CHAIN LEMMA FOR SPLITTING FIELDS OF SYMBOLS

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`preliminary version`

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1. An invariant

We assume $\operatorname{char} k = 0$ (to have available resolution of singularities).

For a proper variety X we denote

$$I(X) = \deg\bigl(\mathrm{CH}_0(X)\bigr) \subset \mathbb{Z}$$

Let X be an proper variety of dimension d, let $X_0 \subset X$ be closed subvariety, and let $U = X \setminus X_0$. We assume that U is smooth. Let $W \to U$ be a μ_p-torsor. We define

$$\eta(W, X, X_0) \in \mathbb{Z}/I(X_0)$$

as follows: Let L/U be the line bundle obtained as the image of $[W/U]$ via

$$[W/U] \in H^1_{\text{ét}}(U, \mu_p) \to H^1_{\text{ét}}(U, \mathbb{G}_{\mathrm{m}}) = H^1_{\mathrm{Zar}}(U, \mathbb{G}_{\mathrm{m}}) = \mathrm{Pic}(U).$$

The degree map induces a map

$$\deg\colon\ \mathrm{CH}_0(U) \to \mathbb{Z}/I(X_0).$$

We define

$$\eta(W, X, X_0) = \deg\bigl(c_1(L)^d\bigr).$$

Example 1.1. Let X be smooth, let $F = k(X)$ and let K/F be a Kummer extension of degree p. Let $\bar{W} \to X$ be the normal closure of K/F. Then $\bar{W} \to X$ is etale over an open subset $U = X \setminus X_0$ and we have an invariant

$$\eta'(K/F, X) = \eta(\bar{W}|U, X, X_0) \in \mathbb{Z}/I(X_0).$$

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Passing to the limit over all modells X of F one may define an invariant of K/F in $\mathbb{Z}/...$ where ... can be expressed in terms of valuations on F. But this is not important at the moment.

Proposition 1.2 (Degree formula)**.** *Let $W \to U = X \setminus X_0$ as above with X irreducible and let Y be proper and irreducible of dimension $\dim Y = \dim X = d$. Let $f\colon Y \to X$ be morphism, let $Y_0 = f^{-1}(X_0)$, let $U' = Y \setminus Y_0$ and $W' = W \times_U U'$. Then*

$$\eta(W', Y, Y_0) = (\deg f)\eta(W, X, X_0) \bmod I(X_0).$$

Note that

$$I(Y_0) \subset I(X_0).$$

Proof. This is pretty obvious: Let $\hat{f}\colon U' \to U$ be the restriction of f. Then the line bundle L' given by W'/U' is $\hat{f}^*L$, whence

$$c_1(L')^d = \hat{f}^* c_1(L)^d$$

and

$$\hat{f}_*\big(c_1(L')^d\big) = (\deg \hat{f}) c_1(L)^d.$$

Now apply the degree map. □

Example 1.3. Suppose that in Proposition 1.2 one has $I(X_0) \subset pZ$ and suppose further $\eta(W', Y, Y_0) \not\equiv 0 \bmod p$. Then $\deg f$ is prime to p.

2. Preliminaries, Conventions, and Notations

- The ground field k has characteristic 0. We fix a prime p. We assume $\mu_p \subset k$.
- By a scheme or a variety X (over k) we mean a separated scheme of finite type $\pi_X\colon X \to \operatorname{Spec} k$.
- If X is a smooth variety, then TX denotes the tangent bundle of X.
- Let V be vector bundle over X. We denote by $\pi_V\colon \mathbb{P}(V) \to X$ the projective bundle associated to V. Moreover

  $$\mathbb{L}(V) \to \pi_V^* V$$

  denotes the tautological line bundle on $\mathbb{P}(V)$.

  For the fiber tangent bundle $T\big(\mathbb{P}(V)/X\big)$ one has

  $$T\big(\mathbb{P}(V)/X\big) = \pi_V^* V \otimes \mathbb{L}(V)^\vee / \mathcal{O}_{\mathbb{P}(V)}.$$
- Let V be vector (or an affine) bundle over X. We denote by $\mathbb{A}(V) \to X$ the associated scheme V.
- By a *form* we understand a triple $(T/S, L, \alpha)$ where $T \to S$ are schemes, L is line bundle on T and $\alpha \in H^0(T, L^{\otimes -p})$ is a form of degree p on L.

  There is a natural homomorphism $\mu_p \to \operatorname{Aut}(T/S, L, \alpha)$ induced from the standard action of $\mathbb{G}_{\mathrm{m}}$ on L.
- Let $(\operatorname{Spec} k, L, \alpha)$ be a nonzero form and let $u \in L$ be a basis vector. Then the p-power class

  $$\{\alpha\} = \{\alpha(u)\} \in K_1 k/p = k^*/(k^*)^p$$

  is independent on the choice of u.

- Let $(T/S, L, \alpha)$ and let Γ be a finite group acting on $(T/S, L, \alpha)$ (i.e., there is given a homomorphism $\Gamma \to \operatorname{Aut}(T/S, L, \alpha)$). We say that $(T/S, L, \alpha)$ is an *admissable* Γ*-form* if the following conditions hold:
  - α is nonzero on an open dense subscheme of T.
  - Γ has only finitely many fixed points on T (a fixed point is a point $P \in T$ with $gP = P$ for all $g \in G$).
  - At each fixed point P the form α is nonzero.
- For vector bundles V, V' on schemes X/S resp. X'/S we denote by $V \boxplus_S V'$ the *exterior direct sum*, given by the sum of the pull backs to $X \times_S X'$. Similarly we denote by $V \boxtimes_S V'$ the *exterior tensor product*, given by the tensor product of the pull backs.
- For forms $(T/S, L, \alpha)$ and $(T'/S, L', \alpha')$ we denote by
$$(T/S, L, \alpha) \boxtimes_S (T'/S, L', \alpha') = \big((T \times_S T')/S, L \boxtimes_S L', \alpha \boxtimes_S \alpha'\big)$$
their *exterior product*, with the form defined by
$$(\alpha \boxtimes_S \alpha')(u \boxtimes_S u') = \alpha(u)\alpha'(u')$$
for sections u, u' of L, L', respectively.

  If $(T/S, L, \alpha)$ and $(T'/S, L', \alpha')$ are admissable Γ-forms, then $(T/S, L, \alpha) \boxtimes_S (T'/S, L', \alpha')$ is an admissable Γ-form.
- Let (S, H_i, α_i), $i = 1, \ldots, n$, be admissable Γ-forms and let $P \in S$ be a k-rational fixed point. We say that P is *twisting* for the family $(S, H_i, \alpha_i)_i$, if the homomorphism
$$\Gamma \to \mu_p^n = \prod_{i=1}^{n} \operatorname{Aut}(H_i|P, \alpha_i|P)$$
is surjective.
- By a *cellular* variety we mean a variety which admits a stratification by affine spaces. The motive of a cellular variety is the direct sum of powers of the Tate motive L, with a summand $L^{\otimes i}$ for each i-cell. If X and Y are cellular, then $X \times Y$ is cellular and one has
$$\mathrm{CH}_*(X \times Y) = \mathrm{CH}_*(X) \otimes_{\mathbb{Z}} \mathrm{CH}_*(Y).$$
- Let L be a line bundle L on a smooth and proper variety X over k of dimension $d \geq 0$. We write
$$\delta(L) = \deg\big(c_1(L)^d\big) \in \mathbb{Z}.$$
Here
$$\deg\colon \mathrm{CH}_0(X) \to \mathrm{CH}_0(\operatorname{Spec} k) = \mathbb{Z}$$
is the degree map. If $d = 0$ we understand by $\delta(L)$ the degree of X as a finite extension of k.

