Winter term 2023/2024
Lecture: Prof. Dr. Michael Röckner
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Exercises to Introduction to Stochastic Partial Differential Equations II

Sheet 9
Total points: 13
Submission before: Friday, 15.12.2023, 12:00 noon

Problem 1. (3 Points)

Use Lemma 5.1.7 to prove assertion (3) of Lemma 5.1.6 in the lecture notes.

Problem 2 (Semilinear SPDE)**.** (3 Points)

Let $A_g : V \to V^*$ be defined by

$$A_g(u) = A(u) + g(u)$$

as in Example 5.1.8. Prove that the operator A_g is hemicontinuous, i.e. it satisfies (H1).

Problem 3 (Interpolation inequality)**.** (3 Points)

Let $q \in [1,\infty], p \in [1,\infty], \lambda \in (0,1)$. Let $p' := \frac{p}{p-1} (\in [1,\infty])$. Prove that for all measurable functions $u : \mathbb{R}^d \to \mathbb{R}$ we have

$$||u||_{L^q} \leqslant ||u||^{\lambda}_{L^{\lambda qp}} ||u||^{(1-\lambda)}_{L^{(1-\lambda)qp'}}.$$

Problem 4 (Stochastic 2D Navier–Stokes equation)**.** (2+2 Points)

Consider the setting of Example 5.1.10: Let $\Lambda \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. We set

$$V := \{v \in H^{1,2}_0(\Lambda; \mathbb{R}^2) : \nabla \cdot v = 0 \text{ a.e. in } V\}.$$

Let H be the closure of V in $L^2(\Lambda; \mathbb{R}^2)$. We have the Gelfand triple

$$V \subset H \cong H^* \subset V^*.$$

We define the Stokes operator A with viscosity constant ν through

$$A : H^{2,2}(\Lambda; \mathbb{R}^2) \cap V \to H, \quad u \mapsto \nu P_H \Delta u,$$

where $P_H : L^2(\Lambda; \mathbb{R}^2) \to H$ denotes the Helmholtz–Leray projection.

(i) Show that the Stokes operator extends by continuity to a continuous map $A : V \to V^*$, so that for some $C \in (0,\infty)$, $||Au||_{V^*} \leqslant C||u||_V, u \in V$, and that (via integration by parts) ${}_{V^*}\langle Au, u\rangle_V = -\nu ||u||^2_V$.

Furthermore, the external force $F : V \times V \to V^*$ is defined as follows. For $u, v \in V$ we set

$${}_{V^*}\langle F(u,v), w\rangle_V := \int_\Lambda ((u \cdot \nabla)v) \cdot w d\xi, \quad w \in V.$$

(ii) Show that F is indeed well-defined and continuous *Hint: Use Lemma 5.1.6 (0).*