Chapter 9: Weak approximation of SODEs

9.1 Motivation

In the previous chapter we analyzed the strong convergence of the stochastic theta method to the solution X of a stochastic differential equation

$$(9.1) \quad \begin{aligned} dX(t) &= f(t, X(t))\,dt + g(t, X(t))\,dW(t) \\ X(0) &= X_0 \end{aligned}$$

In particular, under suitable conditions on f, g, X_0 there exists a constant $C > 0$ such that

$$(9.2) \quad \left(\mathbb{E}\left[\max_{0 \le i \le M_h} |X_h(t_i) - X(t_i)|^2 \right] \right)^{1/2} \le C |h|^{1/2} \quad \forall h \text{ with } |h| < \bar{h},$$

where X_h denotes the grid fct. generated by the stochastic theta method. From (9.2) it follows that X_h gives a good pathwise approximation of the exact solution.

However, in many applications we are only interested in approximating an expected value of some quantity of $X(T)$, that is

$$\mathbb{E}[\varphi(X(T))]$$

for a fct. $\varphi: \mathbb{R}^m \to \mathbb{R}$.

Example 9.1

In the Black-Scholes model we encountered the problem to give an approximation of the discounted expected payoff

$$\mathbb{E}[e^{-rT} \max(0, S(T))], \qquad (\varphi(x) = e^{-rT} \max(0, x))$$

where $S(t)$ is the solution to the SODE

$$dS(t) = r S(t)\,dt + \sigma S(t)\,dW(t), \quad S(0) = S_0.$$

In this chapter we will work with following assumptions:

(~~PEAK~~) $\varphi \in C_p^2(\mathbb{R})$, that is $\varphi \in C^2(\mathbb{R})$ and $\exists\, C, q > 0$
$\hat{}$ polynomial growth

such that $|\varphi^{(j)}(x)| \le C(1+|x|^q) \quad \forall x \in \mathbb{R}$

Further, we only consider scalar SODEs $(m=d=1)$ with the usual Lipschitz- and linear growth conditions on $f: [0,T] \times \mathbb{R} \to \mathbb{R}$, $g: [0,T] \times \mathbb{R} \to \mathbb{R}$.

Our aim is to prove, that $\forall \varphi \in C_p^2(\mathbb{R})\ \exists\, C(\varphi) > 0$ s.t

$$(9.3) \quad |\mathbb{E}[\varphi(X(T)) - \varphi(X_h(T))]| \le C(\varphi)\, |h|^\gamma .$$

We refer to (9.3) as the weak error of convergence of the stochastic theta method (with order $\gamma = 1$)

Remark 9.2

From (9.2) it directly follows the weak convergence with order $\gamma = \frac{1}{2}$:

$$|\mathbb{E}[\varphi(X(T)) - \varphi(X_h(T))]| \underset{\text{Taylor}}{=} \left|\mathbb{E}\left[\int_0^1 \varphi'(X_h(T) + s(X(T)-X_h(T)))\,ds\,(X(T)-X_h(T))\right]\right|$$

$$\underset{\text{Cauchy-Schwarz}}{\le} \left(\mathbb{E}\left[\left(\int_0^1 \varphi'(X_h(T)+s(X(T)-X_h(T)))\,ds\right)^2\right]\right)^{1/2} \underbrace{\left(\mathbb{E}[|X(T)-X_h(T)|^2]\right)^{1/2}}_{\le C|h|^{1/2} \text{ strong error}}$$

Need to estimate

$$\mathbb{E}\left[\left(\int_0^1 \varphi'(X_h(T)+s(X(T)-X_h(T)))\,ds\right)^2\right]$$

$$\underset{\text{Jensen}}{\le} \mathbb{E}\left[\int_0^1 \left(\varphi'(X_h(T)+s(X(T)-X_h(T)))\right)^2 ds\right] \underset{\text{polynomial growth } 0 \le s \le 1}{\le} \mathbb{E}\left[\int_0^1 C(\varphi)\left(1+|X_h(T)|^{2q}+|X(T)|^{2q}\right) ds\right]$$

$$\leq C(\varphi)\left(1+\mathbb{E}[|X_h(T)|^{2q}]+\mathbb{E}[|X(T)|^{2q}]\right)$$