  If V is a vector space of dimension n, then
$$\delta\big(\mathbb{L}(V)\big) = \deg\big(c_1\big(\mathbb{L}(V)\big)^{n-1}\big) = (-1)^{n-1}.$$
- The *index* I_X of a proper variety is
$$I_X = \deg\big(\mathrm{CH}_0(X)\big) \subset \mathbb{Z}$$
- If p is a prime, a field k is called *p-special* if $\operatorname{char} k \neq p$ and if k has no finite field extensions of degree prime to p.

- Let (S, L, α) be a form. We consider the bundle of algebras
$$A = A(S, L, \alpha) = TL/I$$
over R. Here TL is the tensor algebra of L and I is the ideal subsheaf generated by
$$\lambda^{\otimes p} - \alpha(\lambda)$$
for local sections λ of L. A a is bundle of commutative algebras of degree p. Note that
$$A = \bigoplus_{i=0}^{p-1} L^{\otimes i}$$
as vector bundles. We denote by
$$N_A \colon A \to \mathcal{O}_S$$
the norm of the algebra A.
- We use the notation
$$\mathrm{Cyclic}^p(Z) = (Z^p)/(\mathbb{Z}/p).$$

3. The forms $\mathcal{A}(\alpha_1, \ldots, \alpha_n)$ ("algebras")

Given a scheme S and forms (S, H_i, α_i), $i = 1, \ldots, m$, we define forms
$$\mathcal{A}(\alpha_1, \ldots, \alpha_n) = (P_n/S, K_n, \Phi_n), \qquad 0 \le n \le m.$$
For $n = 0$ we put
$$P_0 = S,$$
$$K_0 = \mathcal{O}_S,$$
$$\Phi_0(t) = t^p.$$
Suppose $(P_{n-1}/S, K_{n-1}, \Phi_{n-1})$ is defined. We consider the 2-dimensional vector bundle
$$V_n = \mathcal{O}_{P_{n-1}} \oplus H_n \boxtimes_S K_{n-1}$$
on P_{n-1}, and the form
$$\varphi_n \colon V_n \to \mathcal{O}_{P_{n-1}}$$
on V_n defined by
$$\varphi_n(t - u \otimes v) = t^p - \alpha_n(u)\Phi_{n-1}(v)$$
for sections t, u, v of $\mathcal{O}_{P_{n-1}}$, H_n, K_{n-1}, respectively.

Let $(P_{n-1,j}, V_{n,j}, \varphi_{n,j})$, $j = 1, \ldots, p-1$ be copies of $(P_{n-1}, V_n, \varphi_n)$. We put
$$(P_n/S, K_n, \Phi_n) = (P_{n-1}/S, K_{n-1}, \Phi_{n-1}) \boxtimes_S \mathop{\boxtimes_S}_{j=1}^{p-1} \big(\mathbb{P}(V_{n,j}), \mathbb{L}(V_{n,j}), \varphi_{n,j}\big).$$

We assume now that $S = \operatorname{Spec} k$ and list the most important properties of the forms (P_n, K_n, Φ_n).

Lemma 3.1. *The variety P_n is smooth, proper, cellular, connected, and of dimension $p^n - 1$.*

Proof. Indeed, P_n is an iterated projective bundle. The computation of the dimension is clear for $n = 0$ and for $n > 0$ we find

$$\begin{aligned}\dim P_n &= \dim P_{n-1} + (p-1)(1 + \dim P_{n-1})\\ &= (p^{n-1} - 1) + (p-1)p^{n-1} = p^n - 1\end{aligned}$$

by induction on n. □

Lemma 3.2. $\delta(K_n) = (-1)^n \bmod p$.

Proof. This is clear for $n = 0$. Let

$$\begin{aligned}u_n &= c_1(K_n) &&\in \mathrm{CH}^1(P_n), && n \geq 0,\\ u_{n-1,j} &= c_1(K_{n-1,j}) &&\in \mathrm{CH}^1(P_{n-1,j}), && n \geq 1,\ j = 1, \dots, p-1,\\ z_{n,j} &= c_1\big(\mathbb{L}(V_{n,j})\big) &&\in \mathrm{CH}^1\big(\mathbb{P}(V_{n,j})\big), && n \geq 1,\ j = 1, \dots, p-1.\end{aligned}$$

For $n \geq 1$ let

$$\widehat{P}_n = P_{n-1} \times \prod_{j=1}^{p-1} P_{n-1,j}$$

Then

$$\mathrm{CH}^*(\widehat{P}_n) = \mathrm{CH}^*(P_{n-1}) \otimes \bigotimes_{j=1}^{p-1} \mathrm{CH}^*(P_{n-1,j})$$

and

$$\mathrm{CH}^*(P_n) = \frac{\mathrm{CH}^*(\widehat{P}_n)[z_{n,j};\ j = 1, \dots, p-1]}{\langle z_{n,j}^2 - z_{n,j}u_{n-1,j};\ j = 1, \dots, p-1\rangle}.$$

Moreover

$$u_n = u_{n-1} + \overline{z}_n, \quad \text{with} \quad \overline{z}_n = \sum_{j=1}^{p-1} z_{n,j}.$$

Note that

$$u_{n-1}^{p^{n-1}} = u_{n-1,j}^{p^{n-1}} = 0, \quad z_{n,j}^{p^{n-1}+1} = 0$$

by dimension reasons. Hence, calculating mod p,

$$u_n^{p^{n-1}} = (u_{n-1} + \overline{z}_n)^{p^{n-1}} = u_{n-1}^{p^{n-1}} + \overline{z}_n^{p^{n-1}} = \overline{z}_n^{p^{n-1}}.$$

One finds (using Lemma 3.3 below)

$$\begin{aligned}u_n^{p^n-1} &= u_n^{p^{n-1}-1} u_n^{p^{n-1}(p-1)} = u_n^{p^{n-1}-1} \overline{z}_n^{p^{n-1}(p-1)}\\ &= u_n^{p^{n-1}-1}\big(z_{n,1}^{p^{n-1}} + z_{n,2}^{p^{n-1}} + \cdots + z_{n,p-1}^{p^{n-1}}\big)^{p-1}\\ &= -u_{n-1}^{p^{n-1}-1} z_{n,1}^{p^{n-1}} z_{n,2}^{p^{n-1}} \cdots z_{n,p-1}^{p^{n-1}}\\ &= -u_{n-1}^{p^{n-1}-1} z_{n,1} u_{n,1}^{p^{n-1}-1} z_{n,2} u_{n,2}^{p^{n-1}-1} \cdots z_{n,p-1} u_{n,p-1}^{p^{n-1}-1}.\end{aligned}$$

It follows that

$$\begin{aligned}\delta(K_n) &= -\delta(K_{n-1})\big(-\delta(K_{n-1,1})\big)\big(-\delta(K_{n-1,2})\big) \cdots \big(-\delta(K_{n-1,p-1})\big)\\ &= -\delta(K_{n-1}) \bmod p.\end{aligned}$$

whence the claim. □

Lemma 3.3. *Let R be a ring over $\mathbb{F}_p$ and let $v_1, v_2, \ldots, v_{p-1} \in R$, be elements with $v_1^2 = v_2^2 = \cdots = v_{p-1}^2 = 0$. Then*

$$(v_1 + v_2 + \cdots + v_{p-1})^{p-1} = -v_1 v_2 \cdots v_{p-1}.$$

Proof. Note that $(p-1)! = -1 \bmod p$. □

The construction of (P_n, K_n, Φ_n) is functorial in the forms (S, H_i, α_i). In particular the group

$$\Gamma_n = \mu_p^n \subset \prod_{i=1}^{n} \mathrm{Aut}(S, H_i, \alpha_i)$$

acts on (P_n, K_n, Φ_n).

