Winter term 2023/2024
Lecture: Prof. Dr. Michael Röckner
Exercises: Dr. Sebastian Grube

Exercises to Introduction to Stochastic Partial Differential Equations II

Sheet 12
Total points: 12+2*
Submission before: Friday, 19.12.2023, 12:00 noon

Problem 1. (2 Points)

In the proof of Lemma 5.2.14 show that

$$t\,{}_{V^*}\langle A_0(z), u-v\rangle_V - (1-t)\,{}_{V^*}\langle A_0(u_n), u_n-u\rangle_V \geqslant t\,{}_{V^*}\langle A_0(u_n), u_n-v\rangle_V - {}_{V^*}\langle A_0(z), u_n-u\rangle_V$$

holds.

Problem 2. (2 Points)

Remark 5.2.8: Let A_1, A_2 be as in the assertion Theorem 5.2.7, let Y be a solution (in the sense of Definition 5.1.2) to

$$dY(t) = A_1(t, Y(t))dt + B(t)dW(t), \quad t \in [0,T], Y(0) = 0,$$

(which exists due to Theorem 5.1.3). Let X be as in Theorem 5.2.7 a local solution up to a stopping time τ to

$$dX(t) = [A_1(t, X(t)) + A_2(t, X(t))]dt + B(t)dW(t), \quad t \in [0,T], X(0) = X_0.$$

Set $u := X - Y$. Show that (on Ω) u satisfies the deterministic evolution equation

$$u(t) = u_0 + \int_0^t \tilde{A}(s, u(s))ds, \quad t \in [0, \tau],$$

where $u_0 := X_0$ and

$$\tilde{A}(t,v) := A_1(t, v + \bar{Y}(t)) - A_1(t, \bar{Y}(t)) + A_2(t, v + \bar{Y}(t)), \quad v \in V, t \in [0,T],$$

where $\bar{Y}$ is a V-valued progressively measurable version of Y.

Problem 3. (2 Points)

In the proof of Theorem 5.2.7: Let A_1, A_2 be as in the assertion of Theorem 5.2.7, let Y be a solution (in the sense of Definition 5.1.2) to

$$dY(t) = A_1(t, Y(t))dt + B(t)dW(t), \quad t \in [0,T], Y(0) = 0,$$

$\bar{Y}$ a V-valued progressively measurable version of Y and

$$\tilde{A}(t,v) := A_1(t, v + \bar{Y}(t)) - A_1(t, \bar{Y}(t)) + A_2(t, v + \bar{Y}(t)), \quad v \in V, t \in [0,T].$$

If u satisfies

$$u(t) = u_0 + \int_0^t \tilde{A}(s, u(s))ds, \quad t \in [0, \tau],$$

with $u_0 := X_0$ up to some stopping time $\tau : \Omega \to [0, T]$, show that $X := u + Y$ is a solution to

$$dX(t) = [A_1(t, X(t)) + A_2(t, X(t))]dt + B(t)dW(t), \quad t \in [0, T], X(0) = X_0,$$

on $[0, \tau]$ in the sense of Definition 5.2.6.

Problem 4. (2+(2+1*)+(2+1*) Points)

(i) Compare each of the four main conditions (hemicontinuity, (local) (weak) monotonicity, (generalised) coercivity, boundedness/growth) among Chapter 4, Chapter 5, Section 5.1 and Section 5.2. Argue which condition is the strongest and weakest among the respective category (i.e. compare the monotonicity conditions, etc.). Which of the three chapters has the strongest and which the weakest set of conditions? Discuss if there is a 'price to pay' when considering a weaker set of these conditions with regard to the main existence and uniqueness results in the respective chapters (i.e. Theorem 4.2.4 Theorem 5.1.3, Theorem 5.2.7).

(ii) Summarise concisely the idea of proof of the main existence and uniqueness results in Chapter 5.1 (Theorem 5.1.3). (Presentation of this part of the exercise gives 1 extra point.)

(iii) Summarise concisely the idea of proof of the main existence and uniqueness result in Chapter 5.2 (Theorem 5.2.7). (Presentation of this part of the exercise gives 1 extra point.)