On Hilbert Satz 90 for K_3 for quadratic extensions

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I. Preliminaries

Notation: $K_nF = K_n^MF$ (for convenience)

1) For a variety X/F denote by $A^p(X, K_n)$ the homology of

$$\bigoplus_{v\in X^{(p-1)}} K_{n-p+1}K(v) \xrightarrow{d} \bigoplus_{v\in X^{(p)}} K_{n-p}K(v) \xrightarrow{d} \bigoplus_{v\in X^{(p+1)}} K_{n-p-1}K(v).$$

2) For X projective, the norm homomorphism in Milnor K-theory induces a map

$$N : A_0(X, K_n) \longrightarrow K_nF, \qquad N = \sum_{v\in X_{(0)}} N_{K(v)/F},$$

where $A_0(X, K_n)$ denotes the cokernel of

$$\bigoplus_{v\in X_{(1)}} K_{n+1}K(v) \xrightarrow{d} \bigoplus_{v\in X_{(0)}} K_nK(v).$$

3) Given a fibration $\pi : X \to Y$, one has a filtration of the complex 1) by codimension in Y which induces a spectral sequence

$$E_1^{p,q} = \bigoplus_{v\in Y_{(p)}} A^q(\pi^{-1}(v), K_{n-p}) \Longrightarrow A^{p+q}(X, K_n).$$

4) For a quadratic form $\varphi : F^k \to F$ (which may singular) I denote by $X_\varphi \subset \mathbb{P}^{k-1}$ the corresponding quadric.
Moreover I put

$$D_n(\varphi) = N(A_0(X_\varphi, K_n)) \subset K_nF$$

If φ is singular, then $D_n(\varphi) = K_nF$.
One has

$$D_0(\varphi) = \begin{cases} K_0F & \text{if } \varphi \text{ is isotropic} \\ 2K_0F & \text{if } \varphi \text{ is non-isotropic.} \end{cases}$$

If φ represents 1, then $D_1(\varphi)$ is the subgroup of F^* generated by all nonzero $\varphi(x)$.

II. The results

Theorem A

Let $X = X_\varphi$ with $\varphi = \langle\langle a, b \rangle\rangle - \langle c \rangle$. Then there are natural isomorphisms

$$A^2(X; K_2) = D_0(\langle\langle a, b\rangle\rangle) \oplus K_0F/D_0(\langle\langle a, b, c\rangle\rangle)$$
$$A^2(X, K_3) = D_1(\langle\langle a, b\rangle\rangle) \oplus K_1F/D_1(\langle\langle a, b, c\rangle\rangle)$$

compatible with multiplication.

Consequences:

Theorem B

Let $Y = X_\varphi$ with $\varphi = \langle 1, -a, -b \rangle$. Then, for $n \leq 2$,

$$N : A^1(Y, K_{n+1}) \longrightarrow K_nF \qquad \text{is injective.}$$

Theorem C

a) Nrd : $K_2D \to K_2F$ is injective for quaternion algebras D

b) $K_3L \xrightarrow{1-\sigma} K_3L \xrightarrow{N} K_3F$ is exact $(L = F(\sqrt{a}); \mathrm{Gal}\,(L/F) = (\sigma))$

c) $K_3F/2 \longrightarrow H^3(F)$ is bijective.

Proof of Thm B $\Rightarrow$ Thm C

a) One has a commutative diagram

$$\begin{tikzcd} A^1(Y, K_3) \arrow[r, "r"] \arrow[d, "N"] & A^1(Y; K_3^Q) \arrow[r, equal] & H^1(Y; K_3) \arrow[r, equal] & K_2D \arrow[dlll, "\mathrm{Nrd}"'] \\ K_2F & & & \end{tikzcd}$$

Since r is surjective and N is injective one has Ker Nrd $= 0$.

b) This follows from Theorem B as shown in my first preprint on Hilbert 90 for K_3.

c) This follows from b) by Merkuriev's arguments.

