Andrzej Skowroński and the Tame Algebras.

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ARTA conference 2021 is devoted to the memory of Andrzej Skowroński. As many know, I have withdrawn nearly completely from public mathematical life (I still do some research, but just on a somewhat private scale). In particular, since 2017, I have stopped to participate in conferences or to give mathematical lectures. But when Ibrahim Assem asked me for an address in honour of Andrzej I felt that I could not refuse it. By now, most of my (fading) activities are restricted to my family — Andrzej, as one of the first Humboldt fellows coming to Birep, really may be considered as part of my former mathematical family ... .

I.

1) The Humboldt family. These were the first two Humboldt fellows coming to Birep: Ibrahim Assem and Andrzej Skowroński. They arrived nearly at the same time. They had cooperated before at a distance, and now enjoyed very much working at the same place. They had completely different attitudes towards mathematics, but in my opinion this was the basic factor for the big success of their cooperation. Whereas Andrzej always had weird, but challenging ideas of what could and should be true, it was Ibrahim who had to bring him back to reality, who insisted to formulate precise expections and then to look at examples. I remember, when we spoke about the current research, Andrzej would point out one (or usually several) results which they were going to prove, whereas Ibrahim would stress that this was just a hope, that many details still had to be checked. At that time, Andrzej reminded me of the old Italian geometers with their interest in general behaviour, and their neglect of all the nasty special cases.

Also afterward, Andrzej visited quite often Bielefeld, sometimes accompanied by his wife Mirka and his daughter. So he was present in 1985, when Liu Shao-Xue came to Bielefeld in order to establish cooperations between China and the West. Liu gave a fourfold lecture, with one part devoted to a Toruń theme: the pure semisimplicity conjecture, thus there was a vivid discussion. Later, I tried via the Humboldt foundation to initiate some better arrangements for Andrzej's visits to Bielefeld. Also, Andrzej sent some of his students, in particular Bobiński, in order to foster the relationship (or, one may say: he kindly allowed Bobiński to spent some time at Bielefeld).

As you know, the collaboration of Ibrahim and Andrzej has continued, see for example the first volume of the *Elements*. There is also the picture for the conference homepage, showing Andrzej with Jan Karski, chosen by the organizing committee. As I was told, it was taken quite recently, when Andrzej and Mirka showed Ibrahim and Sonja parts of Krakow.

2) Poland. The picture of Andrzej with Jan Karski draws the attention to the worst period of Polish history, the time of the German occupation (and one should be aware that really for 200 years, the Prussian kings, the Habsburg monarchy and the Russian Empire tried to suppress Poland in all respects). We have to appreciate the present friendship between Poland and Germany. The Humboldt foundation definitely has its share in improving the relationship.

3) Toruń. I always felt very happy being in Toruń. There was a satellite conference for the Warsaw ICM at 1983, with a charming sightseeing tour, very professionally handled by Bogumila Klemp (fighting well ongoing interjections by Gabriel on mutual suppression of minorities). Looking at the history, the old city of Toruń seems to reflect that during many centuries, German and Polish people tried to live peaceful and prosperous side-by-side.

When I visited Toruń, I liked to stay in a tiny old hotel with name *Petite Fleur*. The French name should not be considered as a curiosity — afterall, in old days, the Polish aristocracy was strongly in favour of French life style (as the Prussian king Friedrich II was). The long standing cooperation of Andrzej and Ibrahim fits into this pattern.

My first visit to Toruń was in december 1981, just few days before martial low was invoked in Poland. It was a quite miserable time for Poland, not much was available (I remember, I lived mainly on bread, milk and pickled cucumbers), and already my travel via East Germany and Berlin was overshadowed by the events to come.

