega^{n}(J P(M)) \xrightarrow{\Omega^{n} p} \Omega^{n}(J M) \rightarrow 0\right) .
$$

with $n \geq 0$. By induction on $n$ we show that $\eta_{M}(n)$ is $t$-exact, and that all modules $\Omega^{n} S$ is aligned.

Proof of the induction. First, let $n=0$. We know that $\eta_{M}(0)=\eta_{M}$ is $t$-exact. Also, the module $\Omega^{0} S=S$ is always aligned. Now assume that for some $n \geq 0$ the sequences $\eta_{M}(n)$ is $t$-exact and the modules $\Omega^{n} S$ is aligned. Since $M$ is a Koszul module, the module $\Omega^{n} \Omega M$ is aligned. Since $J M$ is semisimple, the modules $\Omega^{n}(J M)$ is aligned. Thus, we can apply Lemma 4.3 in order to conclude that $\Omega^{n}(J P(M))$ is aligned and that $\eta_{M}(n+1)$ is $t$-exact. Let $m=\mid$ top $M \mid$. Then $J P(M)=\Omega\left(S^{m}\right)$. Since $\Omega^{n} J P(M)=\Omega^{n+1} S^{m}$ is aligned and $m \geq 1$, we see that $\Omega^{n+1} S$ is aligned. This completes the induction step.

Altogether, we see that $\Omega^{n} S$ is aligned for all $n \geq 0$, thus $S$ is a Koszul module.
4.7. Finally, let us draw the attention again to the simple module $S$. By definition, $A$ is a left Koszul algebra iff $S$ is Koszul. What does it mean that $S$ is a Koszul module?

If $e, a$ are real number, one may define recursively the sequence $b_{n}=b(e, a)_{n}$ with $n \geq-1$ as follows: $b_{-1}=0, b_{0}=1$ and

$$
\begin{equation*}
b_{n+1}=e b_{n}-a b_{n-1} \tag{*}
\end{equation*}
$$

for $n \geq 0$. By induction, one sees that $\left(b_{n}, a b_{n-1}\right)=\left(\omega_{a}^{e}\right)^{n}(1,0)$.
Proposition. Let $A$ be a short local algebra of Hilbert type $(e, a)$. The module $S$ is Koszul iff $\operatorname{dim} \Omega^{n} S=\left(b(e, a)_{n}, a \cdot b(e, a)_{n-1}\right)$ for all $n \geq 0$.

Proof. Write $b_{n}=b(e, a)_{n}$ for all $n \geq-1$. According to 4.1, $S$ is a Koszul module iff $\operatorname{dim} \Omega^{n} S=\left(\omega_{a}^{e}\right)^{n} \operatorname{dim} S$ for all $n \geq 0$. Of course, $\operatorname{dim} S=(1,0)=\left(b_{0}, b_{-1}\right)$, and therefore $\left(\omega_{a}^{e}\right)^{n} \operatorname{dim} S=\left(\omega_{a}^{e}\right)^{n}(1,0)=\left(b_{n}, a b_{n-1}\right)$.

Remark. Avramov-Iyengar-Şega have shown: if $a<\frac{1}{4} e^{2}$, then for all $n \geq 0$

$$
b(e, a)_{n}=\frac{1}{2^{n}} \sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n+1}{2 j+1}\left(e^{2}-4 a\right)^{j} e^{n-2 j},
$$

see Appendix B of [RZ].
4.8. Historical remark. The notion of a "Koszul" module is motivated by the classical Koszul duality between the polynomial algebras and the exterior algebras. Various kinds of such "Koszul modules" or "modules with linear resolution" have been introduced: one wants to generalize some important features of the simple modules over certain wellbehaved algebras which Priddy [P] called Koszul algebras. It is now customary to call a local ring "Koszul" in case its simple module is a "Koszul module". One of the aims was to obtain duality theorems for module categories. A more modest aim (and this is also our target, here and in the previous paper [RZ]) is to understand projective resolutions with nice (namely "linear") behaviour. In the paper [GM] by Green and Martínez (see also [MZ]) a finitely generated module $M$ over a semi-perfect, noetherian $k$-algebra $A$ with radical $J$, was said to have a linear resolution provided $J \Omega^{n+1} M=J^{2} P\left(\Omega^{n} M\right) \cap \Omega^{n+1} M$, for all $n \geq 0$ (and Theorem 4.4 of [GM] asserts that the modules with a linear resolution are just the "quasi-Koszul" modules defined by some Ext-condition). We use in this paper the definition of a Koszul module $M$ as formulated by Herzog and Iyengar [HI], see 4.1
above; it has the advantage that not only the module $M$, but also its associated graded module $\bigoplus_{t \geq 0} J^{t} M / J^{t+1} M$ as a module over the graded algebra $\bigoplus_{t \geq 0} J^{t} / J^{t+1}$ has a linear resolution, see Proposition 1.5 of $[\mathrm{HI}]$. But we should note that in our case, where $A$ is a short local algebra, a module $M$ with Loewy length at most 2 with a linear resolution is already a Koszul module in the sense of [HI]: see the equivalence of (i) and (v) in Proposition 3.3 and the equivalence of (i) and (iv) in Proposition 4.1.

## 5. Again: The $\Omega$-growth of a module.

Let $A$ be a short local algebra and $M$ a module of Loewy length at most 2. What are the possible values for $\gamma(M)$ ? First, we assume that $M$ is not a Koszul module. The following observation was mentioned already in the introduction, see 1.3.
5.1. Proposition. Let $A$ be a short local algebra. Let $M$ be a module of Loewy length at most 2. If $M$ is not Koszul, then $\gamma(M)=\gamma_{A}$.