III. The basic result

Let $f \in \mathcal{O}_{\mathbb{A}^N}$ be a polynomial and let ψ be a Pfister form over F. We are concerned with the following subcomplex of the usual Milnor complex for $\mathbb{A}^N$:

$$\bigoplus_{v\in(\mathbb{A}^N)^{(p-2)}} D_2(\psi\otimes\ll f(v)\gg) \xrightarrow{d} \bigoplus_{v\in(\mathbb{A}^N)^{(p-1)}} D_1(\psi\otimes\ll f(v)\gg)$$
$$\xrightarrow{d} \bigoplus\nolimits_{v\in(\mathbb{A}^N)^{(p)}} D_0(\psi\otimes\ll f(v)\gg) \longrightarrow 0.$$

The homology groups of this complex are denoted by

$$A^{p-1}(\mathbb{A}^N, D_p(\psi\otimes\ll f\gg)) \quad\text{and}\quad A^{p}(\mathbb{A}^N, D_p(\psi\otimes\ll f\gg)).$$

Theorem D
Let $\varphi = <1, -a, -b, abc>$. Then

$$N : A_0(X_\varphi, K_1) \longrightarrow K_1F$$

is injective. Its image is $D_1(\ll a, b\gg_{F(\sqrt{c})}) \cap K_1F \subset K_1F(\sqrt{c})$.

The injectivity of N is proved in [Merkuriev, Suslin; On the norm homomorphism in degree 3]. There is a proof without using Quillen-K-Theory similar to Merkuriev's proof of $A_0(Y, K_1) \hookrightarrow K_1F$ or a conic Y. I will consider this elsewhere.

The main technical result in the proof of Hilbert Satz 90 for K_3 is the following:

Theorem E:

i) For any quadratic from φ over F:

$$A^N(\mathbb{A}^N, D_n(\varphi)) = 0$$

ii) Let $a, b \in F^*$, $\varphi = <1>$, $d \in F$; Then for $n = 0, 1$:

$$A^1(\mathbb{A}^1, D_{n+1}(\ll a, b\hat{\varphi} - abd\gg) = \tfrac{D_n(\ll a,b\gg_K)\cap K_nF}{D_n(\ll a,b\gg)}$$

where $K = F(\sqrt{d})$ and $\hat{\varphi} \in \mathcal{O}_{\mathbb{A}^1}$ is the polynomial corresponding to φ.
(so $\hat{\varphi}(t) = t^2$)

iii) $A^0(\mathbb{A}^1, D_1(\ll a, b\hat{\varphi} - abd\gg) = D_1(\ll a\gg) + N_{K/F}(D_1(\ll a, b\gg_K))$

iv) Let $\psi = \langle\!\langle a \rangle\!\rangle$ and $c \in F^*$. Then
$A^1(\mathbb{A}^2; D_2(\langle\!\langle a, b\hat{\psi} + c \rangle\!\rangle)) = 0$,
where $\hat{\psi} \in \mathcal{O}_{\mathbb{A}^2}$ is the polynomial corresponding to ψ.

We need the following (well known?) lemma:

Lemma

a) $D_1(\langle\!\langle a \rangle\!\rangle_{F(\sqrt{e})}) \cap K_1F = D_1(\langle\!\langle a \rangle\!\rangle) + D_1(\langle\!\langle ae \rangle\!\rangle)$

b) Let ψ be a Pfister form; then

$$D_1(\psi) \cap D_1(\langle\!\langle e \rangle\!\rangle) = 2K_1F + N_{F(\sqrt{e})}(D_1(\psi_{F(\sqrt{e})})).$$

Proof of a)
Let $u \in F(\sqrt{a}, \sqrt{e})^*$ such that $N_{F(\sqrt{a},\sqrt{e})(F(\sqrt{e})}(u) \in F^*$. Multiplying u by an element from $F(\sqrt{a})^*$ we may assume $u = \alpha + \beta\sqrt{a} + \gamma\sqrt{e}$; $\alpha, \beta, \gamma \in F$. One must have $\alpha \cdot \gamma = 0$. ...