From now on we suppose that $\alpha_i \neq 0$ for $i = 1, \ldots, n$.

Lemma 3.4. *The triple (P_n, K_n, Φ_n) is an admissable Γ_n-form. All fixed points are k-rational.*

Proof. By induction on n. Suppose that $(P_{n-1}, K_{n-1}, \Phi_{n-1})$ is an admissable Γ_{n-1}-form. It suffices to show that $(\mathbb{P}(V_n), \mathbb{L}(V_n), \varphi_n)$ is an admissable Γ_n-form. It is easy to see that φ_n is generically nonzero. Every Γ_n-fixed point on $\mathbb{P}(V_n)$ lies over a Γ_{n-1}-fixed point $P \in P_{n-1}$. It suffices to show that the fibre $(\mathrm{Spec}\, \kappa(P), \mathbb{L}(V_n)|P, \varphi_n|P)$ is an admissable Γ-form where

$$\Gamma = \mathrm{Aut}(S, H_n, \alpha_n) = \ker(\Gamma_n \to \Gamma_{n-1}).$$

This is easy to see: If $(\mathrm{Spec}\, k, H, \alpha)$ is a nonzero form over k, then

$$\mu_p = \mathrm{Aut}(\mathrm{Spec}\, k, H, \alpha)$$

has in $\mathbb{P}(k \oplus H)$ only the two fixed points $\mathbb{P}(0 \oplus H)$ and $\mathbb{P}(k \oplus 0)$. The form $\varphi(t-u) = t^p - \alpha(u)$ is nonzero on the lines $t = 0$ and $u = 0$. □

Lemma 3.5. *Let $\eta_n \in P_n$ be the generic point. Then*

$$\{\alpha_1, \ldots, \alpha_n, \Phi_n(\eta_n)\} = 0 \in K_{n+1}^M k(P_n)/p.$$

Proof. By induction on n. Suppose that

$$\{\alpha_1, \ldots, \alpha_{n-1}, \Phi_{n-1}(\eta_{n-1})\} = 0 \in K_n^M k(P_{n-1})/p.$$

One has

$$\Phi_n(\eta_n) = \Phi_{n-1}(\eta_{n-1}) \cdot \prod_{j=1}^{p-1} \big(1 - \alpha_n \Phi_{n-1,j}(\eta_{n-1,j})\big).$$

Hence it suffices to show

$$\{\alpha_1, \ldots, \alpha_n, 1 - \alpha_n \Phi_{n-1,j}(\eta_{n-1,j})\} \in K_{n+1}^M k(P_n)/p$$

for each $j = 1, \ldots, p-1$. This follows from $\{a, 1-ab\} = -\{b, 1-ab\}$. □

Remark 3.6. Given the forms $(\mathrm{Spec}\, k, H_i, \alpha_i)$, form the vector space

$$A_n = \bigoplus_{j_1, \ldots, j_n = 0}^{p-1} H_1^{\otimes j_1} \otimes \cdots \otimes H_n^{\otimes j_n}.$$

One has $\dim A_n = p^n$. On A_n there is the form

$$\Theta_n = \bigoplus_{j_1,\ldots,j_n=0}^{p-1} (-\alpha_1)^{\otimes j_1} \otimes \cdots \otimes (-\alpha_n)^{\otimes j_n}$$

Consider the form $(\mathbb{P}(A_n), \mathbb{L}(A_n), \Theta_n)$. If $p = 2$, this form satisfies all the properties of (P_n, K_n, Φ_n) listed above (up to a sign in the computation of $\delta\big(\mathbb{L}(A_n)\big)$). If $p > 2$, all properties of (P_n, K_n, Φ_n) are also valid, except for the splitting of the symbol. If $n = 1$, $n = 2$, or $n = p = 3$, one may define on A_n an algebra structure with norm form Θ'_n in such a way that $(\mathbb{P}(A_n), \mathbb{L}(A_n), \Theta'_n)$ satisfies all the properties. The (P_n, K_n, Φ_n) form an approximation to these algebras, with the advantage, that (P_n, K_n, Φ_n) can be constructed for all p and n.

4. The forms $\mathcal{B}(\alpha_1, \ldots, \alpha_n)$ ("relative algebras")

Let $n \geq 1$. Given forms (S, H_i, α_i), $i = 1, \ldots, n-1$, and $(S'/S, L, \beta)$, we define a form

$$\mathcal{B}(\alpha_1, \ldots, \alpha_{n-1}, \beta) = (P'_n/S', K'_n, \Phi'_n)$$

as follows. Let $(P_{n-1}/S, K_{n-1}, \Phi_{n-1})$ be as in section 3. Put

$$\overline{P}_{n-1} = S' \times_S P_{n-1}$$

We consider the 2-dimensional vector bundle

$$\overline{V}_n = \mathcal{O}_{\overline{P}_{n-1}} \oplus L \boxtimes_S K_{n-1}$$

on $\overline{P}_{n-1}$, and the form

$$\overline{\varphi}_n \colon \overline{V}_n \to \mathcal{O}_{\overline{P}_{n-1}}$$

on $\overline{V}_n$ defined by

$$\overline{\varphi}_n(t - u \otimes v) = t^p - \beta(u)\Phi_{n-1}(v)$$

for sections t, u, v of $\mathcal{O}_{\overline{P}_{n-1}}$, L, K_{n-1}, respectively.

Let

$$(\overline{P}_{n-1,j}, \overline{V}_{n,j}, \overline{\varphi}_{n,j}, K_{n-1,j}, P_{n-1,j}),\ j = 1, \ldots, p-1$$

be copies of $(\overline{P}_{n-1}, \overline{V}_n, \overline{\varphi}_n, K_{n-1}, P_{n-1})$. We put

$$(P'_n/S', K'_n, \Phi'_n) = \mathop{\boxtimes_{S'}}_{j=1}^{p-1} \big(\mathbb{P}(\overline{V}_{n,j}), \mathbb{L}(\overline{V}_{n,j}), \overline{\varphi}_{n,j}\big).$$

We assume now that $S = \operatorname{Spec} k$ and list the most important properties of the forms (P'_n, K'_n, Φ'_n).

