THE MULTIPLICATIVE TRANSFER FOR THE GROTHENDIECK WITT RING

MARKUS ROST

Introduction

In this text we collect some material related with the multiplicative norm for the Grothendieck-Witt ring. It is a first draft.

1. Polynomials on semi groups

Let A be an additive semi group. In other words, we are given a set A and a commutative and associative operation $(x, y) \mapsto x + y$ on A.

We do not assume that A contains a neutral element 0. In any case, there is the semi group with neutral element $A_0 = A \cup \{0\}$ (with $A_0 = A$ if $0 \in A$).

Let $A \to \bar{A}$ be the group completion of A.

Let further B be a commutative group. Then, by the very definition of group completion, a semi group homomorphism $P\colon A \to B$ extends to a unique group homomorphism $\bar{P}\colon \bar{A} \to B$.

The purpose of this section is to note that every polynomial map $P\colon A \to B$ extends to a unique polynomial map $\bar{P}\colon \bar{A} \to B$. This is perhaps well known (indeed, see [2]); it becomes almost a triviality after relating group completions and polynomial maps with the corresponding (semi) group rings.

Let $\mathbf{Z}[A]$ be the semi group ring of A. As a group, $\mathbf{Z}[A]$ is the free abelian group on A. For $x \in A$ we denote by $[x]$ the corresponding generator of $\mathbf{Z}[A]$. Then the multiplication in $\mathbf{Z}[A]$ is given by $[x][y] = [x + y]$.

If $0 \in A$, then $[0] = 1$ is the unit element of $\mathbf{Z}[A]$. In general, $\mathbf{Z}[A]$ is an ideal of the unital ring $\mathbf{Z}[A_0]$.

We consider A as subset of $\mathbf{Z}[A]$ in the natural way, via $x \mapsto [x]$. It is clear that any map $P\colon A \to B$ extends to a unique group homomorphism $\widehat{P}\colon \mathbf{Z}[A] \to B$.

The subset A of $\mathbf{Z}[A]$ is a multiplicative subset. It is tautological that the group ring of the group completion $\bar{A}$ is localization of $\mathbf{Z}[A]$ at A:

$$\mathbf{Z}[\bar{A}] = A^{-1}\mathbf{Z}[A]$$

Definition. Let $P\colon A \to B$ be a map.

(1) P is called a polynomial map of degree ≤ -1 if $P = 0$.
(2) For $n \geq 0$, P is called a polynomial map of degree $\leq n$ if for each $x \in P$ the map

$$\Delta_x P\colon A \to B$$
$$\Delta_x P(y) = P(y + x) - P(y)$$

is a polynomial map of degree $\leq n - 1$.

Date: January 28, 2003.

(3) P is called a polynomial map if it is a polynomial map of degree $\leq n$ for some n.

We denote by

$$\epsilon_A \colon \mathbf{Z}[A_0] \to \mathbf{Z}$$
$$\epsilon_A([x]) = 1 \quad (x \in A_0)$$

the augmentation homomorphism and by $I_A = \ker \epsilon_A$ its kernel.

For $\alpha \in \mathbf{Z}[A]$ one has

$$\widehat{\Delta_x P}(\alpha) = \widehat{P}\big(([x]-1)\alpha\big)$$

This implies easily by an induction argument the following

Lemma 1. *A map $P \colon A \to B$ is a polynomial map of degree $\leq n$ if and only if*

$$\widehat{P}(I_A^{n+1}\mathbf{Z}[A]) = 0$$

Corollary 1. *There is a bijection*

$$P \mapsto \big(\alpha \bmod I_A^{n+1}\mathbf{Z}[A] \mapsto \widehat{P}(\alpha)\big)$$

from the group of polynomial maps $A \to B$ of degree $\leq n$ to

$$\operatorname{Hom}(\mathbf{Z}[A]/I_A^{n+1}\mathbf{Z}[A], B)$$

Example. The polynomial maps on $A = \mathbf{N}_0^r$ are the linear combinations of the polynomials of the form

$$P(x_1, \dots, x_r) = \prod_{i=1}^{r} \binom{x_i}{n_i}$$

where the n_i are nonnegative integers. (Use Corollary 1.)