Proof of b)
Any element of $D_1(\psi)$ is in $D_1(\langle\!\langle a \rangle\!\rangle)$ for some a such that $\psi_{F(\sqrt{a})} \sim 0$. Hence we may assume $\psi = \langle\!\langle a \rangle\!\rangle$. But

$$N(F(\sqrt{a})^*) \cap N(F(\sqrt{e})^*) = (F^*)^2 \cdot N(F(\sqrt{a}, \sqrt{e})^*);$$

To see this suppose $u \in F(\sqrt{a})^*$, $v \in F(\sqrt{e})^*$ such that $N(u) = N(v)$. One checks easily

$$N(u) = N(v) = (tr(u) + tr(v))^{-2} N(u + v)$$ qed.

Proof of i)
By the norm principle we may assume that φ is isotropic. Then

$$A^N(\mathbb{A}^N, D_n(\varphi)) = A^N(\mathbb{A}^N, K_n) = 0.$$

Proof of ii)
Put $\Omega = A^1(\mathbb{A}^1, D_{n+1}(\langle\!\langle a, b\hat{\varphi} - abd \rangle\!\rangle)$. In view of i) we find that Ω is the cokernel of

$$(*) \qquad \frac{D_{n+1}(\langle\!\langle a, b\hat{\varphi}(\eta) - abd \rangle\!\rangle)}{D_{n+1}(\langle\!\langle a \rangle\!\rangle_{K(\eta)})} \xrightarrow{d} \bigoplus_{v \in \mathbb{A}^{1(1)}} \frac{D_n(\langle\!\langle a, b\hat{\varphi}(v) - abd \rangle\!\rangle)}{D_n(\langle\!\langle a \rangle\!\rangle_{K(v)})}$$

where η is the generic point of $\mathbb{A}^1$.

Let $W = \{x_1^2 - ax_2^2 - bx_3^2 + abd = 0\} \subset \mathbb{A}^3$. Then $W = \bar{W} \backslash Y$, where

$$\bar{W} = X_{<1,-a,-b,abd>}, Y = X_{<1,-a,-b>}.$$

We have an exact sequence

$$A^1(Y; K_{n+1}) \longrightarrow A^2(\bar{W}, K_{n+2}) \longrightarrow A^2(W, K_{n+2}) \longrightarrow 0.$$

By Theorem D and the computation $A^1(Y, K_{n+1}) = D_n(\ll a, b \gg)$ it suffices to show $\Omega = A^2(W, K_{n+2})$.

Consider the projection $\pi : W \to \mathbb{A}^1$, $(x_1, x_2, x_3) \to x_3$. The corresponding spectral sequences yield exact sequences

$$(**) \qquad A^1(\pi^{-1}(\eta), K_{n+2}) \xrightarrow{d} \bigoplus_{v \in \mathbb{A}^{1(1)}} A^1(\pi^{-1}(v), K_{n+1}) \longrightarrow A^2(W, K_{n+2}) \longrightarrow 0.$$

The fibers $\pi^{-1}(v)$ are affine conics given by $x_1^2 - ax_2^2 - (b\hat{\varphi}(v) - abd) = 0$. Hence $\pi^{-1}(v) = X_{<1,-a,-(b\hat{\varphi}(v)-abd)>} \setminus \{\mathrm{Spec} L\}$ and

$$A^1(\pi^{-1}(v), K_{n+1}) = A^1(X_{<1,-a,-(b\hat{\varphi}(v)-abd)>}, K_{n+1})/i_* K_n L.$$

Taking norms gives a map from $(**)$ to $(*)$ which yields the desired isomorphism $A^2(W, K_{n+2}) = \Omega$.

Proof of iii)

We have

$$\begin{array}{lll}
 & A^0(\mathbb{A}^1, D_1(\ll a, b\hat{\varphi} - abd \gg)) = & \\
= & D_1(\ll a, bt^2 - abd \gg) \cap K_1 F & (\text{in } K_1 F(t)) \\
= & \{f \in F^* \mid \{a, bt^2 - abd, f\} = 0 \text{ in } K_3 F(t)/2\} & \\
= & \{f \in F^* \mid \{a, b, f\} = 0 \text{ in } K_3 F/2,\ \{a, f\} = 0 \text{ in } K_2 F(\sqrt{ad})/2\} & \\
= & D_1(\ll a, b \gg) \cap D_1(\ll a \gg_{F(\sqrt{ad})}) & \\
= & D_1(\ll a, b \gg) \cap (D_1(\ll a \gg) + D_1(\ll d \gg)) & \text{by the Lemma a)} \\
= & D_1(\ll a \gg) + (D_1(\ll a, b \gg) \cap D_1(\ll d \gg)) & \\
= & D_1(\ll a \gg) + N_{K/F}(D_1(\ll a, b \gg_K)) & \text{by the Lemma b).}
\end{array}$$

qed.