Proof. Proposition 4.2 asserts that there is $n \geq 1$ such that $\Omega^{n} M$ is not bipartite. Since $\Omega^{n} M$ has Loewy length at most 2 , we see that $S$ is a direct summand of $\Omega^{n} M$. According to Theorem 1, $\gamma(M)=\gamma_{A}$.

It remains to consider the Koszul modules. We will need two elementary considerations from real linear algebra. Given vectors $\mathbf{x}=\left(x_{1}, x_{2}\right)$ and $\mathbf{y}=\left(y_{1}, y_{2}\right) \in \mathbb{R}^{2}$, we write $\mathbf{x} \leq \mathbf{y}$ provided $x_{1} \leq y_{1}$ and $x_{2} \leq y_{2}$ and we write $\mathbf{x}<\mathbf{y}$ provided $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. Let $|\mathbf{x}|=\left|x_{1}\right|+\left|x_{2}\right|$.

If $\omega: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ is a linear transformation, let $\gamma_{\omega}(\mathbf{x})=\lim \sup _{n} \sqrt[n]{\left|\omega^{n}(\mathbf{x})\right|}$.
5.2. Lemma. Let $\omega: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ be a linear transformation. If $\mathbf{x}$ is an eigenvector of $\omega$ with eigenvalue $\lambda$, then $\gamma_{\omega}(\mathbf{x})=|\lambda|$. If $\mathbf{x}$ is non-zero and not an eigenvector of $\omega$, then $\gamma_{\omega}(\mathbf{x})=\rho(\omega)$.

Proof. Of course, if $\omega(\mathbf{x})=\lambda \mathbf{x}$, then $\gamma_{\omega}(\mathbf{x})=|\lambda|$. Thus, let us assume that $\mathbf{x}=\left(x_{1}, x_{2}\right)$ is non-zero and not an eigenvector of $\omega$.

First, let $\omega$ be semisimple with eigenvalues $\lambda, \lambda^{\prime}$, where $\left|\lambda^{\prime}\right| \leq|\lambda|$. Thus we can assume that $\omega=\left[\begin{array}{cc}\lambda & 0 \\ 0 & \lambda^{\prime}\end{array}\right]$. Then $\omega^{n}\left(x_{1}, x_{2}\right)=\lambda^{n}\left(x_{1}+\left(\lambda^{\prime} / \lambda\right)^{n} x_{2}\right)$. Since $\left.0 \leq \mid\left(\lambda^{\prime} / \lambda\right)^{n} x_{2}\right)\left|\leq\left|x_{2}\right|\right.$ and $x_{1} \neq 0$, we see that $\gamma_{\omega}\left(x_{1}, x_{2}\right)=|\lambda| \cdot \limsup \sup _{n} \sqrt[n]{\left|x_{1}\right|+\left|\left(\lambda^{\prime} / \lambda\right)^{n} x_{2}\right|}=|\lambda|=\rho(\omega)$.

Second, let $\omega$ be not semisimple. Let $\lambda$ be its eigenvalue. If $\lambda=0$, then $\gamma_{\omega}(\mathbf{x})=0=$ $\rho(\omega)$. Otherwise, we can assume that $\omega=\lambda\left[\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right]$, thus $\omega^{n}\left(x_{1}, x_{2}\right)=\lambda^{n}\left(x_{1}+n x_{2}, x_{2}\right)$ and $\gamma_{\omega}\left(x_{1}, x_{2}\right)=|\lambda| \cdot \lim \sup _{n} \sqrt[n]{\left|x_{1}+n x_{2}\right|+\left|x_{2}\right|}=|\lambda|=\rho(\omega)$.
5.3. Lemma. Let $\omega: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ be a linear transformation. Let $\mathbf{x}$ be an element of $\mathbb{R}^{2}$ such that $\omega^{n} \mathbf{x}>0$ for all $n \geq 0$. Then $\omega$ has real eigenvalues. If $\mathbf{x}$ is an eigenvector with eigenvalue $\lambda$, then $\lambda$ is positive. If $\mathbf{x}$ is not an eigenvector, then $\rho(\omega)$ is an eigenvalue of $\omega$ (and, of course, positive).

Note that this lemma is a version of the Perron-Frobenius theorem in dimension 2, but in contrast to the classical Perron-Frobenius theorem, we cannot assert hat $\gamma_{\omega}(\mathbf{x})$ is
a simple eigenvalue of $\omega$, as the example of $\omega=\left[\begin{array}{cc}2 & -1 \\ 1 & 0\end{array}\right]$ and $\mathbf{x}=(1,0)$ shows: We have $\omega^{n} \mathbf{x}=(n+1, n)>0$ for all $n \geq 0$, and $\mathbf{x}$ is not an eigenvector of $\omega$; on the other hand, 1 is an eigenvalue of $\omega$ with multiplicity 2 .

Proof of Lemma. Let $\rho=\rho(\omega)$ be the spectral radius of $\omega$. Let $\mathbf{x}$ be a vector in $\mathbb{R}^{2}$ with $\omega^{n} \mathbf{x}>0$ for all $n \geq 0$. The existence of $\mathbf{x}$ shows that $\omega$ cannot be nilpotent, thus $\rho(\omega)>0$. Of course, if $\mathbf{x}$ is an eigenvector with eigenvalue $\lambda$, then $\lambda>0$. Thus, let us assume that $\mathbf{x}$ is not an eigenvector.