That $\mathbb{E}[|X(T)|^{2q}]<\infty$ is shown as in the estimate in Chapter 7. but we need the additional assumption $\mathbb{E}[|X_0|^{2q}]<\infty$.
(for example $X_0 = x_0 \in \mathbb{R}$ constant random variable)
Similar for the term $\mathbb{E}[|X_h(T)|^{2q}]$ (see at the end).

$\square$

9.2 Kolmogorov backward equation

For the analysis of the weak error of convergence we will make use of a link between SODEs and PDEs

For this we consider the PDE (Kolmogorov backward equation)

$$(9.4)\quad \begin{cases} \frac{\partial}{\partial t}u(t,x)+\frac{1}{2}a(t,x)\frac{\partial^2}{\partial x^2}u(t,x)+f(t,x)\frac{\partial}{\partial x}u(t,x)=0, & x\in\mathbb{R},\ t\in[0,T] \\ u(T,x)=\varphi(x) \end{cases}$$

where $a:[0,T]\times\mathbb{R}\to[0,\infty)$, $f:[0,T]\times\mathbb{R}\to\mathbb{R}$, $\varphi:\mathbb{R}\to\mathbb{R}$ are continuous.
Define

$$g(t,x)=\sqrt{a(t,x)}$$

and assume further that f,g satisfy a global Lipschitz condition. Then the SODE

$$(9.5)\quad \begin{aligned} dZ(s)&=f(s,Z(s))\,ds+g(s,X(s))\,dW(s) \quad \text{on } s\in[t,T] \\ Z(t)&=x \end{aligned}$$

has a unique solution, which we denote by $Z(s;t,x)$.

The link between (9.4) and (9.5) is given by the following special case of the Feynman-Kac formula:

Theorem 9.3

Assume that there exists a smooth solution u to (9.4), that is $u \in C^{2,1}([0,T] \times \mathbb{R})$ and $u(t,\cdot) \in C_p^2(\mathbb{R})$ (uniformly in t)

Then for all $t \le s \le T$

$$u(t,x) = \mathbb{E}[\, u(s, Z(s;t,x))] \qquad \forall x \in \mathbb{R}$$

In particular

$$(9.6) \qquad u(t,x) = \mathbb{E}[\, \varphi(Z(T;t,x))] \qquad \forall x \in \mathbb{R}$$

Proof:

Apply Itô's formula to $u(s, Z(s;t,x)) =: u(s, Z(s))$

$$du(s,Z(s)) = \underbrace{\Big(\frac{\partial}{\partial t} u(s,Z(s)) + \frac{\partial}{\partial x} u(s,Z(s))\, f(s,Z(s)) + \frac{1}{2}\frac{\partial^2}{\partial x^2} u(s,Z(s))\, g^2(s,Z(s)) \Big)}_{\underset{(9.4)}{\equiv} 0 \ \forall s} ds$$

$$+ \frac{\partial}{\partial x} u(s,Z(s))\, g(s,Z(s))\, dW(s)$$

or in integral form

$$u(T,Z(T)) - u(s,Z(s)) = \underbrace{\int_s^T \frac{\partial}{\partial x} u(\eta, Z(\eta))\, g(\eta, Z(\eta))\, dW(\eta)}$$

Well-defined using polynomial growth of u_x, linear growth of g, and existence of all moments of $Z(s)$!

