Summer term 2023
Prof. Dr. Michael Röckner and Sebastian Grube

Exercises to Introduction to Stochastic Partial Differential Equations I

Sheet 13
Total points: 8+4*
Submission before: Friday, 07.07.2023, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Problem 1 (cf. [BS88, Chapter 3]). (2*+4+1* Points)

Let $(V_i, ||\cdot||_{V_i}), i = 1, 2$, be Banach spaces and V a Hausdorff topological vector space such that

$$V_i \subset V \text{ continuously}, i = 1, 2. \tag{I}$$

(i*) Prepare to present that $(V_1 \cap V_2, ||\cdot||_{V_1 \cap V_2}))$ and $(V_1 + V_2, ||\cdot||_{V_1+V_2})$ are Banach spaces, where

$$||v||_{V_1 \cap V_2} := \max\{||v||_{V_1}, ||v||_{V_2}\}, \quad v \in V_1 \cap V_2,$$
$$||v||_{V_1+V_2} := \inf\{||v_1||_{V_1} + ||v_1||_{V_2} : v = v_1 + v_2\}, \quad v \in V_1 + V_2.$$

Hint: You may consult [BS88, Chapter 3][1].

(ii) Assume that $V_1 \cap V_2 \subset V_i$ is dense for $i = 1, 2$. Prove that

$$(V_1 \cap V_2)^* = V_1^* + V_2^*.$$

Here '=' means that both sides are isomorphic.

(iii*) What changes in (i*) and (ii) if we replace V by a different Hausdorff topological vector space $\bar{V}$ such that (I) holds?

Problem 2. (4 Points)

Consider the situation of the proof of Theorem 4.2.5. Assume that $[0, T] \ni t \mapsto ||X(t)||_H$ is lower-semicontinuous. Let $I \subset [0, T]$ be a dense set. Show that

$$\sup_{t\in[0,T]} ||X(t)||_H = \sup_{t\in I} ||X(t)||_H.$$

Problem 3 (Presentation). (1 extra point for presentation)

Prepare to present the proof of the following statement. Let μ be a Gaussian measure on a Hilbert space U with mean zero and covariance operator Q. For $h \in Q^{\frac{1}{2}}(U)$, define the map $T_h(x) := x + h, x \in U$. Then the measure $\mu \circ T_h^{-1}$ is absolutely continuous with respect to μ with Radon–Nikodym derivative

$$\frac{d(\mu \circ T_h^{-1})}{d\mu}(x) = \exp(\langle Q^{-\frac{1}{2}}h, Q^{-\frac{1}{2}}x\rangle_U - \frac{1}{2}||Q^{-\frac{1}{2}}h||_U^2).$$

Here $\langle Q^{-\frac{1}{2}}h, Q^{-\frac{1}{2}}x\rangle_U$ is defined as the sum of the series

$$\sum_{j=1}^{\infty} \frac{\langle x, e_j\rangle_U \langle h, e_j\rangle_U}{\lambda_j}$$

for $Qe_j = \lambda_j e_j$, which converges in $L^2(U; \mu)$.

Hint: You may consult [DPZ92, Proposition 2.20][2]

[1] available for free within Bielefeld University's network (also via VPN) on https://www.sciencedirect.com/bookseries/pure-and-applied-mathematics/vol/129/suppl/C

[2] available for free within Bielefeld University's network (also via VPN) on https://www.cambridge.org/core/books/stochastic-equations-in-infinite-dimensions/17CB9F06965F01647D576C62D28049F6

Literatur

[BS88] Colin Bennett and Robert Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[DPZ92] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.