A “common slot” counterexample in degree 3

Notation: For a, b nonzero elements in a field F containing a primitive cube root of unity ω, the symbol (a,b) denotes the element of the Brauer group of F represented by the F-algebra generated by elements α, β subject to

$$\alpha^3 = a, \qquad \beta^3 = b, \qquad \beta\alpha = \omega\alpha\beta.$$

Let a_1, b_1, $a_2 \in F^\times$. If there exist x, $y \in F^\times$ such that

$$(a_1,b_1) = (a_1,x) + (a_1,y), \quad (a_1,x) = -(a_2,x) \text{ and } (a_1,y) = (a_2,y), \qquad (*)$$

then the additivity of symbols yields $(a_1,b_1) = (a_2, x^{-1}y)$. However, the next example shows that when (a_1,b_1) is split by $F(\sqrt[3]{a_2})$, there need not exist elements x, y satisfying $(*)$.

Example: *A global field F containing a primitive cube root of unity and elements a_1, b_1, a_2, b_2 such that $(a_1,b_1) = (a_2,b_2)$, but no couple of elements x, y satisfying $(*)$. In particular (taking $x = 1$), the field F does not contain any element y such that*

$$(a_1,b_1) = (a_1,y) = (a_2,y) = (a_2,b_2).$$

Let $F = \mathbb{F}_7(t)$, where t is an indeterminate, $a_1 = t$ and $a_2 = t(1-t)$. Note that $(a_1,a_2) = 0$. Therefore, for all place v of F, the local invariant $(a_1,a_2)_v$ is trivial. It follows that in the completion F_v of F at v we have either $a_1 \in F_v^{\times 3}$ or $a_1 \equiv a_2 \bmod F_v^{\times 3}$ or $a_1 \equiv a_2^2 \bmod F_v^{\times 3}$ or $a_2 \in F_v^{\times 3}$, since the (generalized) Hilbert symbol $(\ ,\ )_v$: $(F_v^\times/F_v^{\times 3}) \times (F_v^\times/F_v^{\times 3}) \to \frac{1}{3}\mathbb{Z}/\mathbb{Z}$ is a nondegenerate alternating pairing.

Consider in particular v_1 the t-adic place and v_2 the $(t+3)$-adic place. Since a_1, a_2 are uniformizing parameters at v_1, we have a_1, $a_2 \notin F_{v_1}^{\times 3}$; but $a_1 \equiv a_2 \bmod F_{v_1}^{\times 3}$. On the other hand, a_1 and a_2 have non-cube residues at v_2, hence a_1, $a_2 \notin F_{v_2}^{\times 3}$ but $a_1 \equiv a_2^{-1} \bmod F_{v_2}^{\times 3}$.

Let now A be the central simple F-algebra with local invariants $1/3$ at v_1, $2/3$ at v_2 and 0 everywhere else. If v is a place of F where $a_1 \in F_v^{\times 3}$, then $v \neq v_1$, v_2 hence $[A]_v = 0$. It follows that A is split by $F(\sqrt[3]{a_1})$, hence we may find $b_1 \in F^\times$ such that $[A] = (a_1,b_1)$ in the Brauer group of F. Similarly, A is split by $F(\sqrt[3]{a_2})$ hence we may find $b_2 \in F^\times$ such that $[A] = (a_2,b_2)$; thus,

$$(a_1,b_1) = (a_2,b_2).$$

Suppose now x, $y \in F^\times$ satisfy $(*)$. Since $a_1 \equiv a_2 \bmod F_{v_1}^{\times 3}$, the relation $(a_1,x)_{v_1} = -(a_2,x)_{v_1}$ implies $(a_1,x)_{v_1} = 0$. On the other hand, since $a_1 \equiv a_2^{-1} \bmod F_{v_2}^{\times 3}$, it follows from $(a_1,y)_{v_2} = (a_2,y)_{v_2}$ that $(a_1,y)_{v_2} = 0$, hence $(a_1,x)_{v_2} = (a_1,b_1)_{v_2} = 2/3$.

For $v \neq v_1$, v_2, we consider four cases, according to the relation between a_1 and a_2 in the group of cube classes:

- if $a_1 \in F_v^{\times 3}$, then clearly $(a_1, x)_v = 0$.
- if $a_1 \equiv a_2 \bmod F_v^{\times 3}$, then $(a_1, x)_v = 0$ as for $v = v_1$ above.
- if $a_1 \equiv a_2^{-1} \bmod F_v^{\times 3}$, then $(a_1, x)_v = (a_1, b_1)_v$ as for $v = v_2$ above, hence $(a_1, x)_v = 0$.
- if $a_2 \in F_v^{\times 3}$, then $(a_1, x)_v = 0$ follows from $(a_1, x) = (a_2, x^{-1})$.

Thus, the invariants of (a_1, x) are:

$$(a_1, x)_{v_2} = 2/3, \qquad \text{and } (a_1, x)_v = 0 \text{ for } v \neq v_2,$$

a contradiction to the reciprocity law.

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