ON THE CASSELS-PFISTER THEOREM

MARKUS ROST

`preliminary version`

Introduction

In this extremely preliminary notes we discuss proofs of the Cassels-Pfister Theorem (in short CPT; it is formulated here as Proposition 2) and a variant of it for division algebras (Proposition 1).

The "standard" argument to the CPT is by means of an explicit sequence of reflections which finally transform the given rational vector into a polynomial vector. This proof can be found in [3] (and where else?).

This argument appears also in Serre's book [4] (and where else?) in the proof that an integer is a sum of 3 integer squares if it is a sum of 3 rational squares.

The sequence of reflections appearing in this argument are defined by a very straight algorithm. What is however not obvious a priori is that this algorithm actually succeeds. To me, it looks like a miracle.

We have included the standard computation used for this below as *"Other Proof of Proposition 2"*.

Another proof of showing that the algorithm succeeds is contained here as the first proof of Proposition 2. It uses the CPT for the trivial case when the ground field is algebraically closed. (According to Jean-Pierre Tignol, I had presented that argument in Oberwolfach in the evening of May 21, 1992). This argument involves Clifford algebras and applies also to division algebras.

The case of division algebras is treated in Proposition 1.

There are also other approaches which are less miracolous.

One can prove the CPT using Harder's theorem (this seems to go back to Gerstein [1]): The given rational vector lies in a maximal $F[t]$-order. Any maximal order is regular (see [6]). Finally, by Harder's theorem (see [3] and where else?), any regular quadratic form over the affine line is constant (extended from the ground field). Thus there is an isometry over the rational function field which transforms the maximal order into the standard order.

The latter argument is very appealing because it deduces the CPT from a general fact: The triviality of G-bundles over the affine line (at least if F is perfect, see [2]).

It is perhaps worthwhile to look at this argument for the representation of integers as sums of 3 squares. It should work also for central simple algebras. I have not looked into this.

For the case of central simple algebras there is also another further argument in Tignol's paper [6] which "was probably known to Hasse" (according to Tignol).

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Details

Proposition 1. *Let A be a division algebra finite-dimensional over its center F. Let $x \in A(t)$ and suppose that there exists an extension K/F such that x is conjugate in $A \otimes K(t)$ to an element in $A \otimes K[t]$. Then x is conjugate in $A(t)$ to an element in $A[t]$.*

Proof. Let u, $\alpha \in A \otimes K[t]$ with

$$x\alpha = \alpha u$$

Write x as

$$x = y + \frac{r}{P} \qquad y, r \in A[t],\ P \in F[t],\ \deg r < \deg P$$

If $r = 0$, then x is a polynomial. Otherwise let

$$x' = rxr^{-1}$$
$$\alpha' = \frac{r\alpha}{P}$$

Then

$$x'\alpha' = \alpha' u$$

The claim follows by induction on $\deg \alpha$, since

$$\alpha' = \frac{r\alpha}{P} = \alpha u - y\alpha$$

is a polynomial with $\deg \alpha' < \deg \alpha$. □

Proposition 2. *Let $\varphi\colon V \to F$ be a quadratic form over a field F with $\operatorname{char} F \neq 2$. Let $x \in V(t)$ with $\varphi(x) \in F[t]$. Then there exists $y \in V[t]$ with $\varphi(x) = \varphi(y)$.*

Proof. Basically the same argument as for Proposition 1 can be used.

We work in the Clifford algebra $C(V)$. Recall that for an anisotropic vector $z \in V$ the reflection at z can be written in $C(V)$ as

$$v \mapsto zvz^{-1}$$

Proposition 2 is clear if φ is isotropic. Let us assume φ is anisotropic. There exists an extension K/F and an element $u \in V \otimes K[t]$ with $\varphi(x) = \varphi(u)$. For instance, one may choose any field extension K over which φ is isotropic.

Since $\varphi(x) = \varphi(u)$, there exists a product of (at most two) reflections which transforms u into x. Hence there exist $\alpha \in C\big(V \otimes K[t]\big)$ such that

$$x\alpha = \alpha u$$

Write x as

$$x = y + \frac{r}{P} \qquad y, r \in V[t],\ P \in F[t],\ \deg r < \deg P$$

If $r = 0$, then x is a polynomial. Otherwise let

$$x' = rxr^{-1}$$
$$\alpha' = \frac{r\alpha}{P}$$

Then

$$x'\alpha' = \alpha' u$$

The claim follows by induction on $\deg \alpha$, since

$$\alpha' = \frac{r\alpha}{P} = \alpha u - y\alpha$$

is a polynomial with $\deg \alpha' < \deg \alpha$. $\square$

Here is a version more close to the standard proof of Proposition 2.

Other Proof of Proposition 2. Let $\langle v, w\rangle$ be the symmetric bilinear form determined by $\varphi(v) = \langle v, v\rangle$.

Write x as

$$x = y + \frac{r}{P} \qquad y, r \in V[t],\ P \in F[t],\ \deg r < \deg P$$

We have

$$\langle x, x\rangle = \langle y, y\rangle + 2\frac{\langle y, r\rangle}{P} + \frac{\langle r, r\rangle}{P^2}$$

Therefore

$$P' = \frac{\langle r, r\rangle}{P} = P(\langle x, x\rangle - \langle y, y\rangle) - 2\langle y, r\rangle$$

is a polynomial with $\deg P' < \deg P$.

If $r = 0$, then x is a polynomial. Otherwise consider the reflection

$$s_r(v) = v - 2\frac{\langle v, r\rangle}{\langle r, r\rangle} r$$

We get for $x' = s_r(x)$ the expression

$$\begin{aligned} x' &= s_r(y) - \frac{r}{P} \\ &= y - \left(2\frac{\langle y, r\rangle}{\langle r, r\rangle} + \frac{1}{P}\right) r \\ &= y - \frac{\langle x, x\rangle - \langle y, y\rangle}{P'} r \end{aligned}$$

It is now clear how to proceed by induction on $\deg P$. $\square$

Bibliography

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Department of Mathematics, The Ohio State University, 231 W 18th Avenue, Columbus, OH 43210, USA

E-mail address: `rost@math.ohio-state.edu`

URL: `http://www.math.ohio-state.edu/~rost`