Winter term 2023/2024
Lecture: Prof. Dr. Michael Röckner
Exercises: Dr. Sebastian Grube

Exercises to Introduction to Stochastic Partial Differential Equations II

Sheet 10
Total points: 12
Submission before: Friday, 22.12.2023, 12:00 noon

Problem 1. (3 Points)

Consider the proof of 'Claim' in Example 5.1.10. Show that by (5.1.25) and (5.1.17), respectively, it follows that for some constant $C \in]0,\infty[$ and all $v, w \in V$

(i) $|{}_{V^*}\langle F(w), v\rangle_V| \leqslant C||w||^2_{L^4(\Lambda;\mathbb{R}^2)}||v||_V$;

(ii) $|{}_{V^*}\langle F(w), v\rangle_V| \leqslant C||w||_V^{\frac{3}{2}}||w||_H^{\frac{1}{2}}||v||_{L^4(\Lambda;\mathbb{R}^2)}$.

Furthermore, cf. Remark 5.1.11, use (5.1.18) (instead of (5.1.17)) in order to prove that for all $v, w \in V$

(iii) $|{}_{V^*}\langle F(w), v\rangle_V| \leqslant C||w||_V^{\frac{7}{4}}||w||_H^{\frac{1}{4}}||v||_{L^4(\Lambda;\mathbb{R}^3)}$.

Problem 2. (4 Points)

Prove the claimed equivalence in Remark 5.2.4 in the lecture notes.

Problem 3. (2 Points)

Prove Theorem 5.2.2 (ii) for $h \equiv 1$ and $g(x) := C(x+1)^\gamma$ for some $C > 0$ and $\gamma > 1$, i.e. show that for $T_0 \in]0,T]$ we have $L_{x_0}(T_0) < \sup_{x\in(0,\infty)} G_{x_0}(x)$ if and only if

$$T_0 < \frac{1}{C(\gamma-1)\left(||u_0||_H^2 + \int_0^{T_0} f(s)ds + 1\right)^{\gamma-1}}.$$

Furthermore, take the limit $\gamma \downarrow 1$.

Problem 4 (Bihari's inequality)**.** (3 Points)

Consider the situation of Lemma 5.2.8. Let g and h be as below. Check if g and h are admissible. If yes, calculate G and its inverse G^{-1}. For which $T_0 > 0$ is inequality (5.2.8) still satisfied?

(i) $g(x) := C(x+1)^\gamma, x \in [0,\infty[$, for some $C > 0$, $\gamma > 1$ and $h \equiv 1$;

(ii) $g(x) := Cx^\gamma, x \in [0,\infty[$, for some $C > 0$, $0 < \gamma < 1$ and $h(t) = t^{-\frac{1}{2}}, t \in]0,\infty[$;

(iii) $g(x) := Cx^\gamma, x \in [0,\infty[$, for some $C > 0$, $\gamma > 1$ and $h(t) := t^{-1}, t \in]0,\infty[$.