Proof of iv)
Consider the projection $\pi : \mathbb{A}^2 \to \mathbb{A}^1$, $(x,y) \to y$ where x,y are coordinates such that $\hat{\psi} = x^2 - ay^2$. π induces the following exact sequence (where $d = y^2 - abc \in F[y] = \mathcal{O}_{\mathbb{A}^1}$)

$$A^0(\mathbb{A}^1_{F(y)}; D_2(\langle\langle a, b\hat{\varphi} - abd\rangle\rangle)) \xrightarrow{d'} \bigoplus_{v\in\mathbb{A}^{1(1)}} A^0(\mathbb{A}^1_{K(v)}, D_1(\langle\langle a, b\hat{\varphi} - abd(v)\rangle\rangle)) \xrightarrow{i_*}$$

$$A^1(\mathbb{A}^2, D_2(\langle\langle a, b\hat{\psi} + c\rangle\rangle)) \xrightarrow{\pi^*}$$

$$A^1(\mathbb{A}^1_{F(y)}, D_2(\langle\langle a, b\hat{\varphi} - abd\rangle\rangle)) \xrightarrow{d''} \bigoplus_{v\in\mathbb{A}^{1(1)}} A^1(\mathbb{A}^1_{K(v)}; D_1(\langle\langle a, b\hat{\varphi} - abd\rangle\rangle)).$$

We show that d' is surjective and that d'' is injective.

Surjectivity of d'
Consider the following diagram

$$\begin{tikzcd}
K_2L(y) \arrow[r, "d"] \arrow[d, "N_{L/F}"] & \bigoplus_{v\in\mathbb{A}^{(1)}} K_1L\otimes_F K(v) \arrow[r] \arrow[d, "N_{L/F}"] & 0 \\
A^0(\mathbb{A}^1_{F(y)}, D_2(\langle\langle a, b\hat{\varphi} - abd\rangle\rangle) \arrow[r, "d'"] & \bigoplus_{v\in\mathbb{A}^{1(1)}} A^0(\mathbb{A}^1_{K(v)}, D_1(\langle\langle a, b\hat{\varphi} - abd(v)\rangle\rangle) & \\
D_2(\langle\langle a,b\rangle\rangle_{F(Z)}) \arrow[r, "d"] \arrow[u, "p_*"] & \bigoplus_{W\in Z^{(1)}} D_1(\langle\langle a,b\rangle\rangle_{K(w)}) \arrow[r] \arrow[u, "p_*"] & 0
\end{tikzcd}$$

The top row is the surjective tame symbol for $\mathbb{A}^1_L$.
Clearly $D_n(\langle\langle a\rangle\rangle) \subset A^0(\mathbb{A}^1, D_n(\langle\langle a, b\hat{\varphi} - abd(v)\rangle\rangle)$ hence $N_{L/F}$ is well defined.

To describe the bottom row let

$$\bar{Z} = \{x^2 - y^2 + abcz^2 = 0\} \subset \mathbb{P}^2$$

and

$$Z = \bar{Z}\backslash\{z = 0\}.$$

Clearly $\bar{Z} \simeq \mathbb{P}^1$ and $Z \simeq \mathbb{A}^1\backslash\{\text{rational point}\}$. By i) the bottom row is exact.
The maps p_* are induced by the double cover $p : Z \to \mathbb{A}^1$, $[x,y,1] \to [y,1]$. It has $y^2 = abc$ as branching point and one has $K(p^{-1}(v)) = K(v)(\sqrt{d(v)})$ for $v \in \mathbb{A}^1$. Note that (with $v = p(w)$) $p_*(D_n(\langle\langle a,b\rangle\rangle_{K(w)})) \subset A^0(\mathbb{A}^1_{K(v)}, D_n(\langle\langle a, b\hat{\varphi} - abd(v)\rangle\rangle)$ because $D_n(\langle\langle a,b\rangle\rangle) \subseteq A^0(\mathbb{A}^1, D_n(\langle\langle a, b\hat{\varphi} - abd\rangle\rangle)$ if d is a square. By iii) we know that $p_* \oplus N_{L/F}$ is surjective on the right side (degree 1). Consequently d' is surjective.