Given a set $\mathcal{X}$ of vectors in $\mathbb{R}^{2}$, let $C(\mathcal{X})$ be the cone in $\mathbb{R}^{2}$ of all vectors which are linear combinations of the elements in $\mathcal{X}$ using positive coefficients. Let $C=C\left(\left\{\omega^{n}(\mathbf{x}) \mid n \geq 0\right\}\right)$. Then all non-zero vectors $\mathbf{y} \in C$ satisfy $\mathbf{y}>0$ and $\omega(C) \subseteq C$. If $C$ is a ray, then any non-zero element in $C$ is an eigenvector of $\omega$, thus $\mathbf{x}$ is an eigenvector, a contradiction.

Thus, $C$ is not a ray, and there is a basis $\mathbf{y}, \mathbf{y}^{\prime}$ of $\mathbb{R}^{2}$ such that the topological closure $\bar{C}$ of $C$ is the cone $\bar{C}=C\left(\left\{\mathbf{y}, \mathbf{y}^{\prime}\right\}\right)$. We have $\omega(\bar{C}) \subseteq \bar{C}$, in particular $\omega(\mathbf{y}), \omega\left(\mathbf{y}^{\prime}\right) \in \bar{C}$.

If $\omega(\mathbf{y}) \in \mathbb{R}_{+} \mathbf{y}$, say $\omega(\mathbf{y})=\lambda \mathbf{y}$ with $\lambda \in \mathbb{R}_{+}$, let $\omega\left(\mathbf{y}^{\prime}\right)=c \mathbf{y}+d \mathbf{y}^{\prime}$, thus $c \geq 0$ and $d>0$. Now $\omega$ is similar to the matrix $\left[\begin{array}{ll}\lambda & c \\ 0 & d\end{array}\right]$, thus its eigenvalues are $\lambda$ and $d$, and both are positive. Therefore $\rho(\omega)=\max \{\lambda, d\}$ is an eigenvalue.

Next, assume that $\omega(\mathbf{y})=\lambda \mathbf{y}^{\prime}$. Let $\mathbf{y}^{\prime}=c \mathbf{y}+d \mathbf{y}^{\prime}$. Then $\omega$ is similar to a matrix of the form $\left[\begin{array}{ll}0 & c \\ 1 & d\end{array}\right]$, its eigenvalues are $\frac{1}{2} d \pm \frac{1}{2} \sqrt{d^{2}+4 c}$, thus the spectral radius is the eigenvalue $\frac{1}{2} d+\frac{1}{2} \sqrt{d^{2}+4 c}$.

Finally. it remains to consider the case that both $\omega(\mathbf{y}), \omega\left(\mathbf{y}^{\prime}\right)$ belong to the interior of $\bar{C}$, then $\omega$ is similar to a matrix with positive coefficients, thus the usual Perron-Frobenius theorem asserts that the spectral radius of $\rho(\omega)$ is an eigenvalue of $\omega$.
5.4. Lemma. The transformation $\omega_{a}^{e}$ has real eigenvalues iff $a \leq \frac{1}{4} e^{2}$. In this case, both eigenvalues are non-negative and $\rho\left(\omega_{a}^{e}\right)=\frac{1}{2}\left(e+\sqrt{e^{2}-4 a}\right)$.

If $\lambda \neq 0$ is an eigenvalue of $\omega_{a}^{e}$, then $(\lambda, a)$ is eigenvector of $\omega_{a}^{e}$ with eigenvalue $\lambda$.
Proof: The eigenvalues of $\omega_{a}^{e}$ are $\frac{1}{2}\left(e \pm \sqrt{e^{2}-4 a}\right)$, thus they are real iff $e^{2} \geq 4 a$. Also, since $e^{2}-4 a \leq e^{2}$, it follows from $e^{2} \geq 4 a$ that $\sqrt{e^{2}-4 a} \leq e$.

Let $\lambda$ be an eigenvalue of $\omega_{a}^{e}$. The characteristic polynomial of $\omega_{a}^{e}$ is $T^{2}-e T+a$, thus $\lambda^{2}=e \lambda-a$, therefore $\omega_{a}^{e}(\lambda, a)=(e \lambda-a, a \lambda)=\lambda(\lambda, a)$.
5.5. Proposition. Assume that $A$ is a short local algebra of Hilbert type ( $e, a$ ) with $e \geq 2$. If there exists a non-projective Koszul module $M$, then $a \leq \frac{1}{4} e^{2}$ and $\gamma(M)$ is a positive eigenvalue of $\omega_{a}^{e}$.

Proof. Let $M$ be a non-projective Koszul module. Replacing, if necessary, $M$ by $\Omega(M)$, we can assume that $M$ has Loewy length at most 2 . Let $\mathbf{x}=\operatorname{dim} M$. Since $M$ is Koszul, we have $\left(\omega_{a}^{e}\right)^{n} \mathbf{x}=\operatorname{dim} \Omega^{n} M>0$ for all $n \geq 0$. Thus, 5.3 assert that $\gamma(M)$ is a positive real eigenvalue. The existence of a real eigenvalue shows that $a \leq \frac{1}{4} e^{2}$, see 5.4.

Corollary. Let $A$ be a short local algebra of Hilbert type ( $e, a$ ) with $a>\frac{1}{4} e^{2}$. If $M$ is a non-projective module, then $M$ is not Koszul, thus $\gamma(M)=\gamma_{A}$.