Lemma 4.1. *The variety P'_n is smooth and proper over S', and of relative dimension $p^n - p^{n-1}$. If S' is cellular, so is P'_n. The fibres of S/S' are connected.*

Proof. Note that P'_n/S' is an iterated projective bundle. Moreover

$$\dim P'_n/S' = (p-1)(\dim P_{n-1} + 1) = p^n - p^{n-1}$$

by Lemma 3.1. ∎

Let

$$\begin{aligned} u_n' &= c_1(K_n') &&\in \mathrm{CH}^1(P_n'),\\ u_{n-1,j} &= c_1(K_{n-1,j}) &&\in \mathrm{CH}^1(P_{n-1,j}),\\ v_n &= c_1(L) &&\in \mathrm{CH}^1(S'). \end{aligned}$$

Lemma 4.2. *One has*

$$u_n'^{\,p^n} = u_n'^{\,p^{n-1}} v_n^{p^n-p^{n-1}} \bmod p.$$

If $S' = \operatorname{Spec} k$, *then*

$$\delta(K_n') = \deg(u_n'^{\,p^n-p^{n-1}}) = -1 \bmod p.$$

Proof. Let

$$\widehat{\overline{P_n}} = S' \times \prod_{j=1}^{p-1} P_{n-1,j}$$

Then

$$\mathrm{CH}^*(\widehat{\overline{P_n}}) = \mathrm{CH}^*(S') \otimes \bigotimes_{j=1}^{p-1} \mathrm{CH}^*(P_{n-1,j})$$

and

$$\mathrm{CH}^*(P_n') = \frac{\mathrm{CH}^*(\widehat{\overline{P_n}})[z_{n,j};\, j=1,\dots,p-1]}{\langle z_{n,j}^2 - z_{n,j}(v_n+u_{n-1,j});\, j=1,\dots,p-1\rangle}.$$

Moreover

$$u_n' = \overline{z}_n, \quad \text{with} \quad \overline{z}_n = \sum_{j=1}^{p-1} z_{n,j}.$$

Recall that $u_{n-1,j}^{p^{n-1}} = 0$. Calculating mod p, one finds

$$\begin{aligned} u_n'^{\,p^n} &= \overline{z}_n^{p^n}\\ &= z_{n,1}^{p^n} + \dots + z_{n,p-1}^{p^n}\\ &= z_{n,1}^{p^{n-1}}(v_n+u_{n-1,1})^{p^{n-1}(p-1)} + \dots + z_{n,p-1}^{p^{n-1}}(v_n+u_{n-1,p-1})^{p^{n-1}(p-1)}\\ &= z_{n,1}^{p^{n-1}}(v_n^{p^{n-1}}+u_{n-1,1}^{p^{n-1}})^{(p-1)} + \dots + z_{n,p-1}^{p^{n-1}}(v_n^{p^{n-1}}+u_{n-1,p-1}^{p^{n-1}})^{(p-1)}\\ &= z_{n,1}^{p^{n-1}} v_n^{p^{n-1}(p-1)} + \dots + z_{n,p-1}^{p^{n-1}} v_n^{p^{n-1}(p-1)}\\ &= \overline{z}_n^{p^{n-1}} v_n^{p^{n-1}(p-1)} = u_n'^{\,p^{n-1}} v_n^{p^{n-1}(p-1)}. \end{aligned}$$

This proves the first claim.

Suppose $v_n = 0$. Then $z_{n,j}^{p^{n-1}+1} = 0$. One finds mod p (using Lemma 3.3)

$$\begin{aligned} u_n'^{\,p^{n-1}(p-1)} &= \bigl(z_{n,1}^{p^{n-1}} + z_{n,2}^{p^{n-1}} + \dots + z_{n,p-1}^{p^{n-1}}\bigr)^{p-1}\\ &= -z_{n,1}^{p^{n-1}} z_{n,2}^{p^{n-1}} \cdots z_{n,p-1}^{p^{n-1}}\\ &= -z_{n,1}u_{n-1,1}^{p^{n-1}-1} z_{n,2}u_{n-1,2}^{p^{n-1}-1} \cdots z_{n,p-1}u_{n-1,p-1}^{p^{n-1}-1} \end{aligned}$$

Since $\delta(K_{n-1}) \neq 0 \bmod p$, it follows that

$$\begin{aligned}\delta(K'_n) &= -\big(-\delta(K_{n-1,1})\big)\big(-\delta(K_{n-1,2})\big)\cdots\big(-\delta(K_{n-1,p-1})\big)\\ &= -1 \bmod p,\end{aligned}$$

whence the second claim. ☐

From now on we suppose that $\alpha_i \neq 0$ for $i = 1, \ldots, n-1$. Let Γ be a finite group, let $\Gamma \to \Gamma_{n-1}$ be an epimorphism and let $\Gamma \to \operatorname{Aut}(S', L, \beta)$ be a homomorphism. Thus Γ acts on all the forms $(\operatorname{Spec} k, H_i, \alpha_i)$, $i = 0, \ldots, n-1$, and (S', L, β).

Lemma 4.3. *Suppose that (S', L, β) is an admissable Γ-form with all fixed points k-rational. Moreover suppose that each fixed point is twisting for the forms*

$$(S, H_i, \alpha_i),\ i = 1, \ldots, n-1, \text{ and } (S', L, \beta).$$

Then (P'_n, K'_n, Φ'_n) is an admissable Γ-form with all fixed points k-rational.

Proof. This follows as for Lemma 3.4. ☐

Lemma 4.4. *Suppose that S' is irreducible. Let $\eta_n \in P_n$ be the generic point. Then*

$$\{\alpha_1, \ldots, \alpha_{n-1}, \beta(\eta_n), \Phi_n(\eta_n)\} = 0 \in K^M_{n+1}k(P_n)/p.$$

Proof. This follows as for Lemma 3.5. ☐

Remark 4.5. Given the form (S', L, β) one may define the "Kummer algebra"

$$A = A(S', L, \beta) = L^{\otimes 0} \oplus L^{\otimes 1} \oplus \cdots \oplus L^{\otimes p-1}$$

with the product given by the natural multiplication in the tensor algebra using the form $\beta\colon L^{\otimes p} \to L^{\otimes 0}$ to reduce the degree mod p. One finds

$$\mathrm{CH}^*\big(\mathbb{P}(A)\big) \otimes \mathbb{F}_p = \mathrm{CH}^*(S') \otimes \mathbb{F}_p[x]/\langle x^p - x^{p-1}y\rangle$$

with $x = c_1\big(\mathbb{L}(A)\big)$ and $y = c_1(L)$.

Hence we have a homomorphism

$$R = \mathbb{F}_p[x]/\langle x^p - x^{p-1}y\rangle \to \mathrm{CH}^*\big(\mathbb{P}(A)\big) \otimes \mathbb{F}_p$$

Lemma 4.2 shows that there is a homomorphism

$$R \to \mathrm{CH}^*(P'_n) \otimes \mathbb{F}_p, \qquad x \mapsto {u'_n}^{p^{n-1}},\ y \mapsto v_n^{p^{n-1}}.$$

If one thinks in terms of the (in general nonexisting) algebras

$$A_n = A(\alpha_1, \ldots, \alpha_{n-1}, \beta)$$

with "subalgebras"

$$A_{n-1} = A(\alpha_1, \ldots, \alpha_{n-1}),$$

and one imagines to form something like the projective space $\mathbb{P}_{A_{n-1}}(A_n)$, then one may think of P'_n as an approximation $P'_n \to \mathbb{P}_{A_{n-1}}(A_n)$ with the homomorphism $R \to \mathrm{CH}^*(P'_n) \otimes \mathbb{F}_p$ being the pull back on the Chow rings (if say $S' = \mathbb{P}^\infty$ and with L the universal line bundle).