Injectivity of d''

One has the following diagram

$$\begin{tikzcd}
 & 0 & 0 & \\
 & A^1(\mathbb{A}^1_{F(y)}, D_2(\ll a, b\hat{\varphi} - abd\gg)) \arrow[u] \arrow[r, "d''"] & \bigoplus\limits_{v \in \mathbb{A}^{1(1)}} A^1(\mathbb{A}^1_{K(v)}, D_1(\ll a, b\hat{\varphi} - abd\gg)) \arrow[u] & \\
D_1(\ll a, b\gg) \arrow[r] & D_1(\ll a, b\gg_{F(y)(\sqrt{d})}) \cap K_1F(y) \arrow[u] \arrow[r, "d"] & \bigoplus\limits_{v \in \mathbb{A}^{1(1)}} (D_1(\ll a, b\gg_{K(v)(\sqrt{d(v)})}) \cap K_1K(v)) \arrow[u] & \\
D_1(\ll a, b\gg) \arrow[u, "||"] \arrow[r] & D_1(\ll a, b\gg_{F(y)}) \arrow[u] \arrow[r, "d"] & \bigoplus\limits_{v \in \mathbb{A}^{1(1)}} D_1(\ll a, b\gg_{K(v)}) \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

Here the columns are exact and given by ii). The bottom row is exact, because $D_1(\ll a, b\gg_{F(y)}) \cap K_1F = D_1(\ll a, b\gg)$ and by i). The middle row is exact, because $\operatorname{Ker} d = D_1(\ll a, b\gg_{F(y)(\sqrt{d})}) \cap K_1F$ and $F(y)(\sqrt{d}) = F(y)(\sqrt{y^2 - abc})$ is rational over F. Now an easy diagram chase does the job.

IV. Proof of Thm A

Proposition 1
Let $Z = X_{\ll a,b \gg}$. Then

$$A^1(Z, K_2) = D_1(\ll a, b\gg) \oplus K_1 F.$$

Proof
Let $X = X_{<1,-a,-b>}$. Then the spectral sequences for $Y \times Z \to Z$, $Y \times Z \to Y$ yield exact sequences

$$\begin{array}{ccccccc} 0 \longrightarrow A^1(Z, A^0(Y, K_2)) & \longrightarrow & A^1(Z \times Y, K_2) & \longrightarrow & A^0(Z, A^1(Y, K_2)) & \xrightarrow{d_2} & \dots \\ & & \| & & & & \\ 0 \longrightarrow A^1(Y, A^1(Z, K_2)) & \longrightarrow & A^1(Y \times Z, K_2) & \longrightarrow & A^0(Y, A^1(Z, K_2)) & \longrightarrow & 0. \end{array}$$

Because Y is trivial over Z and Z is trivial over Y we find

$$\begin{aligned} A^1(Z, A^0(Y, K_2)) &= A^1(Z, K_2) \\ A^0(Z, A^1(Y, K_2)) &= A^0(Z, K_1) = K_1 F \\ A^1(Y, A^0(Z, K_2)) &= A^1(Y, K_2) = D_1(\ll a, b\gg) \\ A^0(Y, A^1(Z, K_2)) &= A^=(Y, K_1 F \oplus K_1 F) = K_1 F \oplus K_1 F. \end{aligned}$$

The result follows immediately (consider e.g. the situation one degree lower and use multiplicativity) qed.