Proof. Let $M$ be non-projective. Since $a>\frac{1}{4} e^{2}$, Proposition 5.5 asserts that $M$ cannot be Koszul. According to 5.1 we have $\gamma(M)=\gamma_{A}$.
5.6. Proof of Theorems 2 and 3. Let $A$ be a short local algebra of Hilbert type $(e, a)$.

First, let $a=0$. Then $\Omega^{S}=S^{e}$ shows that $S$ is a Koszul module ( $S$ is always aligned) and $\gamma_{A}=e$. For any non-zero module $M$, the module $\Omega M$ is semisimple and not zero, thus $\gamma(M)=\gamma_{A}$.

Now let $a \neq 0$. If $e=1$, then $\Omega J=S$, thus $\Omega^{2} S=S$ shows that $S$ is not Koszul, thus there are no non-projective Koszul modules.

Thus, let $e \geq 2$ and $a \neq 0$. Let $M \neq 0$ be a Koszul module of Loewy length at most 2. According to 4.6, $A$ is left Koszul. According to 5.5 we know that $a \leq \frac{1}{4} e^{2}$ and that $\gamma(M)$ and $\gamma_{A}$ are positive eigenvalues of $\omega_{a}^{e}$. Since $a \neq 0$, we see that $(1,0)$ is not an eigenvector of $\omega_{a}^{e}$, thus 5.2 asserts that $\gamma_{A}=\gamma_{\omega}(1,0)=\rho(\omega)=\frac{1}{2}\left(e+\sqrt{e^{2}-4 a}\right)$. Assume that $\gamma(M) \neq \gamma_{A}$, then $\gamma(M) \neq \rho(\omega)$, thus 5.2 asserts that $\operatorname{dim} M$ is an eigenvector of $\omega$ and $\gamma(M)$ is the corresponding eigenvalue, thus equal to $\frac{1}{2}\left(e-\sqrt{e^{2}-4 a}\right)$. This completes the proof of Theorem 2.

Now assume that there is a non-zero module $M$ of Loewy length at most 2 with $\gamma(M)<\gamma_{A}$. As we have seen, $\operatorname{dim} M$ is an eigenvector of $\omega$ and the corresponding eigenvalue is $\gamma(M)$. But this means that $\gamma(M) \operatorname{dim} M$ is a vector with integral coefficients, thus $\gamma(M)$ has to be rational and therefore $e^{2}-4 a$ has to be the square of an integer. Since $\operatorname{dim} M$ is an eigenvector of $\omega$ with eigenvalue $\gamma(M)$, and all eigenvectors have multiplicity $1, \operatorname{dim} M$ is a multiple of $(\gamma(M), a)=\left(\gamma(M), \gamma(M) \gamma_{A}\right.$, and thus a multiple of $\left(1, \gamma_{A}\right)$.

Let us assume that $e=\gamma(M)+\gamma_{A}$. Since $0<\gamma(M)<\gamma_{A}$, we have $2 \gamma(M)<$ $\gamma(M)+\gamma_{A}=e$, thus $\gamma(M)<\frac{1}{2} e$. Since $\gamma(M)<\frac{1}{2} e$, we have $\frac{1}{2} e=e-\frac{1}{2} e<e-\gamma(M)=\gamma_{A}$. Since $0<\gamma(M)$, we have $\gamma_{A}<\gamma(M)+\gamma_{A}=e$. This shows that $0<\gamma(M)<\frac{1}{2} e<\gamma_{A}<e$. Since $S$ is a Koszul module, Theorem 2 asserts that $\gamma_{A}=\frac{1}{2}\left(e+\sqrt{e^{2}-4 a}\right)$ and $\gamma(M)=$ $\frac{1}{2}\left(e-\sqrt{e^{2}-4 a}\right)$. It follows that $\gamma_{A}-\gamma(M)=\sqrt{e^{2}-4 a}$, thus $\left(\gamma_{A}-\gamma(M)\right)^{2}=e^{2}-4 a$, so that $e^{2}-4 a$ is the square of a positive integer.

## 6. Left Conca ideals.

6.1. Let $A$ be a local algebra and $U$ an ideal of $A$. We say that $U$ is a left Conca ideal provided $U^{2}=0$ and $J^{2} \subseteq J U$. If $A$ has a left Conca ideal $U$, then $A$ is short (namely, $\left.J^{3} \subseteq J^{2} U \subseteq J U^{2}=0\right)$.

Remark. The name corresponds to the considerations in [AIS]. Following [AIS] (but dealing also with non-commutative local algebras), an element $x$ may be called a left Conca generator of $J$ provided $x^{2}=0 \neq x$ and $J^{2}=J x$. If $x$ is a left Conca generator of $J$, then clearly $A x$ is a left Conca ideal (note that $A x$ is a twosided ideal, since $x A=k x+x J \subseteq$ $\left.k x+J^{2} \subseteq A x\right)$. Obviously, the existence of a left Conca generator for $J$ implies that $a \leq e-1$. As we will see in 7.1, for any pair $(e, a)$ with $a \leq \frac{1}{4} e^{2}$, there are short local algebras of Hilbert type $(e, a)$ with a left Conca ideal.
6.2. Proof of Theorem 4. Let $U$ be a left Conca ideal in $A$. Let $N$ be a local module annihilated by $U$, thus, $N \simeq^{A} / V$ for some proper left ideal $V$ of $A$ and $U \subseteq V$, since $U N=0$. Since $J^{2} \subseteq J U \subseteq U \subseteq V \subseteq J$, the factor module $V / U$ is a subquotient of ${ }_{A} J / J^{2}$, thus semisimple. in addition, $J^{2} \subseteq J U \subseteq J V \subseteq J^{2}$ shows that the embedding
$u: U \rightarrow V$ yields the equality $J U=J V$. Thus, the Snake Lemma applied to

shows that $U$ is a $t$-submodule of $V$.
We show by induction on $n \geq 0$ : If $N$ is a local module annihilated by $U$, then $\Omega^{i} N$ is aligned, for $0 \leq i \leq n$.