4) The Toruń school in representation theory. The Toruń school was created by Daniel Simson (a "grandchild" of J. Loś). He started to work on the pure semisimplicity conjecture, then on representations say of posets. Andrzej was one of his first PhD-students. Andrzej's interests at that time were devoted to Hopf algebras. Of course, this helped him later when he was dealing with group algebras, or, more generally, with self-injective algebras. The representation group in Toruń grow bigger and bigger. Let me mention just some names: Dowbor, Kasjan, and Pogorzaly, also Meltzer, then Zwara (with his famous module-theoretical characterization of degeneration) and Bobiński (looking at module varieties). And there is Kosakowska with her interest in Hall algebras, linked to Bielefeld also via her cooperation with Schmidmeier.

5) The blue books. For a long time there was a tremendous lack of text books (and partly this is still the case). I remember a photo showing Andrzej in a swimming pool with my Lecture Notes in one of his hands — carefully avoiding that it got wet.

He is the coauthor of five big volumes, three with the modest title *Elements*, covering some basic concepts, but still many topics are missing (not even the string-and-band classification is included), the first volume in collaboration with Assem and Simson, volumes II and III just with Simson.

And there are two volumes about Frobenius Algebras, written jointly with Kunio Yamagata (with a third volume announced; it is supposed to include the principles of covering theory). The title directs the attention to the algebras which finally are the target. But the books include a lot of preliminary results: In Volume I, this is the first half. Volume II deals first with hereditary and tilted algebras (405 pages), then with Auslander-Reiten components in general (70 pages); only the last chapter (134 pages) concerns self-injective algebras, namely Hochschild extension algebras using duality modules. This is, of course, the focus of the volume. It provides for the first time a concise presentation of a series of important results by Yamagata. The cooperation of Skowroński and Yamagata started in 1991, when Andrzej was full professor at Tsukuba University for a year, by looking at self-injective orbit algebras.

6) Lectures. I have to admit that usually I did not like Andrzej's lectures: with 30 slides and may-be 40 theorems; so fast that only he himself (and perhaps Happel) were able to follow — conversely, he told me that he did not like my lectures, either! But his vision of mathematics is, of course, the correct one: To stress the results is what really matters. In mathematics, there are definitions, results and proofs. And the separation between results and proofs usually should mean that a lecture will have to present the results, whereas the reception of proofs has to be done afterwards at home. To communicate ideas, or to illustrate assertions by pictures or mind maps (as I usually tried) is a dangerous endeavor and often seems to fail.

7) Conferences. Skowroński has always fostered the communication between mathematicians. He was involved in organizing several conferences, some directly at Toruń, but many at various places throughout the world.

By now there are three regular series of meetings devoted to representation theory, say of finite-dimensional algebras and related topics. There is ICRA, initiated by Vlasta Dlab, following the lines of the old representation theory, as outlined in the books of Curtis and Reiner, dealing with representations of finite-dimensional algebras, of finite groups and of orders, and emphasizing for a yet undeveloped area the three essential E's: **E**xamples. **E**xamples. **E**xamples. But ICRA moved more and more away from its original setting to more general themes, stressing applications in geometry and topology, and focusing the attention to triangulated categories. Then, there is a corresponding Chinese(-American) series established by Jianpan Wang from East China Normal University, devoted mainly to topics related to Lie theory and algebraic groups. Third, there is ARTA, established by Skowroński, Happel and others, as a revival of the old concept of ICRA, promoting again the discussion of finite-dimensional algebras and their module categories. It has to be appreciated that ARTA devotes the present conference to the memory of Andrzej.

II.

Let me mention at least a few topics Andrzej was interested in. I will concentrate on his contributions to the representation theory of a finite-dimensional k-algebra A; always k will be assumed to be an algebraically closed field. Since his interest centered around indecomposable representations, we may assume that A is sincere (this means that there exists an indecomposable A-module having any simple module as a composition factor).

