Exercises to Introduction to Stochastic Partial Differential Equations II

Sheet 3
Total points: 10
Submission before: Friday, 03.11.2023, 12:00 noon

Problem 1. (4 Points)

We have

$$\int_H \int_{\mathbb{W}_0} \int_{\mathbb{B}} \mathbb{1}_A(x, w_1, w)\delta_{F_\mu(x,w)}(dw_1)P^Q(dw)\mu(dx) = \int_H \int_{\mathbb{W}_0} \int_{\mathbb{B}} \mathbb{1}_A(x, w_1, w)\delta_{F_{\delta_x}(x,w)}(dw_1)P^Q(dw)\mu(dx)$$

for all $A \in \mathcal{B}(H) \otimes \mathcal{B}(\mathbb{W}_0) \otimes \mathcal{B}(\mathbb{B})$ (cf. with the equation after (E.0.10) in the lecture notes).

(a) Show that for all $A_1 \times A_3 \in \mathcal{B}(\mathbb{W}_0) \otimes \mathcal{B}(\mathbb{B})$ there exists a μ-zero set N such that for all $x \in N^\complement$

$$\int_{\mathbb{W}_0} \int_{\mathbb{B}} \mathbb{1}_{A_1\times A_3}(w_1, w)\delta_{F_\mu(x,w)}(dw_1)P^Q(dw) = \int_{\mathbb{W}_0} \int_{\mathbb{B}} \mathbb{1}_{A_1\times A_3}(w_1, w)\delta_{F_{\delta_x}(x,w)}(dw_1)P^Q(dw).$$

Note that the zero set N might depend on A_1 and A_3!

(b) Prove that the zero set of (a) can be chosen independently of A_1 and A_3.

Hint: Use that the involved σ-algebras are countably generated.

(c) Conclude from (b) that for all $f \in C_b(\mathbb{B})$

$$\int_{\mathbb{W}_0} \int_{\mathbb{B}} f(w_1)\mathbb{1}_{A_3}(w)\delta_{F_\mu(x,w)}(dw_1)P^Q(dw) = \int_{\mathbb{W}_0} \int_{\mathbb{B}} f(w_1)\mathbb{1}_{A_3}(w)(w_1, w)\delta_{F_{\delta_x}(x,w)}(dw_1)P^Q(dw)$$

for all $x \in N^\complement$.

(d) Assume that for the set $\mathbb{B}$ there exists a countable set of bounded continuous functions f which is a point separating set (exists, see e.g. [MR92, Chapter IV.4.b]). (This means that whenever $w \neq w'$ there exists at least one of such f with $f(w) \neq f(w')$). Conclude from this and part (c) that for all $x \in N^\complement$: $F_\mu(x, w) = F_{\delta_x}(x, w)$ P^Q-a.s.

Definition. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ a filtered probability space. A process $((X_t)_{t\in\mathbb{R}_+}, (\mathcal{F}_t)_{t\in\mathbb{R}_+})$ (with state space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$) is called an *elementary Markov process* if

$$\mathbb{P}[X_t \in B|\mathcal{F}] = \mathbb{P}[X_t \in B|\sigma(X_s)]$$

for every $s, t \in \mathbb{R}_+$ with $s \leqslant t$ and every $B \in \mathcal{B}(\mathbb{R})$.

Problem 2. (3 Points)

Show that: A process $(X_t)_{t\in\mathbb{R}_+}$ (w.r.t its canonical filtration) is a Markov process if and only if for every set $B \in \mathcal{B}(\mathbb{R})$ and all $s_1, ..., s_n, t \in \mathbb{R}_+$ (form some $n \in \mathbb{N}$) with $s_1 \leqslant ... \leqslant s_n \leqslant t$

$$\mathbb{P}[X_t \in B|\sigma(X_{s_1}, ..., X_{s_n})] = \mathbb{P}[X_t \in B|\sigma(X_{s_n})].$$

Problem 3. (3 Points)

Let $(X_t)_{t\in\mathbb{R}_+}$ be a stochastic process (w.r.t its canonical filtration) with finite dimensional distribution given by a Markov semigroup $(p_t)_{t\in\mathbb{R}_+}$ and an initial distribution μ, i.e.

$$\mathbb{P}[X_0 \in B_0, X_{t_1} \in B_1, ..., X_{t_n} \in B_n] = \int_{B_0}\int_{B_1} ... \int_{B_n} p_{t_n - t_{n-1}}(x_{n-1}, dx_n)...p_{t_1}(x_0, dx_1)\mu(dx_0).$$

Show that $(X_t)_{t\in\mathbb{R}_+}$ is an elementary Markov process. Additionally, prove that

$$\mathbb{P}[X_t \in B|\mathcal{F}_s] = p_{t-s}(X_s, B).$$

Literatur

[MR92] Zhi-Ming Ma Ma and Michael Röckner. *Introduction to the Theory of (non-symmetric) Dirichlet Forms.* Springer-Verlag, 1992.