Taking expectation: $\forall\, t \le s \le T$:

$$\Rightarrow \quad \mathbb{E}[u(T,Z(T))] = \mathbb{E}[u(s,Z(s))] \quad \Big(\underset{t=s}{=} \mathbb{E}[u(t, \underbrace{Z(t;t,x)}_{x})] = u(t,x) \Big)$$

$$\| $$

$$\mathbb{E}[\varphi(Z(T;t,x))]$$

$\square$

(Start here with lecture on Friday)

Example 9.4

Feynman-Kac formula gives explicit ~~..~~ representation of solutions to PDEs:
Consider the heat equation

$$\begin{cases} \frac{\partial}{\partial t} v = \frac{1}{2} \frac{\partial^2}{\partial x^2} v, & \text{on } [0,T] \times \mathbb{R} \\ v(0,x) = \varphi(x) & \forall x \in \mathbb{R} \end{cases} \qquad (9.7)$$

Transform (9.7) into a backward-equation by setting

$$u(t,x) = v(T-t,x) \quad (u(T-\tilde{t},x) = v(\tilde{t},x))$$

Then

$$\frac{\partial}{\partial t} u(t,x) = (-1) \frac{\partial}{\partial t} v(T-t,x) \underset{\substack{\uparrow \\ (9.7)}}{=} (-1) \frac{\partial^2}{\partial x^2} v(T-t,x) = (-1) \frac{\partial^2}{\partial x^2} u(t,x)$$

that is, u solves

$$\begin{cases} \frac{\partial}{\partial t} u + \frac{1}{2} \frac{\partial^2}{\partial x^2} u = 0 \\ u(T,x) = \varphi(x). \end{cases}$$

The corresponding SODE is

$$dZ(s) = 1 \, dW(s), \quad Z(t) = x$$

with solution $\quad Z(s;t,x) = W(s) - W(t) + x.$

Hence

$$u(t,x) \underset{\text{Th. 9.3}}{\Longrightarrow}= \mathbb{E}[\varphi(Z(T;t,x))] = \mathbb{E}[\varphi(\underbrace{W(T)-W(t)}_{\sim N(0,T-t)}+x)]$$

$$\underset{(T-t>0)}{=} \int_{\mathbb{R}} \varphi(y+x) f_{0,T-t}(y)\, dy$$

$$= \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} \varphi(y+x) e^{-\frac{1}{2}\frac{y^2}{(T-t)}} dy$$

$$= \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} \varphi(y) e^{-\frac{1}{2}(x-y)^2/(T-t)} dy \qquad \square$$

or $v(t,x) = u(T-t,x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \varphi(y) e^{-\frac{(x-y)^2}{2t}} dy$

Example 9.5 (perhaps skip this example)

Consider the geo. B. M.

$$dS(t) = \mu \; S(t)\, dt + \sigma S(t)\, dW(t), \quad S(0) = S_0,$$

where $S_0 \in [0,\infty)$ is deterministic, $\mu, \sigma > 0$,

We want to approximate $\mathbb{E}[\varphi(S(T))]$ for some $\varphi: \mathbb{R} \to \mathbb{R}$

The corresponding K.B. Eq is

$$(9.8) \quad \begin{cases} \frac{\partial}{\partial t} u(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} u(t,x) + \mu x \frac{\partial}{\partial x} u(t,x) = 0 \\ u(T,x) = \varphi(x) \end{cases}$$

Since we do not know if there exists a solution to (9.8) we apply the ansatz:

$$(9.9) \quad \hat{u}(t,x) = \mathbb{E}[\varphi(S(T;t,x))]$$

Since (general initial data):

$$S(T;t,x) = x \exp\left((\mu - \tfrac{1}{2}\sigma^2)(T-t) + \sigma (W(T) - W(t)) \right)$$

we get

$$\hat{u}(t,x) = \mathbb{E}\left[\varphi\left(x \cdot \exp\left((\mu - \tfrac{1}{2}\sigma^2)(T-t) + \sigma (W(T) - W(t)) \right)\right)\right]$$

$$= \int_{\mathbb{R}} \varphi\left(x \cdot \exp\left((\mu - \tfrac{1}{2}\sigma^2)(T-t) + \sigma \sqrt{T-t}\, y \right)\right) f_{0,1}(y)\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi\left(x \cdot \exp\left((\mu - \tfrac{1}{2}\sigma^2)(T-t) + \sigma \sqrt{T-t}\, y \right)\right) \exp\left(-\tfrac{1}{2}y^2\right) dy$$

