Winter term 2023/2024
Lecture: Prof. Dr. Michael Röckner
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Exercises to Introduction to Stochastic Partial Differential Equations II

Sheet 1
Total points: 10
Submission before: Friday, 20.10.2023, 12:00 noon

Problem 1. (4 Points)

The aim of this exercise is to show that $b(t, X)$ and $\sigma(t, X)$ of equation (E.0.1) only depend on $\{X_s | s \leqslant t\}$.

(i) (2 Points) Let $(\Omega, \mathcal{A})$, $(\Omega', \mathcal{A}')$ be two measure spaces. Let $T : \Omega \to \Omega'$ be $\mathcal{A}/\mathcal{A}'$-measurable. Let $X : \Omega \to \mathbb{R}$ be a random variable. Show that X is measurable w.r.t. $\sigma(T) := T^{-1}(\mathcal{A}')$ if and only if there exists a map $\varphi : \Omega' \to \mathbb{R}$ which is $\mathcal{A}'/\mathcal{B}(\mathbb{R})$-measurable such that $X = \varphi \circ T$.

Hint: Prove the statement first if X is an indicator function.

(ii) (2 Points) Use (i) (you are allowed to assume that (i) holds if $\mathbb{R}$ is replaced by E) to conclude that for every $t \geqslant 0$ the map $b(t, X) : \omega \mapsto b(t, X(\omega))$ of equation (E.0.1) only depends on $\{X_s | s \leqslant t\}$.

Problem 2. (3 Points)

Let $(\Omega, \mathcal{A})$, $(\Omega', \mathcal{A}')$ be two measure spaces. Let μ be a measure on $(\Omega, \mathcal{A})$ and $X : \Omega \to \Omega'$ be $\mathcal{A}/\mathcal{A}'$-measurable. Denote by $\overline{\mathcal{A}}^{\mu}$ and $\overline{\mathcal{A}'}^{\mu \circ X^{-1}}$ the completion of $\mathcal{A}$ w.r.t. μ and $\mathcal{A}'$ w.r.t. $\mu \circ X^{-1}$. Show that X is $\overline{\mathcal{A}}^{\mu} / \overline{\mathcal{A}'}^{\mu \circ X^{-1}}$-measurable.

Problem 3. (3 Points)

For fixed b and σ as in Appendix E assume that the SDE (E.0.1)

$$dX(t) = b(t, X)dt + \sigma(t, X)dW(t), \quad t \in [0, \infty),$$

has a unique strong solution as in Definition E.0.6. Prove that weak uniqueness for the above SDE holds.

Hint: See Remark E.0.7.