Let $U = X \backslash Z$, where X is as in Theorem A and $Z \subset X$ is considered as hyperplane section. There is an exact sequence

$$A^1(Z, K_2) \xrightarrow{i_*} A^2(X, K_3) \longrightarrow A^2(U, K_3).$$

One finds that the kernel of i_* is the image of

$$\begin{aligned} D_1(\ll a, b, c\gg) &\longrightarrow D_1(\ll a, b\gg) \oplus K_1 F \\ U &\longrightarrow (2u, -u) \end{aligned}$$

I omit the proof here. Clearly the hard point in the proof of Theorem A is the surjectivity of i_*. I show $A^2(U, K_3) = 0$.

Compactification of *U*
Let $\bar{U} \subset \mathbb{A}^2 \times \mathbb{P}^2$ be the variety defined by

$$0 = x_1^2 - a x_2^2 - x_3^2[(y_1^2 - a y_2^2) b + c] \ , \ [x_1, x_2, x_3] \in \mathbb{P}^2 \ , \ (y_1, y_2) \in \mathbb{A}^2,$$

and let $V = \bar{U} \cap \{x_3 = 0\} \subset \mathbb{A}^2 \times_F \mathbb{P}^1$. Note that $U = \bar{U} \setminus V$ and $V = \mathbb{A}^2 \times_F \operatorname{Spec} L$. We have an exact sequence

$$A^2(\bar{U}, K_3) \longrightarrow A^2(U, K_3) \longrightarrow A^2(V, K_2).$$

Because $A^2(V, K_2) = 0$ it suffices to show:

$A^2(\bar{U}, K_3) = 0$

Let $\pi : \bar{U} \to \mathbb{A}^2$ be induced by the projection $\mathbb{A}^2 \times \mathbb{P}^2 \to \mathbb{A}^2$. π induces a spectral sequence

$$E_1^{p,q} = \bigoplus_{v \in \mathbb{A}^{2(p)}} A^q(\pi^{-1}(v), K_{3-p}) \Rightarrow A^{p+q}(\bar{U}, K_3).$$

It suffices to show $E_2^{p,q} = 0$ for $p + q = 2$. Note that the fiber over v is the projective conic $Y_{<1,-a,-f(v)>}$ where $f = (y_1^2 - a y_2^2) b + c \in \mathcal{O}_{\mathbb{A}^2}$. It is singular over $v \in W = \{y_1^2 - a y_2^2 + b^{-1} c = 0\} \subset \mathbb{A}^2$.

Proof of $E_2^{2,0} = 0$

We have for $n \leq 2$:

$$A^0(\pi^{-1}(v), K_n) = \begin{cases} K_n(K(v)) & \text{if } v \notin W \\ K_n(L \otimes_F K(v)) & \text{if } v \in W \end{cases}$$

Consider the diagram

$$\begin{array}{ccccccc}
\bigoplus_{v \in \mathbb{A}^{2(1)}} K_2 K(v) & \xrightarrow{d} & \bigoplus_{v \in \mathbb{A}^{2(2)}} K_1 K(v) & \longrightarrow & 0 & & \\
\downarrow r & & \downarrow r & & & & \\
\bigoplus_{v \in \mathbb{A}^{2(1)}} A^0(\pi^{-1}(v), K_2) & \xrightarrow{d_1^{2,0}} & \bigoplus_{v \in \mathbb{A}^{2(2)}} A^0(\pi^{-1}(v), K_1) & \longrightarrow & E_2^{2,0} & \longrightarrow & 0 \\
\uparrow r' & & \uparrow r'' & & & & \\
\bigoplus_{v \in W^{(0)}} K_2(L \otimes_F K(v)) & \xrightarrow{d} & \bigoplus_{v \in W^{(1)}} K_1(L \otimes_F K(v)) & \longrightarrow & 0 & &
\end{array}$$

Here r is induced by restriction and r' is induced by identifying $L \otimes_F K(v)$ with the algebraic closure of $K(v)$ in the function field of $\pi^{-1}(v)$ for $v \in W$.
The top row is exact, and so is the bottom row, because
$W \times \operatorname{Spec} L \simeq \mathbb{P}^1 \times \operatorname{Spec} L \backslash \{2 \text{ L-rational points}\}$.

Since $r \oplus r''$ is surjective we find $E_2^{2,0} = 0$.