First, let $n=0$. We have $N={ }_{A} A / V$ for some left ideal $V$, thus $\Omega N=V$, and as we have mentioned already, $J^{2} \subseteq J V=J \Omega N$, thus condition (vi) of 3.3 asserts that $N$ is aligned.

Now, assume that we know for some $n \geq 0$, that for all local modules $N^{\prime}$ annihilated by $U$ the modules $\Omega^{i} N^{\prime}$ with $0 \leq i \leq n$ are aligned. According to Corollary (a) in 4.5, this implies that for all modules $M$ annihilated by $U$, the modules $\Omega^{i} M$ with $0 \leq i \leq n$ are aligned. Let $N$ be a local module annihilated by $U$, say $N={ }_{A} A / V$ for some left ideal $V$. By induction assumption, we know that the modules $\Omega^{i} N$ are aligned for $0 \leq i \leq n$. It remains to be seen that $\Omega^{n+1} N$ is aligned. As we have mentioned, $U$ is a $t$-submodule of $V$. Now $U$ is annihilated by $U$. Also, $V / U$ is annihilated by $U$ (since it is semisimple). Thus, all the modules $\Omega^{i} U$ and $\Omega^{i}(V / U)$ are aligned, for $0 \leq i \leq n$. We apply Lemma 4.4 (a) in order to conclude that $\Omega^{n} V$ is aligned. Thus, $\Omega^{n+1} N=\Omega^{n} V$ is aligned.

Remark. This improves Theorem 3.2 of [AIS]. In addition, we should stress that the proof yields the following stronger assertion: If $A$ has a left Conca ideal $U$, then any module with a t-filtration with factors annihilated by $U$ is a Koszul module.

One should be aware that given any ideal $U$, there may be modules which are not annihilated by $U$, but which have a $t$-filtration with factors annihilated by $U$. For example, if $A$ is of Hilbert type $(2,0)$ (thus, $A$ is the local 3 -dimensional algebra with radical square zero) and $U$ is one-dimensional, then there are just two indecomposable modules annihilated by $U$, namely $S$ and $I={ }_{A} A / U$, but infinitely many indecomposable modules which have a $t$-filtration with factors of the form $I$ and $S$, namely the modules in the Auslander-Reiten component which contains $I$ as well as the preinjective modules.
6.3. Remark. A short local Koszul algebra A may not have any left Conca ideal. Also, A may not have a left Conca ideal, whereas its opposite algebra has a left Conga ideal.

Here is an example: Let $A$ be generated by $x, y, z$ with relations

$$
x^{2}, y x, z x, z y, y^{2}-x z, y z, z^{2},
$$

so that $J^{2}$ has the basis $x y, y^{2}=x z$. One easily checks that $A$ has no left Conga ideal (namely, any ideal $U$ with $U^{2}=0$ is contained in $A x+A z$, thus $J U \subseteq k x z$ ). But $x A$ (with basis $x, x y, x z$ is a right Conga ideal.

Since the opposite algebra of $A$ has a left Conga ideal, $A$ is a right Koszul algebra. In order to see that $A$ is also left Koszul, write ${ }_{A} J=S \oplus W$ where $W=A y+A z$. Then
$\Omega W=W^{2}$. Therefore $\Omega^{n} S=S \oplus W^{n}$ has dimension vector $(2 n+1,2 n)=\left(\omega_{2}^{3}\right)^{n}(1,0)$ and this shows that $S$ is a Koszul module.

## 7. Construction of Koszul algebras.

7.1. Proposition. If $0 \leq a \leq \frac{1}{4} e^{2}$, there are short local algebras of Hilbert type ( $e, a$ ) (even commutative ones) with a left Conca ideal.

Proof. Assume that $0 \leq a \leq \frac{1}{4} e^{2}$. We are going to construct a commutative short local algebra $A$ of Hilbert type ( $e, a$ ) which is Koszul.

Let $c=\left\lfloor\frac{1}{2} e\right\rfloor$ and $d=e-c$. Since $0 \leq a \leq \frac{1}{4} e^{2}$, we have $a \leq c d$ (namely, for $e$ even, $c=d=\frac{1}{2} e$ and $a \leq c^{2}=c d$, whereas for $e$ odd, we have $d=c+1$ and $a \leq \frac{1}{4}(2 c+1)^{2}$ implies that $\left.a \leq c^{2}+c=c d\right)$. Thus we can write $a=\sum_{j=1}^{d} a(j)$ with $0 \leq a(j) \leq c$.

Let $A=\Lambda(c ; a(1), \ldots, a(d))$ be the commutative algebra generated by the elements $x_{i}, y_{j}$ with $1 \leq i \leq c$ and $1 \leq j \leq d$ and the relations $x_{i} x_{i^{\prime}}, y_{j} y_{j^{\prime}}$ for all $i, i^{\prime} \in\{1, \ldots, c\}$ and $j, j^{\prime} \in\{1, \ldots, d\}$, as well as $x_{i} y_{j}$ for all pairs $(i, j)$ with $1 \leq j \leq d$ and $a(j)<i \leq c$. It follows that $J^{2}$ has the basis $x_{i} y_{j}$ with $1 \leq j \leq d$ and $1 \leq i \leq a(j)$.