As one can see, $\hat{u}(t,x)$ is differentiable w.r.t (t,x):

$$\frac{\partial}{\partial t} \hat{u}(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi'\left(x \cdot \exp\left((\mu - \tfrac{1}{2}\sigma^2)(T-t) + \sigma \sqrt{T-t}\, y \right)\right) \cdot \left(-(\mu - \tfrac{1}{2}\sigma^2) + \tfrac{1}{2}\sigma y \cdot \frac{1}{\sqrt{T-t}} \right) \exp\left(-\tfrac{1}{2}y^2\right) dy$$

and similar for $\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$. It is an exercise to verify that $\hat{u}(t,x)$ is a solution to (9.8).

(Example 9.5 continued:)

Remember that we want to approximate

$$\mathbb{E}[\varphi(S(T;t,x))]$$

By using (9.9) we can do this by approximating (9.8), using numerical methods for deterministic PDEs.

$\square$

The following theorem (which we quote from Friedman [1975]) gives sufficient conditions on f, g, such that the K.B. Eq has a smooth solution.

Theorem 9.6

Let $f, g \in C([0,T]\times\mathbb{R})$, such that $f(t,\cdot), g(t,\cdot) \in C^2(\mathbb{R})$ with bounded derivatives and $\varphi \in C_p^2(\mathbb{R})$.

Then the function (Z is the solution to (9.5))

$$u(t,x) = \mathbb{E}[\varphi(Z(T;t,x))]$$

has continuous derivatives $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$ in $[0,T]\times\mathbb{R}$ and is a solution to (9.4).

Moreover, $\exists\, C, q > 0$ independent of t such that

$$\left|\frac{\partial}{\partial x} u(t,x)\right| + \left|\frac{\partial^2}{\partial x^2} u(t,x)\right| \le C(1+|x|^q).$$

(The proof relies on differentiation of $Z(s;t,x)$ w.r.t x)

9.3 Weak convergence of Euler-Maruyama

As above we consider the SODE

$$dX(t) = f(t, X(t))\,dt + g(t, X(t))\,dW(t) \qquad (9.10)$$
$$X(0) = x_0$$

with $x_0 \in \mathbb{R}$, f, g as in Th. 9.6

Then the corresponding K.B.Eq

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) + \frac{1}{2} g^2(t,x) \frac{\partial^2}{\partial x^2} u(t,x) + f(t,x) \frac{\partial}{\partial x} u(t,x) = 0 \\ u(T,x) = \varphi(x) \end{cases} \qquad (9.11)$$

has a smooth solution for every $\varphi \in C_p^2(\mathbb{R})$

Theorem 9.7 In addition to the assumptions of Th. 9.6 let the mappings $\begin{cases} (t,x) \mapsto u_{xx}(t,x)(g^2(s,y) - g^2(t,x)) \\ (t,x) \mapsto u_x(t,x)(f(s,y) - f(t,x)) \end{cases}$ be sufficiently smooth such that all derivatives which appear in the proof have polynomial growth.

Then the Euler-Maruyama method is weakly convergent with order $\gamma = 1$.

Proof: By Th. 9.3 we have

$$u(0, x_0) = \mathbb{E}[\varphi(X(T))] \qquad (9.12)$$

where u solves (9.11).

