Winter term 2023/2024
Lecture: Prof. Dr. Michael Röckner
Exercises: Dr. Sebastian Grube

Exercises to Introduction to Stochastic Partial Differential Equations II

Sheet 13
Total points: 12
Submission before: Friday, 26.01.2024, 12:00 noon

Problem 1. (3 Points)

Consider Section 5.2, Subsection '3D Navier–Stokes equation'. Let us hypothetically consider the following: Assume we are given a smooth classical solution $u \in C^{1,2}([0,T] \times \overline{\Lambda})$ to the 3D Navier–Stokes equation with pressure term $p \in C^1(\overline{\Lambda})$, for an initial condition u_0 and external force f fulfilling at least $u_0 \in C(\overline{\Lambda})$ and $f \in C(\overline{\Lambda})$. Prove that the following *energy equality* holds.

$$\frac{1}{2}||u(t)||_{L^2}^2 + \nu \int_0^t ||\nabla u(t)||_{L^2}^2 dt = \frac{1}{2}||u_0||_{L^2}^2 + \int_0^t \langle f(s), u(s)\rangle_{\mathbb{R}^3} ds, \ \forall t \in [0,T]. \tag{E}$$

Hint: Take the inner product of the 3D Navier–Stokes equation and u. Then integrate over Λ.

Problem 2. (3 Points)

Consider Section 5.2, Subsection '3D Navier–Stokes equation'. Recall that $C_{0,\sigma}^\infty(\Lambda;\mathbb{R}^3)$ denotes the set of all divergence free smooth vector fields from Λ into $\mathbb{R}^3$ with compact support and for $u \in C_{0,\sigma}^\infty(\Lambda;\mathbb{R}^3)$ we set

$$||u||_{H^{1,2}} := \left(\int_\Lambda |\nabla u|^2 d\xi\right)^{\frac{1}{2}}; \ ||u||_{H^{2,2}} := \left(\int_\Lambda |\Delta u|^2 d\xi\right)^{\frac{1}{2}}.$$

Prove *in detail* that for all $u \in C_{0,\sigma}^\infty(\Lambda;\mathbb{R}^3)$ there exits a finite constant $C > 0$ such that

$$||u||_{H^{1,2}} \leqslant C||u||_{H^{2,2}}.$$

Problem 3 (Prove the details)**.** (1+2 Points)

(i) (cf. proof of Example 5.2.24) Let $F : V \times V \to V*$ be bilinear. Show that F is hemicontinuous.

(ii) (cf. proof of Theorem 5.2.7) Fill in the details of the proof of the last assertion in Theorem 5.2.7, i.e. show that if A_2 satisfies $(H3')$ with g satisfying $\int_{x_0}^\infty (g(s)+s)^{-1} ds = \infty$, and if $\alpha\beta \leqslant 2$, then all assertions in Theorem 5.2.7 hold for $\tau \equiv T$.

See also the newest version of Theorem 5.2.2 (at least available in the lecture recordings).

Problem 4. (3 Points)

Let $A : D(A) \to C([0,1])$ be a linear operator defined by

$$D(A) := C^1([0,1])$$

and

$$Au := \frac{d}{dt}u, \ \forall u \in D(A).$$

Is $A : D(A) \to C([0,1])$ a closed operator, i.e. continuous with respect to the graph norm?