If $U=\sum_{j=1}^{d} A y_{j}$, then $U^{2}=0$ and $J^{2} \subseteq J U$, thus $U$ is a left Conca ideal.
7.2. Proof of Theorem 5. If $A$ is a short local left Koszul algebra of Hilbert type $(e, a)$, then Theorem 2 asserts that $0 \leq a \leq \frac{1}{4} e^{2}$. Conversely, 7.1 shows that for $0 \leq a \leq \frac{1}{4} e^{2}$, there are commutative short local algebras $A$ of Hilbert type $(e, a)$ with a left Conca ideal. According to Theorem 4, these algebras $A$ are left Koszul algebras.
7.3. For any pair $c, d$ of natural numbers, there exists a commutative short local algebra $\Lambda(c, d)$ of Hilbert type $(e, a)$, where $e=c+d$ and $a=c d$, with a module $M$ with dimension vector $(1, c)$ such that $\Omega M \simeq M^{d}$ (thus $\left.\gamma(M)=d\right)$, and such that $\gamma_{A}=\max \{c, d\}=\rho\left(\omega_{a}^{e}\right)$.

Proof. Let $A=\Lambda(c, d)$ be the commutative algebra generated by $x_{1}, \ldots, x_{c}, y_{1}, \ldots, y_{d}$, and with relations $x_{i} x_{i^{\prime}}, y_{j} y_{j^{\prime}}$ for all $i, i^{\prime} \in\{1, \ldots, c\}$ and $j, j^{\prime} \in\{1, \ldots, d\}$. Then $J^{2}$ has the basis $x_{i} y_{j}$ with $1 \leq i \leq c, 1 \leq j \leq d$. Let $M=A y_{1}$; this is a local module of Loewy length 2 with socle $x_{1} y_{1}, \ldots, x_{c} y_{1}$, thus with dimension vector $(1, c)$. All the module $A y_{j}$ with $1 \leq j \leq d$ are isomorphic to $M$ and $J$ is isomorphic to $S^{c} \oplus M^{d}$. Since $A / \bigoplus_{j=1}^{d} A y_{j}$ is isomorphic to $M$, we see that $\Omega M \simeq M^{d}$. It follows that $\gamma(M)=d$.

Here is a similar, but non-commutative example: a non-commutative short local algebra $\Lambda^{\prime}(c, d)$ of the same Hilbert type $(c+d, c d)$ with a module $M$ with dimension vector $(1, c)$ such that $\Omega M \simeq M^{d}$, so that $\gamma(M)=d$, whereas $\gamma_{A}=\max \{c, d\}=\rho\left(\omega_{a}^{e}\right)$. Let $\Lambda^{\prime}(c, d)$ be generated by $x_{1}, \ldots, x_{c}, y_{1}, \ldots, y_{d}$, and with relations $x_{i} x_{i^{\prime}}, y_{j} y_{j^{\prime}}, y_{j} x_{i}$ for all $i, i^{\prime} \in$ $\{1, \ldots, c\}$ and $j, j^{\prime} \in\{1, \ldots, d\}$. Again, $J^{2}$ has the basis $x_{i} y_{j}$ with $1 \leq i \leq c, 1 \leq j \leq d$. Note that the elements $x_{1}, \ldots, x_{d}$ do not belong to $J^{2}$, but to $\operatorname{soc}_{\Lambda^{\prime}(c, d)} J$.
7.4. In particular, let us focus the attention to the case $d=1$. The algebras $\Lambda(c, 1)$ and $\Lambda^{\prime}(c, 1)$ have a non-zero $\Omega$-periodic module $M$.

On the other hand, let us stress that the algebra $A=\Lambda^{\prime}(c, 1)$ is a short local algebra with $J^{2} \subset \operatorname{soc}_{A} A$ as well as $J^{2} \subset \operatorname{soc} A_{A}$. Note that Lescot [L] Prop. 3.9 (2) has pointed out that for a commutative short local algebra with $J^{2} \subset \operatorname{soc} A$ and a non-projective module $M$, the sequence $t_{n}(M)$ is always strictly increasing.

## 8. A lower bound for $\gamma_{A}$.

8.1. Proof of Theorem 7. We assume that $A$ is a short local algebra of Hilbert type (e,a) with $a \leq \frac{1}{4} e^{2}$. Let $\omega=\omega_{a}^{e}$ and $\mathbf{d}(n)=\omega^{n}(1,0)$ for all $n \geq 0$. We have $\omega(0,-1)=(1,0)=\mathbf{d}(0)$, and therefore $\omega(w,-w)=w \mathbf{d}(1)+w \mathbf{d}(0)$ for any $w \in \mathbb{Z}$.
(1) Let us show that $\mathbf{d}(n)>0$ and that

$$
\limsup _{n} \sqrt[n]{|\mathbf{d}(n)|}=\frac{1}{2}\left(e+\sqrt{e^{2}-4 a c}\right)
$$

According to Theorem 5, there exists a short local algebra $A^{\prime}$ of Hilbert type $(e, a)$ which is left Koszul. Let $S^{\prime}$ be the simple $A^{\prime}$-module. Since $S^{\prime}$ is a Koszul module, we have $\operatorname{dim} \Omega_{A^{\prime}} S^{\prime}=\mathbf{d}(n)$, thus $\mathbf{d}(n)>0$. Theorem 2 asserts that $\lim \sup _{n} \sqrt[n]{|\mathbf{d}(n)|}=$ $\limsup _{n} \sqrt[n]{\left|\Omega_{A^{\prime}}^{n} S^{\prime}\right|}=\frac{1}{2}\left(e+\sqrt{e^{2}-4 a c}\right)$.