1) Information stored in the Auslander-Reiten quiver. It is well-known that a representation-finite algebra A can usually be recovered from its Auslander-Reiten quiver; non-standard algebras occur only in case k has characteristic 2. In contrast, if A is wild, then it is very rare that A can be recovered from its

Auslander-Reiten quiver. So what about tame algebras? When Karin Erdmann looked at this question, dealing with group algebras, I used a lot of energy to persuade her that this is a fruitless endeavor — however she was stubborn and she succeeded! One should stress that there are many complications, mainly due to the fact that the simple modules usually live in different components.

Similar to the non-standard representation-finite algebras in characteristic 2, there are non-standard tame components in characteristic 2 and 3. Also, looking at the Auslander-Reiten quiver of a canonical algebra of tubular type $(2, 2, 2, 2)$, the parameter λ gets lost. And, there are many non-isomorphic tame self-injective algebras A and B with $A/\operatorname{soc} A$ isomorphic to $B/\operatorname{soc} B$; thus, the Auslander-Reiten quivers of A and B are of course isomorphic. Are there further obstacles for recovering a tame algebra from the Auslander-Reiten quiver?

2) Tubes and coils. The Bielefeld papers by Assem-Skowroński completed the consideration of tubes in an essential way. Tubes belong to the most basic concepts in mathematics: they are hidden in any linear algebra course as soon as the Jordan normal form is discussed, they form the spine of finite abelian group theory and of the representation theory of hereditary orders. It was Gabriel who pointed out that the four-subspace paper by Gelfand-Ponamarev describes tubes when dealing with the regular representations. And somewhat later, Nazarova and Donovan-Freislich have shown that the regular components of tame hereditary algebras are just tubes.

My joint paper with Gabriella d'Este was dealing with tubes in order to handle tubular extensions. This was a preparation for the study of tubular algebras: tubular algebras are tubular extensions of tame concealed algebras. And, this was seen as a surprise, they are at the same time also tubular coextensions of tame concealed algebras. In general, starting with a stable tube, one needs to look at extensions and coextension, but usually there will be interrelations. What one obtains are no longer tubes, but coils, as studied by Assem-Skowroński. This is definitely a fundamental concept.

An algebra A is said to be a coil algebra if any cycle in mod A lies in a standard coil. The minimal representation-infinite coil algebras are just the tame concealed algebras, but the interesting coil algebras are, of course, those which are not minimal representation-infinite.

Assem and Skowroński also looked at what they called multicoils. These are Auslander-Reiten components which are obtained by gluing together finitely many coils by a directed part (note that multicoils cannot appear in case A is sincere). In the same way, one may discuss the more general (but similar) situation, where finitely many coils are glued together by a finite, but not necessarily directed part. As far as I know, this has not yet been done. But I should stress that during all his further life, multicoil algebras played an important role for Andrzej.

Stable tubes are of interest in representation theory not only when dealing with tame algebras (where they are really fundamental), but also if one looks at τ-periodic modules for wild algebras, since τ-periodic modules usually belong to stable tubes. Is there a difference between one-parameter families of tubes appearing in the module category of a tame or of a wild algebra?

3) The radical and the infinite radical of $\operatorname{mod} A$**.** Let $\mathcal{A}$ be a Hom-finite k-category. The radical $\operatorname{rad} \mathcal{A}$ of $\mathcal{A}$ is the ideal generated by the non-invertible maps between indecomposable objects, the infinite radical is $\operatorname{rad}^\infty \mathcal{A} = \bigcap_n (\operatorname{rad} \mathcal{A})^n$. These are obvious invariants of $\mathcal{A}$ and $\operatorname{rad} \mathcal{A}/\operatorname{rad}^2 \mathcal{A}$ describes the corresponding Auslander-Reiten quiver. Actually, it took a while that these notions got familiar. The generalised standard components considered by Skowroński are just the components with no non-zero maps in $\operatorname{rad}^\infty \mathcal{A}$. A complete description of the possible generalised standard components is not known.

4) Tameness. The notion of tameness was analysed in detail in Toruń. The title of my Springer LNM 1099 refers to *Tame algebras*, however the book does not even provide a definition of tameness: the word tame is just used in order to describe a rather obvious, but not further specified phenomenon.