5. The forms $\mathcal{C}(\alpha_1,\dots,\alpha_n)$ (Chain lemma construction)

Let $n \geq 2$. Given forms (S,H_i,α_i), $i=1,\dots,n-1$, and $(S'/S,L,\beta)$, we define forms

$$\mathcal{C}_r = \mathcal{C}_r(\alpha_1,\dots,\alpha_{n-1},\beta) = (S_r/S_{r-1},L_r,\beta_r), \qquad r \geq -1.$$

For $r=-1$, 0 we put

$$(S_{-1}/S_{-2},L_{-1},\beta_{-1}) = (S/S,H_{n-1},\alpha_{n-1}),$$
$$(S_0/S_{-1},L_0,\beta_0) = (S'/S,L,\beta).$$

Let $r>0$ and suppose $\mathcal{C}_{r-2}$ and $\mathcal{C}_{r-1}$ are defined.

Let

$$(P'_{n-1,r}/S_{r-1},K'_{n-1,r},\Phi'_{n-1,r}) = \mathcal{B}(\alpha_1,\dots,\alpha_{n-1},\beta_{r-1})$$

be the form constructed in section 4, starting from (S,H_i,α_i), $i=1,\dots,n-2$, and $(S_{r-1}/S_{r-2},L_{r-1},\beta_{r-1})$. Put

$$(S_r/S_{r-1},L_r,\beta_r) = (S_{r-2}/S_{r-3},L_{r-2},\beta_{r-2}) \boxtimes_{S_{r-2}} (P'_{n-1,r}/S_{r-1},K'_{n-1,r},\Phi'_{n-1,r}).$$

We assume now that $S=\operatorname{Spec} k$ and list the most important properties of the forms $(S_r/S_{r-1},L_r,\beta_r)$.

Lemma 5.1. *The variety S_r is smooth and proper over S', and of relative dimension $r(p^{n-1}-p^{n-2})$. If S' is cellular, so is S_r. The fibres of S/S' are connected.*

Proof. This follows from Lemma 4.1. For the dimension note

$$\dim S_r/S_{r-1} = \dim P'_{n-1,r}/S_{r-1} = p^{n-1}-p^{n-2}$$

by Lemma 4.1. □

Thus if $\dim S' = (p^l-1)p^n$ for some $\ell \geq 0$, then $\dim S_p = (p^{l+1}-1)p^{n-1}$.

Theorem 5.2. *Let $\ell \geq 0$ and suppose that S' is smooth and proper of dimension $(p^l-1)p^n$. Then*

$$\delta(L_p) = \delta(L) \bmod p.$$

The proof requires some calculations.

Let a, $b \in \mathbb{F}_p$, and let $r \geq 0$ be an integer. In the ring $\mathbb{F}_p[z_1,\dots,z_r]$ let

$$x_{-1} = a,$$
$$x_0 = b,$$
$$x_m = z_m + x_{m-2}, \quad 1 \leq m \leq r.$$

Then

$$x_{2k} = z_{2k} + z_{2k-2} + \cdots + z_4 + z_2 + b,$$
$$x_{2k+1} = z_{2k-1} + z_{2k-3} + \cdots + z_3 + z_1 + a.$$

We denote by I the ideal generated by

$$z_m^p - z_m x_{m-1}^{p-1}, \quad 1 \leq m \leq r$$

and put

$$R_r(a,b) = \mathbb{F}_p[z_1,\dots,z_r]/I.$$

The elements

$$z^J = z_1^{i_1} \cdots z_r^{i_r}, \quad J = (i_1, \ldots, i_r), \quad 0 \le i_j \le p-1$$

form an $\mathbb{F}_p$-basis of $R_r(a,b)$. For $u \in R_r(a,b)$ let $c_m(u)$ be the coefficient of $z_1^{p-1} \cdots z_m^{p-1}$.

Lemma 5.3. *If $1 \le r \le p$ one has $c_r(x_r^{r(p-1)}) = 1$ in $R_r(a,b)$.*

Proof. One has for $1 \le m \le p$:

$$\begin{aligned} x_m^{m(p-1)} &= x_m^{p(m-1)+(p-m)} \\ &= (z_m + x_{m-2})^{p(m-1)+(p-m)} \\ &= (z_m^p + x_{m-2}^p)^{(m-1)} (z_m + x_{m-2})^{(p-m)} \\ &= (z_m x_{m-1}^{p-1} + x_{m-2}^p)^{(m-1)} (z_m + x_{m-2})^{(p-m)}. \end{aligned}$$

Hence for $m \le p$ one has

$$c_m(x_m^{m(p-1)}) = c_{m-1}(x_{m-1}^{(m-1)(p-1)}).$$

The claim follows by induction. $\square$

Proposition 5.4. *If $(a,b) \neq (0,0)$, then $R_r(a,b)$ is isomorphic to a product of rings of the form*

$$\mathbb{F}_p[v_1, \ldots, v_k]/(v_1^p, \ldots, v_k^p), \quad k \ge 0.$$

Proof. By induction on $r \ge 0$. The case $r = 0$ is obvious.

Suppose $b \neq 0$. Then the polynomial

$$z_1^p - z_1 x_0^{p-1}$$

is separable with roots $z_1 = ib$, $i \in \mathbb{F}_p$. It follows that we have isomorphism

$$R_r(a,b) \xrightarrow{\sim} \prod_{i \in \mathbb{F}_p} R_r(a,b)/(z_1 - ib).$$

The ring $R_r(a,b)/(z_1 - ib)$ is the quotient of $\mathbb{F}_p[z_2, \ldots, z_r]$ by the ideal generated by

$$z_m^p - z_m x_{m-1}^{p-1}, \quad 2 \le m \le r$$

with

$$\begin{aligned} x_0 &= b, \\ x_1 &= ib + a, \\ x_m &= z_m + x_{m-2}, \quad 2 \le m \le r. \end{aligned}$$

Hence $R_r(a,b)/(z_1 - ib) \simeq R_{r-1}(b, ib + a)$. The claim follows from the induction hypothesis.