Let $\mathcal{N}(n)$ be the set of linear combinations of $\mathbf{d}(i)$ with $0 \leq i \leq n-1$ using nonnegative coefficients. For $n \geq 1$, we apply the Main Lemma 3.1 to $\Omega^{n-1} S$ and obtain $\operatorname{dim} \Omega^{n} S=\omega\left(\operatorname{dim} \Omega^{n-1} S\right)+\left(w_{n},-w_{n}\right)$ for some integer $w_{n} \geq 0$. In addition, we define $w_{0}=0$.
(2) Using induction on $n \geq 0$, we show that

$$
\operatorname{dim} \Omega^{n} S-\mathbf{d}(n)-\left(w_{n},-w_{n}\right) \in \mathcal{N}(n)
$$

Proof. The assertion holds true for $n=0$, since $\operatorname{dim} S=\mathbf{d}(n)$ and $w_{0}=0$. Now assume that the assertion is true for some $n \geq 0$, thus we have

$$
\operatorname{dim} \Omega^{n} S=\mathbf{d}(n)+\mathbf{x} \quad \text { with } \quad \mathbf{x}=\left(w_{n},-w_{n}\right)+\sum_{i=0}^{n-1} v_{i} \mathbf{d}(i)
$$

with non-negative integers $v_{i}$, where $0 \leq i<n$.
We apply $\omega$ to $\mathbf{x}$ and get

$$
\begin{aligned}
\omega(\mathbf{x}) & =\omega\left(w_{n},-w_{n}\right)+\sum_{i=0}^{n-1} v_{i} \omega(\mathbf{d}(i)) \\
& =w_{n} \mathbf{d}(1)+w_{n} \mathbf{d}(0)+\sum_{i=0}^{n-1} v_{i} \mathbf{d}(i+1)
\end{aligned}
$$

thus $\omega(\mathbf{x})$ belongs to $\mathcal{N}(n+1)$. On the other hand, we have

$$
\begin{aligned}
\operatorname{dim} \Omega^{n+1} S & =\omega\left(\operatorname{dim} \Omega^{n} S\right)+\left(w_{n+1},-w_{n+1}\right) \\
& =\omega(\mathbf{d}(n)+\mathbf{x})+\left(w_{n+1},-w_{n+1}\right) \\
& =\mathbf{d}(n+1)+\left(w_{n+1},-w_{n+1}\right)+\omega(\mathbf{x}) .
\end{aligned}
$$

This shows that $\operatorname{dim} \Omega^{n+1} S-\mathbf{d}(n+1)-\left(w_{n+1},-w_{n+1}\right)=\omega(\mathbf{x})$, and we have seen already that $\omega(\mathbf{x})$ belongs to $\mathcal{N}(n+1)$.
(3) We have $\left|\Omega^{n} S\right| \geq|\mathbf{d}(n)|$ for $n \geq 0$. Namely, the formula (2) implies that $\operatorname{dim} \Omega^{n} S \geq$ $\mathbf{d}(n)+\left(w_{n},-w_{n}\right)($ since $\mathbf{d}(i) \geq 0$ for all $0 \leq i<n)$ and therefore $\left|\Omega^{n} S\right| \geq|\mathbf{d}(n)|$, since $\left|\left(w_{n},-w_{n}\right)\right|=0$.
(4) Altogether, (1) and (3) show that

$$
\gamma_{A}=\limsup _{n} \sqrt[n]{\left|\Omega^{n} S\right|} \geq \limsup _{n} \sqrt[n]{|\mathbf{d}(n)|}=\frac{1}{2}\left(e+\sqrt{e^{2}-4 a c}\right)
$$

this completes the proof.
8.2. Let us show that $\gamma_{A}$ does not only depend on the Hilbert type. Of course, as we have seen, if $A$ is a Koszul algebra, then $\gamma_{A}$ is determined by the Hilbert type ( $e, a$ ), namely $\gamma_{A}=\rho\left(\omega_{a}^{e}\right)$. But we will show that there are algebras $A, A^{\prime}$ which are not Koszul with $\gamma_{A} \neq \gamma_{A^{\prime}}$ (and both $\gamma_{A}, \gamma_{A^{\prime}}$ different from $\rho\left(\omega_{a}^{e}\right)$ ).

Example. Short local algebras $A$ of Hilbert type $(3,2)$ with $\gamma_{A}=2, \psi, 3$, where $\psi=\frac{1}{2}(3+\sqrt{5})$ is the square of the golden ratio. Note that $\rho\left(\omega_{2}^{3}\right)=2$.

First. If a short local algebra has Hilbert type $(3,2)$ and is Koszul, then theorem 2 asserts that $\gamma(S)=\rho\left(\omega_{a}^{e}\right)=\frac{1}{2}(3+\sqrt{9-4 \cdot 2})=2$ (and we know from Theorem 5 that such algebras do exist).

