# Wild Algebras: Two Examples.

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Let k be a field and Λ a finite-dimensional k-algebra (associative, with 1). A recent preprint by Chindris, Kinser and Weyman draws the attention to the present sheer ignorance concerning the possible behavior of wild algebras. The aim of this note is to exhibit two examples which answer questions mentioned in the paper.

1. Wild, Schur-representation-finite algebras of global dimension 2.

Chindris-Kinser-Weyman [CKW] have asked whether there do exists wild, Schur-representation-finite algebras of finite global dimension. Consider the algebra Λ given by the quiver Q
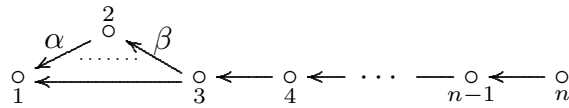
with $n \geq 3$ vertices and the zero relation $\alpha\beta$. Since Q is directed, its global dimension of Λ is finite. Actually, an easy calculation shows that the global dimension is equal to 2.

The vertex 2 is a node in the sense of Martinez [M], thus there is a natural bijection between the indecomposable non-simple representation of Λ, and the indecomposable non-simple representations of the following quiver:
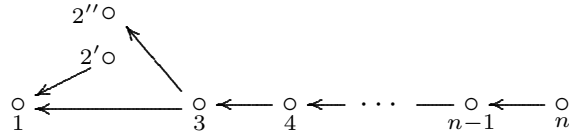
For $n \geq 9$, this is a wild quiver, thus, for $n \geq 9$, the algebra Λ is wild.

Lemma. *If M is an indecomposable representation of Λ such that M_α, M_β both are non-zero, then there is a non-zero endomorphism ϕ of M with $\phi^2 = 0$.*

Proof. If $M_\alpha \neq 0$, then the simple module $S(2)$ is a factor module of M, if $M_\beta \neq 0$, then $S(2)$ is a submodule of M. Thus, if both M_α, M_β are non-zero, we obtain an endomorphism ϕ of M with image $S(2)$. We must have $\phi^2 = 0$, since otherwise $S(2)$ is a direct summand of M; but then $M = S(2)$, impossible.

Corollary. *If M is a representation whose endomorphism ring $\mathrm{End}\, M$ is a division ring, then $M_\alpha = 0$ or $M_\beta = 0$, thus M is a representation of the $\mathbb{D}_n$ quiver obtained from Q by deleting α, or a representation of the $\mathbb{A}_n$ quiver obtained by deleting β.*

This shows that there are only finitely many isomorphism classes of representations M such that $\mathrm{End}(M)$ is a division ring, thus Λ is Schur-representation-finite.

2. A strictly wild algebra without a wild tilted factor algebra.

A conjecture by Yang Han [H] quoted in [CKW] asserts that any strictly wild algebra should have a factor algebra which is a wild tilted algebra.

Let k be an algebraically closed field and Λ the k-algebra with quiver and relations

$$\begin{array}{ccc} \underset{0}{\circ} & \xrightleftharpoons[\alpha_1]{\alpha_0} & \underset{1}{\circ} \\ & \xrightarrow{\beta} & \end{array} \qquad\qquad \beta\alpha_i\beta = 0 \ \text{ for } \ i = 0, 1.$$

Note that the subquiver given by the arrows α_0, α_1 is the Kronecker quiver. The representations of the Kronecker quiver are called Kronecker modules. Since we assume that k is algebraically closed, a Kronecker module is simple regular if and only if it is indecomposable and of length 2, and the simple regular Kronecker modules $R(\lambda)$ are indexed by the elements $\lambda \in \mathbb{P}^1(k) = k \cup \{\infty\}$.

Let $\mathcal{C}$ be the full subcategory of $\operatorname{mod}\Lambda$ given by all Λ-modules M such the restriction of M to the Kronecker quiver is a direct sum of simple regular Kronecker modules. Clearly, $\mathcal{C}$ is an abelian category (with an exact embedding into $\operatorname{mod}\Lambda$). If we endow the simple regular Kronecker module $R(\lambda)$ with β as zero map, we obtain a simple object in $\mathcal{C}$, we denote it again by $R(\lambda)$. Now let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ consisting of all modules M in $\mathcal{C}$ which have a submodule M' which is a direct sum of copies of $R(\infty)$ such that M/M' is a direct sum of modules of the form $R(\lambda)$ with $\lambda \in k$.

The category $\mathcal{D}$ is an abelian category with exact embedding functor into $\operatorname{mod}\Lambda$; its simple objects are the modules $R(\lambda)$ with $\lambda \in \mathbb{P}^1(k)$, and we have $\operatorname{Ext}^1_{\mathcal{D}}(R(\lambda), R(\mu)) = k$ provided $\lambda \in k$ and $\mu = \infty$ and equal to zero, otherwise. For $\lambda \in k$, a non-trivial element of $\operatorname{Ext}^1_{\mathcal{D}}(R(\lambda), R(\omega)) = k$ is given by using β.

Thus, the quiver Δ of $\mathcal{D}$ is a subspace quiver, the number of sources in Δ is equal to $|k|$. This shows that $\mathcal{D}$ and therefore $\operatorname{mod}\Lambda$ is strictly wild. Of course, instead of taking such a large subcategory $\mathcal{D}$, it would be sufficient to ask in the definition of $\mathcal{D}$ that M/M' is a direct sum of modules of the form $R(\lambda)$ with λ belonging to **a fixed 5-element subset** of k (so that one obtains the 5-subspace quiver).

On the other hand, any factor algebra of Λ is given by a quiver with at most 2 vertices (and some relations), and a wild tilted algebra with at most 2 vertices is hereditary (it is a generalized Kronecker algebra with at least 3 arrows). Of course, Λ has no such factor algebra.

References

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