For $x \in A$ one has $1 - [x] \in I_A$ and therefore $[x] \bmod I_A^{n+1}$ is invertible. Thus

$$\mathbf{Z}[\bar{A}]/I_{\bar{A}}^{n+1} = A^{-1}\mathbf{Z}[A]/I_A^{n+1}\mathbf{Z}[A] = \mathbf{Z}[A]/I_A^{n+1}\mathbf{Z}[A]$$

and we have

Corollary 2. *A polynomial map $P \colon A \to B$ extends uniquely to a polynomial map $\bar{P} \colon \bar{A} \to B$. If P is of degree $\leq n$, then $\bar{P}$ is of degree $\leq n$.*

Let $P \colon A \to B$ be a polynomial map of degree $\leq n$. Its extension $\bar{P}$ can be made explicit as follows. For x, $y \in A$ we have

$$[x-y](1-[y])^{n+1} = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} [x+(i-1)y]$$

On the other hand, one has $[x-y](1-[y])^{n+1} \in I_{\bar{A}}^{n+1}$ and therefore

$$\widehat{\bar{P}}\big([x-y](1-[y])^{n+1}\big) = 0$$

This gives

$$\bar{P}(x-y) = \sum_{i=0}^{n} (-1)^i \binom{n+1}{i+1} P(x+iy) \tag{1}$$

expressing any value of $\bar{P}$ in terms of values of P.

2. Multiplicative polynomials on $\mathbf{Z}$

Multiplicative polynomial maps appear from multiplicative transfer maps on cohomology rings, Grothendieck-Witt rings of symmetric bilinear forms, etc. In those cases it is an obvious question to compute the restriction to the subring generated by 1.

Definition. Let A, B be commutative rings. A polynomial map $P\colon A \to B$ is called *multiplicative* if $P(1) = 1$ and if

$$P(xy) = P(x)P(y)$$

for x, $y \in A$.

Let $P\colon A \to B$ be a multiplicative polynomial map. Then $P(0)^2 = P(0)$. If $P(0) = 1$, then P is constant, $P \equiv 1$. If $P(0) = 0$, we call P *proper.* If B is connected, then P is either constant or proper.

A polynomial map $P\colon \mathbf{Z} \to B$ of degree $\leq n$ can be written uniquely as

$$P(k) = a_0 + a_1 k + a_2 \binom{k}{2} + \cdots + a_n \binom{k}{n} \tag{2}$$

with $a_i \in B$. If $P(0) = 0$ and $P(1) = 1$, then $a_0 = 0$ and $a_1 = 1$ and

$$P(k) = k + a_2 \binom{k}{2} + \cdots + a_n \binom{k}{n} \tag{3}$$

Let X_n be the scheme over $\operatorname{Spec} \mathbf{Z}$ representing the multiplicative polynomials on $\mathbf{Z}$. The presentation (2) identifies X_n as a closed subscheme of the affine space $\mathbf{A}^{n+1}$ with coordinates $a_0, \ldots, a_n$.

Note that $a_0 + a_1 = 1$ and $a_0^2 = a_0$ on X_n.

Let $Y_n = X_n \cap \{a_0 = 0, a_1 = 1\}$. Then X_n is the disjoint union of Y_n and a copy of $\operatorname{Spec} \mathbf{Z}$ given by $(a_0, a_1, \ldots, a_n) = (1, 0, \ldots, 0)$. The latter represents $P = 1$ and we may write

$$X_n = Y_n \cup \{1\}$$

The scheme Y_n represents the proper multiplicative polynomials on $\mathbf{Z}$. The presentation (3) identifies Y_n as a closed subscheme of the affine space $\mathbf{A}^{n-1}$ with coordinates $a_2, \ldots, a_n$.

Lemma 2. *The scheme X_n is flat and finite over $\operatorname{Spec} \mathbf{Z}$ of degree $n + 1$.*
Let $T = \mathbf{Z}[(n!)^{-1}]$. Then $(X_n)_T$ is consists of $n + 1$ copies of $\operatorname{Spec} T$.

Proof. If $n!$ is invertible in B, one may parametrize polynomial maps $P\colon \mathbf{Z} \to B$ of degree $\leq n$ as

$$P(k) = b_0 + b_1 k + b_2 k^2 + \cdots + b_n k^n \tag{4}$$

with $b_i \in B$. It is easy to see that P is multiplicative if and only if the b_i are pairwise orthogonal idempotents. This proves the second claim.