Suppose $b = 0$. Then $a \neq 0$. In this case we consider the homomorphism

$$\begin{aligned} \varphi \colon \mathbb{F}_p[z_1, \ldots, z_r] &\to \mathbb{F}_p[z_1]/(z_1^p) \otimes R_{r-1}(0,1), \\ z_m &\mapsto (a + z_1) \otimes z_{m-1}, \quad 2 \le m \le r, \\ z_1 &\mapsto z_1 \otimes 1. \end{aligned}$$

We claim that $\varphi(I)=0$. For this it suffices to show

$$\varphi(z_m^p - z_m x_{m-1}^{p-1}) = 0, \qquad 1 \le m \le r.$$

This is obvious for $m=1$. If $m=2$, then

$$\begin{aligned}
\varphi(z_2^p - z_2 x_1^{p-1}) &= \varphi\big(z_2^p - z_2(z_1+a)^{p-1}\big)\\
&= (a+z_1)^p \otimes z_1^p - \big((a+z_1)\otimes z_1\big)\big(z_1\otimes 1 + 1\otimes a\big)^{p-1}\\
&= (a+z_1)^p \otimes z_1^p - \big((a+z_1)\otimes z_1\big)\big((z_1+a)\otimes 1\big)^{p-1}\\
&= (a+z_1)^p \otimes (z_1^p - z_1) = 0.
\end{aligned}$$

If $m = 2k \ge 2$, then

$$\begin{aligned}
\varphi(z_{2k}^p - z_{2k} x_{2k-1}^{p-1}) &= \varphi\big(z_{2k}^p - z_{2k}(z_{2k-1}+\cdots+z_3+z_1+a)^{p-1}\big)\\
&= (a+z_1)^p \otimes z_{2k-1}^p -\\
-\big((a+z_1)\otimes z_{2k-1}\big)&\big((a+z_1)\otimes z_{2k-2}+\cdots+(a+z_1)\otimes z_2 + z_1\otimes 1 + 1\otimes a\big)^{p-1}\\
&= (a+z_1)^p \otimes z_{2k-1}^p -\\
-\big((a+z_1)\otimes z_{2k-1}\big)&\big((a+z_1)\otimes z_{2k-2}+\cdots+(a+z_1)\otimes z_2 + (a+z_1)\otimes 1\big)^{p-1}\\
&= (a+z_1)^p \otimes \big(z_{2k-1}^p - z_{2k-1}(z_{2k-2}+\cdots+z_2+1)\big)^{p-1}\\
&= (a+z_1)^p \otimes (z_{2k-1}^p - z_{2k-1}x_{2k-2}^{p-1}) = 0.
\end{aligned}$$

If $m = 2k-1 \ge 3$, then

$$\begin{aligned}
\varphi(z_{2k-1}^p - z_{2k-1} x_{2k-2}^{p-1}) &= \varphi\big(z_{2k-1}^p - z_{2k-1}(z_{2k-2}+\cdots+z_2)^{p-1}\big)\\
&= (a+z_1)^p \otimes \big(z_{2k-2}^p - z_{2k-2}(z_{2k-3}+\cdots+z_1)^{p-1}\big)\\
&= (a+z_1)^p \otimes (z_{2k-2}^p - z_{2k-2}x_{2k-3}^{p-1}) = 0.
\end{aligned}$$

It follows that φ induces a homomorphism

$$\begin{aligned}
&\overline{\varphi}\colon R_r(a,b) \to \mathbb{F}_p[z_1]/(z_1^p)\otimes R_{r-1}(0,1),\\
&z_m \mapsto (a+z_1)\otimes z_{m-1}, \quad 2\le m\le r,\\
&z_1 \mapsto z_1\otimes 1.
\end{aligned}$$

$\overline{\varphi}$ is obviously surjective. By dimension reasons, $\overline{\varphi}$ must be an isomorphism. Again the claim follows from the induction hypothesis. □

Corollary 5.5. *$u^{p^2} = u^p$ for all $u \in R_p(0,1)$.* □

Corollary 5.6. *Let $n \ge 2$, and let $u_n = x_p^{p^n-p} \in R_p(0,1)$. Then $c_p(u_n) = 1$.*

Proof. For $n=2$ this is Lemma 5.3. Moreover, by Corollary 5.5, the element u_n does not depend on n. □

We rewrite things in a homogenous form. Let x be a variable and let

$$R' = \mathbb{F}_p[x, z_1, \dots, z_p]/I'$$

where I' is the homogenous ideal generated by

$$z_m^p - z_m x_{m-1}^{p-1}, \quad 1 \le m \le p$$

with

$$\begin{aligned} x_{-1} &= 0, \\ x_0 &= x, \\ x_m &= z_m + x_{m-2}, \quad 1 \le m \le p. \end{aligned}$$

Then $R'/(x-1) = R_p(0,1)$. Corollaries 5.5 and 5.6 yield the following two corollaries:

Corollary 5.7. *$u^{p^2} = u^p x^{p^2-p}$ for all $u \in R'$.* □

Corollary 5.8. *Let $n \ge 2$. Then*

$$x_p^{p^n-p} = z_1^{p-1} z_2^{p-1} \cdots z_p^{p-1} x^{p^n-p^2} \bmod x^{p^n-p^2+1} R'$$

Proof. Recall the basis elements $(z^J)_J$ of $R_p(0,1)$ considered above. The elements $(z^J x^{p^n-p-|J|})_J$ form a basis of the homogenous subspace of R' of degree $p^n - p$. It follows that

$$x_p^{p^n-p} = c_p(x_p^{p^n-p}) z_1^{p-1} z_2^{p-1} \cdots z_p^{p-1} x^{p^n-p^2} \bmod \langle z^J x^{p^n-p-|J|};\, |J| < p^2 - p \rangle.$$

But if $|J| < p^2 - p$ then $z^J x^{p^n-p-|J|} \in x^{p^n-p^2+1} R'$. □

Proof of Theorem 5.2: Let

$$\begin{aligned} x_r &= c_1(L_r)^{p^{n-2}} && \in \mathrm{CH}^{p^{n-2}}(S_r), && r \ge -1, \\ z_r &= c_1(K'_{n-1,r})^{p^{n-2}} && \in \mathrm{CH}^{p^{n-2}}(P'_{n-1,r}), && r \ge 1. \end{aligned}$$

Then, calculating mod p,

$$\begin{aligned} x_{-1} &= 0, \\ x_0 &= c_1(L)^{p^{n-2}} \in \mathrm{CH}^{p^{n-2}}(S') \otimes \mathbb{F}_p, \\ x_r &= x_{r-2} + z_r, \qquad r \ge 1, \end{aligned}$$

since

$$c_1(L_r) = c_1(L_{r-2}) + c_1(K'_{n-1,r}).$$

Moreover

$$z_r^p = z_r x_{r-1}^{p-1}$$

by Lemma 4.2.

We have a homomorphism

$$R'(x) \to \mathrm{CH}^*(S_p) \otimes \mathbb{F}_p, \quad z_m \mapsto z_m, \quad x \mapsto x_0.$$

It follows from Corollary 5.8 that (mod p)

$$x_p^{p^{\ell+2}-p} = z_1^{p-1} z_2^{p-1} \cdots z_p^{p-1} x_0^{p^{\ell+2}-p^2} \bmod \langle x^{p^{\ell+2}-p^2+1} \rangle$$

Now if $\dim S' = (p^l - 1)p^n$, then $x_0^{p^{l+2}-p^2+1} = 0$. Hence

$$x_p^{p^{\ell+2}-p} = \delta(K'_{n-1,1}) \delta(K'_{n-1,2}) \cdots \delta(K'_{n-1,p-1}) \delta(L) = \delta(L) \bmod p,$$

where the last equation follows from Lemma 4.2. □

From now on we suppose that $\alpha_i \neq 0$ for $i = 1, \dots, n-1$. Let Γ be a finite group, let $\Gamma \to \Gamma_{n-1}$ be an epimorphism and let $\Gamma \to \mathrm{Aut}(S', L, \beta)$ be a homomorphism. Thus Γ acts on all the forms $(\mathrm{Spec}\, k, H_i, \alpha_i)$, $i = 0, \dots, n-1$, and (S', L, β).

