Winter term 2023/2024
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Exercises to Introduction to Stochastic Partial Differential Equations II

Sheet 7
Total points: 12
Submission before: Friday, 01.12.2023, 12:00 noon

Problem 1 (Fill in the details)**.** (4 Points)

Consider the situation of Lemma 5.1.5. Provide the details on its proof after (5.1.6), i.e. prove the following to statements.

(i) Show that by (5.1.5), (5.1.6) and Gronwall's lemma we have

$$E\left(\sup_{t\in[0,\tau_R^{(n)}]} ||X^{(n)}(t)||_H^p\right) + E\int_0^{\tau_R^{(n)}} ||X^{(n)}(s)||_H^{p-2}||X^{(n)}(s)||_V^{\alpha} ds$$
$$\leqslant C\left(E||X_0||_H^p + E\int_0^T f^{\frac{p}{2}}(s)ds\right), n \in \mathbb{N},$$

where $C > 0$ is a constant independent of n.

(ii) Show that there exists $C > 0$ such that

$$||A(\cdot, X^{(n_k)})||_{K^*} \leqslant C \quad \forall k \in \mathbb{N}.$$

Problem 2. (3 Points)

Consider the situation of the beginning of the proof of Theorem 5.1.3. We concluded from Lemma 5.1.4. that there is exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and (equivalence classes) $\bar{X} \in K, Y \in K^*, Z \in J$ such that (in particular), as $k \to \infty$,

$$X^{(n_k)} \to \bar{X} \text{ in } K, A(\cdot, X^{(n_k)}) \to Y \text{ in } K^*, P_{n_k}B(\cdot, X^{(n_k)}) \to Z \text{ in } J,$$

where all convergences are meant in the *weak* sense. Argue why $\bar{X}, Y$ and Z have each a progressively measurable representative. Where do we need this property of $\bar{X}, Y$ and Z?

Problem 3 (Application of the Yamada Watanabe)**.** (5 Points)

Let $T > 0$ be a fixed final time, $b : [0, T] \times \mathbb{R} \to \mathbb{R}$ be Borel measurable, bounded and Lipschitz continuous in the second parameter, i.e. there exists $K > 0$ such that

$$|b(t, x) - b(t, y)| \leqslant K|x - y| \quad \forall x, y \in \mathbb{R}, t \in [0, T].$$

For a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ equipped with a continuous $(\mathcal{F}_t)$-Brownian motion $(W(t))_{t\in[0,T]}$ with $W(0) = 0$, we consider the (ordinary) stochastic differential equation

$$dX(t) = b(t, X(t))dt + dW(t), \quad t \in [0, T]. \qquad \text{(SDE)}$$

(i) *(1 Point)* Let (X, W) be a weak solution to (SDE). Show that

$$E\left[\exp\left(\frac{1}{2}\int_0^T |b(t, X(t))|^2 dt\right)\right] < \infty.$$

(ii) *(1 Point)* Prove that (SDE) has a *weak solutions* (in the sense of (E.0.1)).

Hint: Use Girsanov's Transformation.

(iii) *(1 Point)* Let (X, W), (Y, W) be two weak solutions of (SDE) with $X(0) = Y(0)$ P-a.s. on the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ with respect to the same $(\mathcal{F}_t)$-Brownian motion $(W(t))_{t\in[0,T]}$. Show that

$$E[|X(t) - Y(t)|] \leqslant K \int_0^t E[|X(s) - Y(s)|]ds \quad \forall t \in [0, T].$$

(iv) *(1 Point)* Show that *pathwise uniqueness* holds for (SDE).

Hint: Gronwall's inequality.

(v) *(1 Point)* Conclude that (SDE) has a *unique strong solution* (in the sense of E.0.6).

Hint: Use the Yamada–Watanabe theorem.