Proof of $E_2^{1,1} = 0$.
We have the following diagram with exact columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
\bigoplus_{v\in\mathbb{A}^{2(0)}} A^1(\pi^{-1}(v),K_3) \arrow[r] \arrow[d, "N"] & \bigoplus_{v\in\mathbb{A}^{2(1)}} A^1(\pi^{-1}(v),K_2) \arrow[r] \arrow[d, "N"] & \bigoplus_{v\in\mathbb{A}^{2(2)}} A^1(\pi^{-1}(v),K_1) \arrow[d, "N"] \\
\bigoplus_{v\in\mathbb{A}^{2(0)}} D_2(\ll a, f(v)\gg) \arrow[r, "d"] \arrow[d] & \bigoplus_{v\in\mathbb{A}^{2(1)}} D_1(\ll a, f(v)\gg) \arrow[r, "d"] \arrow[d] & \bigoplus_{v\in\mathbb{A}^{2(2)}} D_0(\ll a, f(v)\gg) \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

The homology of the top row is $E_2^{1,1}$. But the bottom row is exact by Theorem E iv) (page 3). $\square$

V. Proof of Thm A $\Longrightarrow$ Thm B

Let $A^1(Y,K_n)^\sim = \operatorname{Ker} N \subset A^1(Y,K_n)$.
Specialization arguments (which will be considered elsewhere) show that it suffices to show that

$$(*) \qquad r_{F(X)/F} : A^1(Y,K_3)^\sim \longrightarrow A^1(Y_{F(X)},K_3)^\sim$$

is surjective (where X is as in Thm A). To prove this I consider the following groups and maps (to be described below) $(n=2,3)$

$$\begin{tikzcd}
\frac{A^1(Y_{F(X)},K_n)^\sim}{r_{F(X)/F}(A^1(Y,K_n)^\sim)} \arrow[d, "\alpha"] & \\
\frac{A^0(X,A^1(Y,K_n))}{r_{F(X)/F}(A^1(Y,K_n))} \arrow[r, "N"] \arrow[d, "\beta"] & \frac{D_{n-1}(\ll a,b\gg_{F(X)})\cap K_{n-1}F}{D_{n-1}(\ll a,b\gg)} \\
\frac{A^0(X,A^1(Y,K_n))}{\pi(A^1(X\times Y,K_n))} \arrow[d, "\delta"] & \\
\operatorname{Ker}\,(A^2(X,K_n) \longrightarrow A^2(X_{F(Y)},K_n)) \arrow[r, "\varepsilon"] & K_{n-2}F/D_{n-2}(\ll a,b,c\gg) \arrow[uu, "\gamma"]
\end{tikzcd}$$

Here ε denotes the isomorphism from Theorem A and N is induced by the norm map. Below I define $\alpha,\beta,\delta,\gamma$ and I show that $\alpha,\beta,\gamma,\delta$ are injective (in fact they are isomorphisms with the exception $\alpha=0$) and that $N=\gamma\varepsilon\delta\gamma$. Clearly this implies $(*)$, because $N\circ\alpha=0$.

Definition and injectivity of $\boldsymbol{\alpha}$
For $n=2$ we know already $A^1(Y,K_n)^\sim=0$. For $n=3$ consider the commutative diagram

$$\begin{tikzcd}
& A^1(Y,K_3)^\sim \arrow[r] \arrow[d, hook] & A^1(Y_{F(X)},K_3)^\sim \arrow[r] \arrow[d, hook] & 0 \arrow[d] \\
& A^1(Y,K_3) \arrow[r] \arrow[d, "N"] & A^1(Y_{F(X)},K_3) \arrow[r, "d'"] \arrow[d, "N"] & \bigoplus\limits_{v\in X^{(1)}} A^1(Y_{K(v)},K_2) \arrow[d, "N"] \\
0 \arrow[r] & K_2F \arrow[r] & K_2F(X) \arrow[r, "d"] & \bigoplus\limits_{v\in X^{(1)}} K_1K(v)
\end{tikzcd}$$

Here d' is the differential $E_1^{0,1}\to E_1^{1,1}$ from the spectral sequence

$$E_1^{p,q} = \bigoplus_{v\in X^{(p)}} A^1(Y_{K(v)},K_{3-p}) \Rightarrow A^{p,q}(X\times Y,K_3).$$

The columns are exact by definition or by the knowledge for the K_2-case. Hence
$$A^1(Y_{F(X)}, K_3)^{\sim} \subset \operatorname{Ker} d' = A^0(X, A^1(Y, K_3)).$$
We define α to be the induced map. It is injective because $K_2F \hookrightarrow K_2F(X)$.