Lemma 5.9. *Suppose that (S', L, β) is an admissable Γ-form, that all fixed points are k-rational and that each fixed point $P \in S'$ is twisting for the forms*

$$(S', H_i, \alpha_i),\ i = 1, \dots, n-1, \text{ and } (S', L, \beta).$$

Then for all $r \geq 0$, (S_r, L_r, β_r) is an admissable Γ-form, all fixed points are k-rational, and each fixed point $P \in S_r$ is twisting for the forms

$$(S_r, H_i, \alpha_i),\ i = 1, \dots, n-2,\ (S_r, L_{r-1}, \beta_{r-1}), \text{ and } (S_r, L_r, \beta_r).$$

Proof. Let $P \in S_r$ be a fixed point. By induction we may assume that P is k-rational and that

$$\Gamma \to \mathrm{Aut}(L_{r-2}|P, \beta_{r-2}|P) \times \mathrm{Aut}(L_{r-1}|P, \beta_{r-1}|P) \times \prod_{i=1}^{n-2} \mathrm{Aut}(H_i|P, \alpha_i|P)$$

is surjective. We claim that

$$\Gamma \to \mathrm{Aut}(L_r|P, \beta_r|P) \times \mathrm{Aut}(L_{r-1}|P, \beta_{r-1}|P) \times \prod_{i=1}^{n-2} \mathrm{Aut}(H_i|P, \alpha_i|P)$$

is surjective. Note that $L_r|P = L_{r-2}|P \otimes K_{n-1,r}|P$. The claim follows now from the fact that $\mathrm{Aut}(L_{r-2}|P, \alpha_{r-2}|P)$ acts trivially on $K_{n-1,r}|P$.

The remaining parts of the statement follow from Lemma 4.3. □

Lemma 5.10. *Suppose that S' is irreducible. Let $\eta_r \in S_r$ be the generic point. Then*

$$\{\alpha_1, \dots, \alpha_{n-2}, \beta_{r-1}(\eta_{r-1}), \beta_r(\eta_r)\} = (-1)^r \{\alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta(\eta_0)\}$$

in $K_n^M k(S_r)/p$.

Proof. We show

$$\{\alpha_1, \dots, \alpha_{n-2}, \beta_{r-1}(\eta_{r-1}), \beta_r(\eta_r)\} = \{\alpha_1, \dots, \alpha_{n-2}, \beta_{r-1}(\eta_{r-1}), \beta_r(\eta_{r-2})\}.$$

We have

$$\beta_r(\eta_r) = \beta_r(\eta_{r-2}) \Phi'_{n-1,r}.$$

The claim follows now from Lemma 4.4. □

We will need the following special case:

Corollary 5.11.

$$\{\alpha_1, \dots, \alpha_{n-2}, \beta_p(\eta_p), \beta_{p-1}(\eta_{p-1})\} = \{\alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta(\eta_0)\}$$

in $K_n^M k(S_p)/p$. □

Remark 5.12. Let $S' = \mathrm{Spec}\, k$. We think of the symbol

$$\{\alpha_1, \dots, \alpha_{n-2}, \beta_p(\eta_p)\}$$

as a family of symbols of weight $n-1$ "between"

$$\{\alpha_1, \dots, \alpha_{n-2}\} \quad \text{and} \quad \{\alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta\},$$

with S_p as parameter space.

Our later considerations indicate that this family is universal over p-special fields. For $n = 2$ we will make this precise, and for $p = 2$ this can be done using Pfister forms. I have no idea how to show this in general. In the case $n = p = 3$ the universality would have important consequences for the classification of groups of type F_4.

6. The forms $\mathcal{K}(\alpha_1, \ldots, \alpha_n)$ (universal families of Kummer splitting fields)

Let $n \geq 1$. Given forms (S, H_i, α_i), $i = 1, \ldots, n$, we define forms

$$\mathcal{K}_i = \mathcal{K}_i(\alpha_1, \ldots, \alpha_n) = (R_i/R_{i+1}, J_i, \gamma_i), \qquad 1 \leq i \leq n,$$
$$\mathcal{K}_i' = \mathcal{K}_i'(\alpha_1, \ldots, \alpha_n) = (R_i/R_{i+1}, J_i', \gamma_i'), \qquad 1 \leq i \leq n.$$

We put

$$(R_n/R_{n+1}, J_n, \gamma_n) = (S/S, H_n, \alpha_n)$$

and

$$(R_n/R_{n+1}, J_n', \gamma_n') = (S/S, \mathcal{O}_S, \tau)$$

with $\tau(t) = t^p$.

Let $i < n$ and suppose that $\mathcal{K}_{i+1}$ is defined.

Recall the forms

$$\mathcal{C}_r = \mathcal{C}_r(\alpha_1, \ldots, \alpha_i, \gamma_{i+1}) = (S_r/S_{r-1}, L_r, \beta_r)$$

defined in section 5. Let $\pi \colon S_p \to S_{p-1}$ be the projection.

We put

$$\mathcal{K}_i = \mathcal{C}_p(\alpha_1, \ldots, \alpha_i, \gamma_{i+1}),$$
$$\mathcal{K}_i' = \pi^* \mathcal{C}_{p-1}(\alpha_1, \ldots, \alpha_i, \gamma_{i+1}).$$

We assume now that $S = \operatorname{Spec} k$ and list the most important properties of the forms $(R_i/R_{i+1}, J_i, \gamma_i)$ and $(R_i/R_{i+1}, J_i', \gamma_i')$.

Lemma 6.1. *The variety R_i is smooth, proper, cellular, and of dimension $p^n - p^i$.*

Proof. This follows from Lemma 5.1. For the dimension note

$$\dim R_i/R_{i+1} = p^{i+1} - p^i, \qquad i < n$$

by Lemma 5.1. □

Lemma 6.2. $\delta(J_i) = 1 \bmod p$.

Proof. By Theorem 5.2 we have

$$\delta(J_i) = \delta(J_{i+1}) \bmod p.$$

Hence $\delta(J_i) = \delta(J_n) = 1 \bmod p$. □

The construction of $(R_i/R_{i+1}, J_i, \gamma_i)$ is functorial in the forms (S, H_i, α_i). In particular the group

$$\Gamma_n = \mu_p^n \subset \prod_{i=1}^n \operatorname{Aut}(S, H_i, \alpha_i)$$

acts on $(R_i/R_{i+1}, J_i, \gamma_i)$.

From now on we suppose that $\alpha_i \neq 0$ for $i = 1, \ldots, n$.

Lemma 6.3. *The forms $(R_i/R_{i+1}, J_i, \gamma_i)$ are admissable Γ_n-forms, all fixed points are k-rational, and each fixed point $P \in R_i$ is twisting for the forms*

$$(R_i, H_m, \alpha_m),\ m = 1,\ \ldots,\ i-1,\ \text{and}\ (R_i, J_i, \gamma_i).$$

Proof. This follows form Lemma 5.9. □

Lemma 6.4. *Let $\eta_i \in R_i$ be the generic point. Then, for $1 \le i < n$,*

$$\{\alpha_1, \ldots, \alpha_{i-1}, \gamma_i(\eta_i), \gamma_i'(\eta_i)\} = \{\alpha_1, \ldots, \alpha_i, \gamma_{i+1}(\eta_{i+1})\}$$

in $K^M_{i+1}k(R_i)/p$.