An early (but quite useless) attempt to deal with representation-infinite algebras was to consider the so-called representation equivalences say between abelian Hom-finite k-categories $\mathcal{A}$. Since the canonical projection $\mathcal{A} \to \mathcal{A}/\operatorname{rad} \mathcal{A}$ is a representation equivalence, the second Brauer-Thrall conjecture shows that all representation-infinite finite-dimensional k-algebras are representation equivalent. A study of representation-infinite algebras always has to deal with families of modules of a fixed dimension, thus one needs tools from algebraic geometry, or one has to look at modules of infinite dimension, say generic modules. One should not (and presumably cannot) avoid tensor functors: functors of the form ${}_AM_{k[T]} \otimes -\colon \operatorname{mod} k[T] \to \operatorname{mod} A$, where ${}_AM_{k[T]}$ is an A-$k[T]$-bimodule. If A is tame, let us denote by $\pi(d)$ the number of primitive 1-parameter families of modules of length d. The growth of the function $\pi(d)$ is the first important invariant of a tame algebra A. If $\sum_d \pi(d)$ is finite, then A is said to be domestic. It seems that there is a strict dichotomy, whether $\pi(d)$ has polynomial growth (and then even linear growth) or non-polynomial growth (and then even exponential growth).

Typical examples of algebras with polynomial growth are the tubular algebras, and one may guess that the relevant parts of module categories of polynomial growth are related to tubular algebras. Many papers by Skowroński and his collaborators are devoted to this problem and verify the guess.

5) The realm $\mathbb{A}$**.** Let us turn now the attention to tame algebras which may be of non-polynomial growth. Let us consider the realm $\mathbb{A}$, where all indecomposable modules are described by strings and bands, thus by walks in the quiver. The first case where such a classification was achieved is the seminal Lorentz-group paper by Gelfand and Ponomarev (and all the subsequent publications in this direction are just minor modifications). As the title of the Gelfand-Ponamarev paper indicates, the authors used the representation theory of algebras in order to deal with representations of the Lorentz group, and many other classification problems in algebra and geometry have turned out to lead to strings and bands, let me mention just the representation-finite blocks of group algebras of finite groups, and of the dihedral groups, but of course also all the hereditary algebras of type $\mathbb{A}_n$ and $\widetilde{\mathbb{A}}_n$.

Several papers of Skowroński are devoted to this setting. There is his joint paper with Waschbüsch, devoted to the representation-finite case, as well his cooperation with Assem on the iterated tilted algebras of type $\widetilde{\mathbb{A}}_n$. They stressed to look at gentle algebras, a small class of $\mathbb{A}$-algebras. At that time, I was reluctant to see the relevance of putting the attention to this very special class of algebras, but it has turned out that this class is definitely important. The main reason is of course that in case A is gentle (and only in this case), its trivial extension, and therefore its derived category can again be described by strings and bands. On the other hand, all the algebras with strings and bands are monitored by gentle algebras, thus the gentle algebras are really the blueprints for the general case.

As I mentioned, it was known for a long time that many classification problems in mathematics lead to $\mathbb{A}$-algebras. The focus on cluster algebras and the cluster tilted algebras has extended this considerably. Namely, it has turned out that some algebras related to Riemannian surfaces and their triangulations are gentle, as observed first by Brüstle and presented at the Toruń-ICRA in 2007. Then there is a paper by Assem, Brüstle, Charbonneau-Jodain, and Plamondon. We see an important link between the representation theory of finite dimensional algebras and topology. For quite a while Andrzej was reluctant to accept the relevance of cluster algebras and cluster tilted algebras, but this has changed completely in the last years.