Next, we interpolate the grid functions X_h to a continuous Itô-process by setting

$$X_h(t) = x_0 + \int_0^t \bar{f}(s)\,ds + \int_0^t \bar{g}(s)\,dW(s)$$

where

$$\bar{f}(s) = \sum_{i=1}^{M_h} \mathbb{1}_{(t_{i-1}, t_i]}(s) f(t_{i-1}, X_h(t_{i-1})), \quad \bar{g}(s) = \sum_{i=1}^{M_h} \mathbb{1}_{(t_{i-1}, t_i]}(s)\, g(t_{i-1}, X_h(t_{i-1}))$$

An application of Itô's formula yields

$$u(T, X_h(T)) - u(0, X_h(0))$$

$$= \int_0^T u_t(s, X_h(s)) + u_x(s, X_h(s))\, \bar{f}(s) + \frac{1}{2} u_{xx}(s, X_h(s))\, \bar{g}^2(s)\, ds$$

$$+ \int_0^T u_x(s, X_h(s))\, \bar{g}(s)\, dW(s).$$

(9.11) gives us the relation

$$u_t(s, X_h(s)) = - f(s, X_h(s)) \frac{\partial}{\partial x} u(s, X_h(s)) - \frac{1}{2} g^2(s, X_h(s)) \frac{\partial^2}{\partial x^2} u(s, X_h(s)).$$

Hence

$$u(T, X_h(T)) - u(0, \underbrace{X_h(0)}_{\overset{\shortparallel}{X_0}}) = \int_0^T u_x(s, X_h(s)) \left(\bar{f}(s) - f(s, X_h(s)) \right) + \frac{1}{2} \frac{\partial^2}{\partial x^2} u(s, X_h(s)) \left(\bar{g}^2(s) - g(s, X_h(s)) \right) ds$$

$$+ \int_0^T u_x(s, X_h(s))\, \bar{g}(s)\, dW(s)$$

Therefore

$$\mathbb{E}[\varphi(X_h(T)) - \varphi(X(T))]$$

$$\overset{(9.12)}{=} \mathbb{E}[u(T, X_h(T)) - u(0, x_0)]$$

$$= \int_0^T \mathbb{E}\left[u_x(s, X_h(s)) \left(\bar{f}(s) - f(s, X_h(s)) \right) + \frac{1}{2} \frac{\partial^2}{\partial x^2} u(s, X_h(s)) \left(\bar{g}^2(s) - g(s, X_h(s)) \right) \right] ds$$

$$= \sum_{i=1}^{M_n} \int_{t_{i-1}}^{t_i} \mathbb{E}[u_x(s, X_h(s)) (f(t_{i-1}, X_h(t_{i-1})) - f(s, X_h(s)))]\, ds$$

$$+ \sum_{i=1}^{M_n} \frac{1}{2} \int_{t_{i-1}}^{t_i} \mathbb{E}[u_{xx}(s, X_h(s)) (g^2(t_{i-1}, X_h(t_{i-1})) - g^2(s, X_h(s)))]\, ds$$

It remains to show

$$(9.13) \quad \sup_{s \in (t_{i-1}, t_i)} \left| \mathbb{E}[u_{xx}(s, X_h(s)) (g^2(t_{i-1}, X_h(t_{i-1})) - g^2(s, X_h(s))] \right| \le C|h|$$

and

$$(9.14) \quad \sup_{s \in (t_{i-1}, t_i)} \left| \mathbb{E}[u_x(s, X_h(s)) (f(t_{i-1}, X_h(t_{i-1})) - f(s, X_h(s)))] \right| \le \underset{\text{indep. of } i}{\underset{\uparrow}{C}} |h|$$

~~and~~

~~(9.15) $\sup_{s\in(t_{i-1},t_i)} (1-\theta) \,|\, \mathbb{E}[u_x(s,X_h(s))(f(t_{i-1},X_h(t_{i-1}))-f(s,X_h(s))]\,| \le Ch$~~

If this is established we have shown

$$|\mathbb{E}[\varphi(X_h(T)) - \varphi(T)]| \le CT\,|h|$$

which is the weak convergence with order $\gamma = 1$.

We only demonstrate the estimate (9.13).

For this define for $t\in[t_{i-1},t_i]$

$$v(t,x,y) := u_{xx}(t,x)\,(g^2(t_{i-1},y) - g^2(t,x))$$

Note that (9.13) has the form

$$|\mathbb{E}[v(s,X_h(s),X_h(t_{i-1}))]|$$

In particular $v(t_{i-1},X_h(t_{i-1}),X_h(t_{i-1})) = 0$.