Definition of $\boldsymbol{\beta}$
Just projection. π is induced by the spectral sequence $E_2^{p,q} = A^p(X, A^q(Y, K_n)) \Rightarrow A^{p+q}(X \times Y, K_n)$.

Injectivity of $\boldsymbol{\beta}$
The spectral sequences for $X \times Y \to X$ and $Y \times X \to Y$ yield exact sequences

$$0 \longrightarrow A^1(X, A^0(Y, K_n)) \xrightarrow{i} A^1(X \times Y, K_n) \xrightarrow{\pi} A^0(X, A^1(Y, K_n)) \xrightarrow{d_2^{0,1}} \ldots$$
$$0 \longrightarrow A^1(Y, A^0(X, K_n)) \xrightarrow{\tilde{i}} A^1(Y \times X, K_n) \xrightarrow{\tilde{\pi}} A^0(Y, A^1(X, K_n)) \longrightarrow 0$$

Because X is trivial over Y we have

i) $\quad A^1(Y, A^0(X, K_n)) = A^1(Y, K_n)$

ii) $\quad A^0(Y, A^1(X, K_n)) = A^0(Y, K_{n-1}) \otimes \operatorname{Pic}(X)$.

We have to show $\operatorname{Im}\pi = \operatorname{Im}\pi \circ \tilde{i}$ by i).
But
$$\frac{\operatorname{Im}\pi}{\operatorname{Im}\pi\circ\tilde{i}} = \frac{\operatorname{Im}\tilde{\pi}}{\operatorname{Im}\tilde{\pi}\circ i} = 0;$$
here the last equation follows from the obvious factorization of the isomorphism in ii) via $A^1(X, A^0(Y, K_n))$.

Definition and injectivity of $\boldsymbol{\delta}$
Consider

$$\begin{tikzcd}
A^1(X \times Y, K_n) \arrow[r, "\pi"] & A^0(X, A^1(Y, K_n)) \arrow[r, "d_2^{0,1}"] & A^2(X, A^0(Y, K_n)) \arrow[r, "i"] & A^2(X \times Y, K_n) \arrow[d, "\tilde{\pi}"] \\
 & & & A^0(Y, A^2(X, K_n)) \\
 & & & \cap \\
 & & A^2(X, K_n) \arrow[uuu, "r"] \arrow[r, "r_{F(X)/F}"] & A^2(X_{F(Y)}, K_n)
\end{tikzcd}$$

Here $\pi, d_2^{0,1}$ and i are from the spectral sequence for $X \times Y \to X$, $\tilde{\pi}$ is from the spectral sequence for $X \times Y \to Y$ and r is induced by multiplication with $A^0(Y, K_0) = CH^0(Y)$. Clearly $d_2^{0,1} \circ \pi = 0$ and $i \circ d_2^{0,1} = 0$. Moreover, r is bijective for $n \leq 3$, because $K_mK = A^0(Y_K, K_m)$ for $m \leq 2$.

Now put $\delta = r^{-1} \circ d_2^{0,1}$. δ is injective because there are no more differentials starting from or landing in $E_2^{0,1}$.

Definition and injectivity of $\boldsymbol{\gamma}$

$\gamma(U \mod D_{n-2}(\ll a, b, c \gg)) = U \cdot \{c\} \mod D_{n-1}(\ll a, b \gg)$. By quadratic from theory γ is well defined and injective ($n \leq 3$).

Proof of $\boldsymbol{N = \gamma\varepsilon\delta\beta}$

We know already that $\gamma\varepsilon\delta\beta$ is injective. If $n = 2$ we know that N is bijective; because the target group is 0 or $\mathbb{Z}/2$ both maps must coincide.
For $n = 3$ use multiplication with K_1 and the injectivity of $\gamma\varepsilon\delta\beta$. Q.E.D.