6) The realm $\mathbb{D}$. In cooperation with Karin Erdmann and others, there are now several papers dealing with algebras which belong to the realm $\mathbb{D}$. These algebras are of a similar nature as the $\mathbb{A}$-algebras, but involve some further symmetries which may lead to Auslander-Reiten components of the form $\mathbb{D}_n$ or $\mathbb{D}_\infty$ (actually, this does not happen always, as the group algebra of the quaternion group shows: here all components just tubes). These cooperations were very fruitful. They provide spectacular results. They culminate in recent preprints which exhibit generalizations of weighted surface algebras (using virtual mutations as well as idempotent closure).

Typical algebras belonging to the realm $\mathbb{D}$ are the group algebras of the semidihedral groups and the generalized quaternion groups. Whereas for special biserial algebras the important relations are paths of length 2, here we deal with the situation that a path of length 2 is equal to a path of length $t \geq 3$; the first interesting case is $t = 3$ (and this is the case for the algebras studied now by Erdmann and Skowroński). Again, the algebras are defined in terms of some Riemann surface, using suitable arcs on the surface. The first algebras of this kind were presented by Schiffler, again in a lecture at ICRA XII in Toruń.

Degeneration arguments relate such algebras to special biserial ones and show in this way the tameness. Unfortunately, an explicit classification of the indecomposable modules is known only in special cases. Crawley-Boevey was able to correct some earlier inadequate attempts, but his approach does not work in general. There were several severe efforts to classify the representations of the quaternion group, but no-one succeeded until now.

And there is a very basic question. When I speak of realm $\mathbb{A}$ and realm $\mathbb{D}$, I have to admit that it is not clear at all, whether these are separate realms, or whether one has to consider the realm $\mathbb{A}$ as part of realm $\mathbb{D}$.

7) Covering theory. A very important tool in representation theory is covering theory, as developed by Gabriel and his school, and, equivalently, the use of group-graded modules, as introduced by Gordon and Green. In commutative ring theory, the use of graded modules was always appreciated. There is the curious fact that covering theory reveals that part of the module theory of a finitely generated commutative k-algebra has to be seen as the shadow, as the projection of a non-commutative world, the representations of the Beilinson quivers. This approach was (at least implicitly) often used when dealing with vector bundles say over projective spaces, but it seems that it was not understood otherwise. It was clear for Gabriel that the use of gradings corresponds to covering theory in topology, as presented in Gabriel-Zisman.

The papers by Dowbor and Skowroński devoted to covering theory are of great importance. The setting was prescribed by Gabriel and his school, but they restricted the attention to representation-finite algebras. It is really the general case which is needed.

As a first application of these considerations, Dowbor and Skowroński have looked at the special biserial algebras. They have shown in which way the band modules arise and that the band modules form controlled one-parameter families (indexed by punctured affine lines and controlled by string modules). Here, we should recall, that dealing with wild algebras, there is the notion of controlled wildness. Similarly, when dealing with tame algebras, one may ask in which way the one-parameter families are controlled.

III.

Andrzej usually was very optimistic. He was convinced that the typical behaviour of tame algebras is known and that at least a rough classification of the tame algebras should be available soon. He could not understand my hesitation, even when I pointed out the still missing knowledge about the simplest tame algebras, the domestic ones. Also, for a long time, the quaternion algebras were considered to be a completely isolated class of algebras (which in my opinion was rather strange). Fortunately, via the cluster approach, a general setting for the quaternion algebras has now been found. But still ...

Skowroński's aim was to develop a general theory of tame algebras. What has to be praised very highly, is his detailed study of some very interesting classes of tame algebras. Also, he has drawn the attention to the infinite radical of a module category and to some module theoretical phenomena which are relevant for dealing with algebras in general: for example directing modules and generalized standard components.

The representation theory of algebras was always proud that many parts of mathematics, of algebra and geometry, even of topology and analysis, can be transformed and reformulated in terms of representations of finite-dimensional algebras. But conversely, this means that representation theory is not just a cake walk, but incorporates and displays many different problems. This concerns first of all the huge world of wild algebras, but may-be also its tame boundary.

Andrzej had the dream to understand all tame algebras. Is it possible that, at present, this dream is still too ambitious?