Apply Itô's formula to $t\mapsto v(t,X_h(t),y)$, $y\in\mathbb{R}$ fixed, to get

$$v(s,X_h(s),y) - v(t_{i-1},X_h(t_{i-1}),y) = \int_{t_{i-1}}^{s} v_t + v_x\bar f + \tfrac12 v_{xx}\bar g^2\,d\eta + \underbrace{\int_{t_{i-1}}^{s} v_x\bar g\,dW(\eta)}_{\text{vanishes!}}$$

$$\Rightarrow |\mathbb{E}[v(s,X_h(s),X_h(t_{i-1}))]| = \left|\int_{t_{i-1}}^{s}\mathbb{E}[v_t(\eta,X_h(\eta),X_h(t_{i-1})) + v_x(\dots)\bar f(\eta) + \tfrac12 v_{xx}(\dots)\bar g^2(\eta)]\,d\eta\right|$$

$$\le \int_{t_{i-1}}^{t_i}\mathbb{E}\Big[|v_t| + |v_x(\dots)\underbrace{\bar f(\eta)}_{f(t_{i-1},X_h(t_{i-1}))}| + \tfrac12|v_{xx}(\dots)\underbrace{\bar g^2(\eta)}_{=g^2(t_{i-1},X_h(t_{i-1}))}|\Big]\,d\eta$$

$$\le C h_i\,\Big(1 + \underbrace{\mathbb{E}[\max_{0\le i\le M_h}|X_h(t_i)|^q]}_{<+\infty}\Big) \le C|h| \qquad \square$$

$\uparrow$ polynomial growth of v_t, v_x etc.

Lemma 9.8

Assume $\mathbb{E}[|X_0|^{2q}]<\infty$ for $q\geq 1$. Then $\exists\, C$ and $\bar{h}>0$ such that $\mathbb{E}\big[\sup_{0\leq i\leq M_h}|X_h(t_i)|^{2q}\big]\leq C$ $\quad\forall h$ with $|h|<\bar{h}$.

Proof

$$\Big(\mathbb{E}\big[\sup_{0\leq i\leq M_h}|X_h(t_i)|^{2q}\big]\Big)^{1/2q}=\|X_h\|_{2q,h,0}$$

$$\leq C_1\|L_h(X_h)\|_{2q,h,-1}=C_1\Big(\|X_0\|_{L^{2q}(\Omega)}+\Big(\mathbb{E}\Big[\max_{1\leq j\leq M_h}\Big|\sum_{i=1}^{j}X_h(t_i)-X_h(t_{i-1})\Big|^{2q}\Big]\Big)^{1/2q}\Big)$$

L_h is stable

and

$$\Big(\mathbb{E}\Big[\max_{1\leq j\leq v}\Big|\sum_{i=1}^{j}X_h(t_i)-X_h(t_{i-1})\Big|^{2q}\Big]\Big)^{1/2q}$$

$$\leq\Big(\mathbb{E}\Big[\max_{1\leq j\leq v}\Big|\sum_{i=1}^{j}(1-\theta)h_i f(t_{i-1},X_h(t_{i-1}))\Big|^{2q}\Big]\Big)^{1/2q}$$

$$+\Big(\mathbb{E}\Big[\max_{1\leq j\leq v}\Big|\sum_{i=1}^{j}\theta h_i f(t_i,X_h(t_i))\Big|^{2q}\Big]\Big)^{1/2q}$$

$$+\Big(\mathbb{E}\Big[\max_{1\leq j\leq v}\Big|\sum_{i=1}^{j}g(t_{i-1},X_h(t_{i-1}))\Delta W_i\Big|^{2q}\Big]\Big)^{1/2q}$$

