Winter term 2023/2024
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Exercises to Introduction to Stochastic Partial Differential Equations II

Sheet 4
Total points: 10
Submission before: Friday, 10.11.2023, 12:00 noon

The aim of this exercise sheet is to repeat the most important facts about Markov processes.

Problem 1. (2 Points)

We are in the situation of Proposition 4.3.5. Let μ be an invariant measure. Show that we have

$$X_0 \sim \mu \implies X_t \sim \mu, \quad \forall t > 0,$$

i.e., if the initial condition is distributed according to μ the law of the solution is μ at every point in time.

Definition. Let E be a polish space. A tuple $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (\mathbb{P}_x)_{x\in E}, (X_t)_{t\in\mathbb{R}_+})$ (with state space $(E, \mathcal{B}(E))$) is called a *Markov process* if

- $(X_t)_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_t)$-adapted stochastic process.
- For all $A \in \mathcal{F}$ we have that $x \mapsto \mathbb{P}_x(A)$ is measurable,
- For all $x \in E$ we have $\mathbb{P}_x(X_0 = x) = 1$,
- For all $s, t \in \mathbb{R}_+$ we have $\mathbb{P}_x(X_{s+t} \in B|\mathcal{F}_s) = \mathbb{P}_{X_s}(X_t \in B)$ $\mathbb{P}_x$-a.s.

Problem 2. (3 Points)

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (\mathbb{P}_x)_{x\in E}, (X_t)_{t\in\mathbb{R}_+})$ be a Markov process. Show that

$$p_t(x, B) := \mathbb{P}_x(X_t \in B)$$

defines a normal (*normal* just means that $p_0(x, \cdot) = \delta_x \ \forall x \in E$) Markov semigroup and show that the finite-time marginal distributions of $(X_t)_{t\in\mathbb{R}_+}$ are given by

$$\mathbb{P}_x(X_{t_1} \in B_1, ..., X_{t_n} \in B_n) = \int_{B_1} ... \int_{B_n} p_{t_n - t_{n-1}}(x_{n-1}, dx_n)...p_{t_1}(x, dx_1)$$

for all $t_1 < ... < t_n, B_i \in \mathcal{F}_{t_i}, x \in E$. (See also the last exercise sheet.)

Problem 3. (2 Points)

Show that for every normal Markov semigroup $(p_t)_{t\in\mathbb{R}_+}$ there exists a Markov process $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (\mathbb{P}_x)_{x\in E}, (X_t)_{t\in\mathbb{R}_+})$ such that

$$p_t(x, B) = \mathbb{P}_x(X_t \in B)$$

for all $x \in E, t \in \mathbb{R}_+$.

Hint: Use Kolmogorov's consistency theorem.

Problem 4. (3 Points)

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (\mathbb{P}_x)_{x\in E}, (X_t)_{t\in\mathbb{R}_+})$, $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\in\mathbb{R}_+}, (\tilde{\mathbb{P}}_x)_{x\in E}, (\tilde{X}_t)_{t\in\mathbb{R}_+})$ be two Markov processes. Assume that the finite-time marginal laws of *both* Markov processes are given by the *same* normal Markov semigroup $(p_t)_{t\in\mathbb{R}_+}$, i.e.

$$p_t(x, B) = \mathbb{P}_x(X_t \in B) = \tilde{\mathbb{P}}_x(\tilde{X}_t \in B),$$

for all $t \in \mathbb{R}_+$ and $x \in E$. For an initial distribution μ define $\mathbb{P}^\mu$ and $\tilde{\mathbb{P}}^\mu$ by

$$\mathbb{P}^\mu(B) := \int_E \mathbb{P}_x(B)\mu(dx)$$

and

$$\tilde{\mathbb{P}}^\mu(B) := \int_E \tilde{\mathbb{P}}_x(B)\mu(dx).$$

Show that $\mathbb{P}^\mu \circ X^{-1} = \tilde{\mathbb{P}}^\mu \circ \tilde{X}^{-1}$ (on $(E, \mathcal{B}(E))$).