Proof. This follows from Lemma 5.11. □

In particular we have

$$\{\alpha_1, \ldots, \alpha_n\} = \{\alpha_1, \gamma_2, \gamma_3', \ldots, \gamma_n'\}, \tag{6.1}$$

$$\{\alpha_1, \gamma_2\} = \{\gamma_1, \gamma_2'\}, \tag{6.2}$$

$$\{\alpha_1, \ldots, \alpha_n\} = \{\gamma_1, \gamma_2', \ldots, \gamma_n'\}. \tag{6.3}$$

We write

$$(R, J, \gamma) = (R_1, J_1, \gamma_1)$$

We denote by $\widetilde{R} \to R$ be the degree p "Kummer extension" corresponding to γ, defined locally by $\mathcal{O}_{\widetilde{R}} = \mathcal{O}_R[t]/\big(t^p - \gamma(\lambda)\big)$ where λ is a local nonzero section of J.

Corollary 6.5. *The symbol $\{\alpha_1, \ldots, \alpha_n\}$ vanishes in the generic point of $\widetilde{R}$.*

Proof. This follows from Lemma 6.4 (see (6.3)). □

7. Proof of the chain lemma

A splitting variety of a symbol is called p-generic, if it is a generic splitting variety over any p-special field.

Let Z be a p-generic splitting variety of $\{\alpha_1, \ldots, \alpha_n\}$ of dimension $p^{n-1} - 1$. We assume $\{\alpha_1, \ldots, \alpha_n\} \neq 0$. It follows that $I_Z \subset p\mathbb{Z}$.

Let (R, J, γ) be the form of defined at the end of section 6.

Note that Z has point of degree prime to p over $k(\widetilde{R})$, hence has a $k(\widetilde{R}')$-rational point where R'/R is of degree prime to p. We have diagram of varieties covered by cyclic extensions of degree p:

$$\begin{tikzcd}
\widetilde{R} \arrow[d] & \widetilde{R}' \arrow[l, "\hat{g}"'] \arrow[r, "\hat{f}"] \arrow[d] & Z^p \arrow[d] \\
R & R' \arrow[l, "g"'] \arrow[r, "f"] & \mathrm{Cyclic}^p(Z).
\end{tikzcd}$$

Let

$$R_0 \subset R$$

be the zero locus of γ. Inspection shows that $I(R_0) \subset p\mathbb{Z}$. We have

$$\eta(\widetilde{R}/R, R, R_0) = c_1(J)^d \bmod p = 1 \bmod p \neq 0 \bmod p$$

by Lemma 6.2.

Let

$$R_0' = \subset R'$$

be the subscheme of ramification of $\widetilde{R}'/R'$. Then $g(R'_0) \subset R_0$ and therefore $I(R'_0) \subset p\mathbb{Z}$. The degree formula tells that

$$\eta(\widetilde{R}'/R', R', R'_0) = (\deg g)^{-1} \bmod p \neq 0 \bmod p.$$

Moreover let

$$\mathrm{Cyclic}^p(Z)_0 = Z \subset \mathrm{Cyclic}^p(Z)$$

be the image of the diagonal. One has $I(\mathrm{Cyclic}^p(Z)_0) = p\mathbb{Z}$. Further, $\mathrm{Cyclic}^p(Z)_0$ contains the subscheme of ramification of $Z^p/\mathrm{Cyclic}^p(Z)$. Therefore $f(R'_0) \subset \mathrm{Cyclic}^p(Z)_0$. The degree formula tells that

$$\deg f \neq 0 \bmod p.$$

Now let $K = k(\sqrt[p]{b})$ be a cyclic extension of degree p which splits $\{\alpha_1, \dots, \alpha_n\}$. We assume that k is p-special. It follows that there is a point $\operatorname{Spec} K \to \widetilde{R}$ lying over a rational point $P\colon \operatorname{Spec} k \to R$. Then $b = \gamma(P)$ in $k^*/(k^*)^p$. It follows that

$$\{\alpha_1, \dots, \alpha_n\} = \{\alpha_1, \gamma_2(P), \gamma'_3(P), \dots, \gamma'_n(P)\}, \tag{7.1}$$

$$\{\alpha_1, \gamma_2(P)\} = \{b, \gamma'_2(P)\}, \tag{7.2}$$

$$\{\alpha_1, \dots, \alpha_n\} = \{b, \gamma'_2(P), \dots, \gamma'_n(P)\}. \tag{7.3}$$

(see (6.1)–(6.3) after Lemma 6.4).

We have proved:

Corollary 7.1. *The chain lemma for cyclic algebras of degree p over p-special fields.*

Corollary 7.2. *The chain lemma for symbols $(a, b, c) \bmod p$ over p-special fields.*

Now let $k(\sqrt[p]{b})$, $k(\sqrt[p]{c})$ be two cyclic extensions of degree p which split the symbol $\{\alpha_1, \dots, \alpha_n\}$. Applying the last arguments twice, one finds first $b_i \in k^*$ such that

$$\{\alpha_1, \dots, \alpha_n\} = \{b, b_1, b_2, \dots, b_n\},$$

and then c_i, $c'_i \in k^*$ such that

$$\{b, b_1, b_2, \dots, b_n\} = \{b, c_1, c_2, \dots, c_n\},$$
$$\{b, c_1\} = \{c, c'_2\}.$$

Let $X(b, c_1)$ be the Brauer-Severi variety associated to the symbol $\{b, c_1\}$. It has rational points over $k(\sqrt[p]{b})$ and over $k(\sqrt[p]{c})$. Morover, since Z is a p-generic spliting field, we have a correspondence $X(b, c_1) \to Z$ lying over $\mathbb{Z} \to \mathbb{Z}$ of degree prime to p.

Corollary 7.3. *Let x, $y \in Z$ be points of degree p and let $\alpha \in \kappa(x)^*$, $\beta \in \kappa(y)^*$. Then there exist $z \in Z$ of degree p and $\gamma \in \kappa(z)^*$, such that*

$$[\alpha] + [\beta] = [\gamma] \quad \textit{in} \quad A_0(Z, K_1).$$

Proof. By the previous considerations, and using that $\mathrm{CH}_0(Z_K) = \mathbb{Z}$ whenever $Z(K) \neq \varnothing$, we may reduce to the case of Brauer-Severi variety. In this case the statement is known [1]. $\square$

Remark 7.4. In the last proof we assumed $\mathrm{CH}_0(Z_K) = \mathbb{Z}$ whenever $Z(K) \neq \varnothing$. This can be shown for $n = 3$ for Z the usual $\mathrm{SL}(p)$-form.

Without this assumption, we get at least the last corollary with $A_0(Z, K_1)$ replaced by

$$\operatorname{coker} A_0(Z^2, K_1) \to A_0(Z, K_1),$$

the group considered in my MSRI-talk.

References

[1] A. S. Merkurjev and A. A. Suslin, *The group of K_1-zero-cycles on Severi-Brauer varieties*, Nova J. Algebra Geom. **1** (1992), no. 3, 297–315.

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