$$\leq T^{\frac{2q-1}{2q}}\Big(\mathbb{E}\Big[\max_{1\leq j\leq v}\sum_{i=1}^{j}(1-\theta)^{2q}h_i L^{2q}(1+|X_h(t_{i-1})|^{2q})\Big]\Big)^{1/2q}$$

$$+T^{\frac{2q-1}{2q}}\Big(\mathbb{E}\Big[\max_{1\leq j\leq M_h}\sum_{i=1}^{j}\theta^{2q}h_i L^{2q}(1+|X_h(t_i)|^{2q})\Big]\Big)^{1/2q}$$

$$+C(q)\,T^{\frac{2q-2}{4q}}\Big(\underbrace{\int_0^{t_v}\mathbb{E}\Big[\Big|\sum_{i=1}^{v}\mathbb{1}_{[t_{i-1},t_i]}(s)\,g(t_{i-1},X_h(t_{i-1}))\Big|^{2q}\Big]ds}_{\leq\sum_{i=1}^{M_h}h_i L^{2q}(1+|X_h(t_{i-1})|^{2q})}\Big)^{1/2q}$$

$$\leq L\,T^{\frac{2q-1}{2q}}(1+C(q))\Big(\sum_{i=1}^{v}h_i\,\mathbb{E}[|X_h(t_i)|^{2q}]\Big)^{1/2q}\leq L\,T^{\frac{2q-1}{2q}}(1+C(q))\Big(\sum_{i=1}^{v}h_i\,\mathbb{E}\big[\max_{0\leq i\leq v}|X_h(t_i)|^{2q}\big]\Big)^{1/2q}$$

Then take $(\ )^{2q}$, bring highest summand on other side, apply discrete Gronwall. $\square$

For example 9.5 (Solution to the exercise) Appendix

$$\frac{d}{dt}\,\varphi\left(x\cdot \exp\left((\mu-\tfrac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\,y\right)\right)$$

$$= \varphi'(x\cdot \ldots)\cdot x\cdot \exp\left((\mu-\tfrac{1}{2}\sigma^2)(T-t)\ldots\right)\cdot\left((-\mu+\tfrac{1}{2}\sigma^2)-\frac{1\sigma y}{2\sqrt{T-t}}\right)$$

(Similar for $\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2} \ldots$)

$\hat{u}(t,x) =$

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \varphi\left(x\cdot \exp\left((\mu-\tfrac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\,y\right)\right)\exp\left(-\tfrac{1}{2}y^2\right)dy$$

$$\frac{\partial}{\partial x}\varphi(\ldots) = \varphi'(\ldots)\cdot\exp(\ldots) \qquad \frac{\partial}{\partial y}\varphi'(\ldots) = \varphi''(\ldots)\, x\, \sigma\sqrt{T-t}\,\exp(\ldots)$$

$$\frac{\partial^2}{\partial x^2}\varphi(\ldots) = \varphi''(\ldots)\,(\exp(\ldots))^2$$

For the second term use integration by parts to obtain

$$\int_{\mathbb{R}} \frac{\partial^2}{\partial x^2}\varphi(\ldots)\exp\left(-\tfrac{y^2}{2}\right)dy = \int_{\mathbb{R}} \varphi''(\ldots)\exp(\ldots)\exp(\ldots)\exp\left(-\tfrac{y^2}{2}\right)dy$$

$$= 0 - \int_{\mathbb{R}} \varphi'(\ldots)\,\frac{1}{x\sigma\sqrt{T-t}}\left(\exp(\ldots)(-y)\exp\left(-\tfrac{y^2}{2}\right)\right)dy$$

$$= \int_{\mathbb{R}} \varphi'(\ldots)\,\frac{1}{x\cancel{\sigma\sqrt{T-t}}}\left(\cancel{\sigma\sqrt{T-t}}\exp(\ldots)\exp\left(-\tfrac{y^2}{2}\right)\right)dy$$

Plug this into (9.8) to obtain:

$\tilde{u}(t,x) = \mathbb{E}[\varphi(S(T;t,x))]$ (is a strong smooth solution to (9.8))

$\square$