We define two algebras $A, A^{\prime}$ of Hilbert type $(3,2)$ with generators $x, y, z$. The relations for $A$ are $y x, z x, y^{2}, z y, x z, y z, z^{2}$. The relations for $A^{\prime}$ are $y x, z x, x y, z y, x z, y z, z^{2}$. The radicals $J$ and $J^{\prime}$, respectively, look as follows:


We will show that $\gamma_{A}=\psi$ and $\gamma_{A^{\prime}}=3$.
First, let us consider $A$. Let $X=A x \simeq A /\left(A x^{2}+A y+A z\right)$ and $Z=A y \simeq A /(A x+A y)$, these are indecomposable modules of length 2 . We claim that for $M \in \operatorname{add}\{S, X, Z\}$, we have $\Omega M \in \operatorname{add}\{S, X, Z\}$ and $t_{2}(M)=3 t_{1}(M)-t_{0}(M)$.

Proof. We can assume that $M$ is indecomposable, thus $M$ is one of $S, X, Z$. We have $\Omega S=X \oplus Z \oplus S$; second, we have $\Omega X=A x^{2} \oplus Z \oplus A z \simeq Z \oplus S^{2}$, and finally $\Omega Z=X \oplus Z$. This shows already that $\Omega M \in \operatorname{add}\{S, X, Z\}$. It follows that $\Omega^{2} S=$ $\Omega(X \oplus Z \oplus S) \simeq X^{2} \oplus Z^{3} \oplus S^{3}$, therefore $t_{2}(S)=t\left(\Omega^{2} S\right)=8$. Since $t_{1}(S)=3$ and $t_{0}(S)=1$, we have $t_{2}(S)=3 t_{1}(S)-t_{0}(S)$. Next, $\Omega^{2} X=\Omega\left(Z \oplus S^{2}\right) \simeq X^{3} \oplus Z^{3} \oplus S^{2}$, therefore $t_{2}(X)=t\left(\Omega^{2} X\right)=8$. Since $t_{1}(X)=3$ and $t_{0}(X)=1$, we have $t_{2}(X)=3 t_{1}(X)-t_{0}(X)$. Finally, $\Omega^{2} Z=\Omega(X \oplus Z) \simeq X^{2} \oplus Z^{2} \oplus S$, therefore $t_{2}(Z)=t\left(\Omega^{2} Z\right)=5$. Since $t_{1}(X)=2$ and $t_{0}(X)=1$, we have $t_{2}(Z)=3 t_{1}(Z)-t_{0}(Z)$.

By induction, we see that $\Omega^{n}(S) \in \operatorname{add}\{S, X, Z\}$ and that the numbers $b_{n}=t_{n}(S)$ satisfy the recursion $b_{n+2}=3 b_{n+1}-b_{n}$ for all $n \geq 0$. Since $b_{0}=1$ and $b_{n}=3$, it follows that the numbers $b_{n}$ are the even-index Fibonacci numbers 1, 3, 8, 21, 55, 144, $\ldots$ and therefore $\gamma_{A}=\gamma(S)=\lim \sup _{n} b_{n}=\phi^{2}=\psi$, where $\phi=\frac{1}{2}(1+\sqrt{5})$ is the golden ratio.

Second, we consider $A^{\prime}$. Let $X=A^{\prime} x \simeq A /\left(A x^{2}+A y+A z\right)$ and $Y=A^{\prime} y \simeq$ $A /\left(A x+A y^{2}+A z\right)$, these are indecomposable modules of length 2 . We claim that for $M \in \operatorname{add}\{S, X, Y\}$, we have $\Omega M \in \operatorname{add}\{S, X, Y\}$ and $t_{1}(M)=3 t_{0}(M)$.

Proof. We can assume that $M$ is indecomposable, thus $M$ is one of the modules $S, X, Y$. We have $\Omega S=J=X \oplus Y \oplus S$, and $|\operatorname{top} \Omega S|=3$. We have $\Omega X=A x^{2} \oplus Y \oplus A z \simeq Y \oplus S^{2}$, thus $\mid$ top $\Omega X \mid=3$. And similarly, $\Omega Y \simeq X \oplus S^{2}$, and thus $\mid$ top $\Omega Y \mid=3$.

By induction, $\Omega^{n} S$ belongs to add $\{S, X, Y\}$ and $t_{n}(S)=\left|\operatorname{top} \Omega^{n}(S)\right|=3^{n}$. Therefore $\gamma_{A^{\prime}}=\gamma(S)=3$.

## References.

[AE] J. L. Alperin, L. Evens. Representations, resolutions, and Quillen's dimension theorem. J. Pure Appl. Algebra 22 (1981), 1-9.
[AIS] L. L. Avramov, S. B. Iyengar, L. M. Şega. Free resolutions over short local rings. J. London Math. Soc. 78 (2008), 459-476.
[BH] W. Bruns, J. Herzog. Cohen-Macaulay Rings. Cambridge Studies in advanced mathematics 39. Cambridge University Press (1993).
[GM] E. L. Green, R. Martínez-Villa. Koszul and Yoneda algebras. In: Representation theory of algebras. CMS conf. Pbl. 18. Amer. Math.Soc. Publ. Providence, RI. (1996) 247-297.
[HI] J. Herzog, S. Iyengar. Koszul modules. J. Pure Appl. Algebra 201 (2005), 154-188.
[L] J. Lescot. Asymptotic properties of Betti numbers of modules over certain rings. J. Pure Appl. Algebra 38 (1985), 287-298.
[MZ] R. Martínez-Villa, D. Zacharia. Approximations with modules having linear resolutions. J. Algebra 266 (2003), 671-697.
[P] S. Priddy. Koszul resolutions. Transactions Amer.Math.Soc. 152 (1970), 39-60.
[RZ] C. M. Ringel, P. Zhang. Gorenstein-projective modules over short local algebras. To appear. arXiv:1912.02081.

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