For the general case observe that X_n is the spectrum of

$$R_n = \mathbf{Z}[a_0, \ldots, a_n]/I$$

where the ideal I is generated by $a_0 + a_1 - 1$ and elements $a_i a_j - l_{ij}$ $(1 \leq i \leq j \leq n)$ with the l_{ij} linear in the a_k. Hence R_n is generated by $a_0, \ldots, a_n$ as a $\mathbf{Z}$-module. Since $\dim_{\mathbf{Q}} R_n \otimes \mathbf{Q} = n + 1$ by the first part of the proof, it follows that R_n a free $\mathbf{Z}$-module of rank $n + 1$. $\square$

Example. The case $n = 2$. The proper multiplicative quadratic polynomials $\mathbf{Z} \to B$ are

$$P(k) = k + a\binom{k}{2}$$

where a is subject to $a^2 = 2a$. This follows easily from

$$\binom{kl}{2} = 2\binom{k}{2}\binom{l}{2} + \binom{k}{2}l + \binom{l}{2}k$$

This describes Y_2 completely.

Example. The case $n = 3$ with $2 = 0$ in B. If $2B = 0$, the proper multiplicative polynomials $\mathbf{Z} \to B$ of degree ≤ 3 are

$$P(k) = k + a\binom{k}{2} + b\binom{k}{2}k$$

where a, b are subject to $a^2 = ab = b^2 = 0$. This describes $(Y_3)_{\mathbf{F}_2}$ completely.

Exercise. Let L/F be a finite field extension and let $\mathcal{N}_{L/F}$ denote the multiplicative norm for the Grothendieck-Witt ring (to be defined later). Describe the polynomial $k \mapsto \mathcal{N}_{L/F}(k)$, $k \in \mathbf{Z}$.

It is clear by Serre that the coefficients of $\mathcal{N}_{L/F}|\mathbf{Z}$ can be expressed in terms of the exterior powers of the trace form. Is there a nice formula for that? Does the Pfister form $P(L/F)$ show up? I have not considered anything here.

Exercise. Let L be a finite field extension of the field of rational numbers $\mathbf{Q}$ of degree d. For $k \in \mathbf{Z}$ let $P(k)$ be the signature of $\mathcal{N}_{L/\mathbf{Q}}(k)$, where $\mathcal{N}_{L/\mathbf{Q}}$ is the multiplicative norm for the Grothendieck-Witt ring (to be defined later).

Show that $P(k) = k^{r+s}$ where r, s are the numbers of real, complex embeddings of L, respectively ($d = r + 2s$).

3. The norm

See also [1].

3.1. **Notations.** The symmetric group on i letters is denoted by $\mathcal{S}_i$. Let

$$\mathrm{sgn}\colon \mathcal{S}_i \to \{\pm 1\}$$

be the signature homomorphism.

Let F be a commutative ring and let M be an F-module. Let

$$T^i M = T^i_F M = M^{\otimes i} = \underbrace{M \otimes_F \cdots \otimes_F M}_{i \text{ factors}}$$

be the i-fold tensor power of M over F. The symmetric group $\mathcal{S}_i$ acts in a natural way on $M^{\otimes i}$ by

$$s(x_1 \otimes \cdots \otimes x_i) = x_{s^{-1}1} \otimes \cdots \otimes x_{s^{-1}i}$$

We write

$$\Lambda^i M = \Lambda^i_F M = M^{\otimes i} / \textstyle\sum_{s \in \mathcal{S}_i} \big(1 - \mathrm{sgn}(s)s\big) M^{\otimes i}$$
$$S_i M = S_i^F M = (M^{\otimes i})^{\mathcal{S}_i} = \{\, \alpha \in M^{\otimes i} \mid s(\alpha) = \alpha \text{ for } s \in \mathcal{S}_i \,\}$$

for the i-th exterior power of M and the module of symmetric i-tensors, respectively.

For $x \in M$ we use the notation

$$[x] = [x]_i = \underbrace{x \otimes \cdots \otimes x}_{i \text{ factors}} \in S_i M$$

We will also use the trace maps

$$T_{i,j} = \sum_{s \in \mathcal{S}_{i+j}/\mathcal{S}_i \times \mathcal{S}_j} s \quad : S_i M \otimes S_j M \to S_{i+j} M$$

3.2. **The norm.** Let F be a commutative ring and let E be a locally free ring extension of F of rank n.

For discussion of geometric interpretations let us write $X = \operatorname{Spec} E$ and $Y = \operatorname{Spec} F$. Thus we have a morphism of schemes $X \to Y$, finite and flat of degree n.

The tensor power $E^{\otimes i}$ is a locally free ring extension of F of rank n^i containing $S_i E$ as subring (all tensors are considered over F). The $S_i E$-module structure on $E^{\otimes i}$ induces an $S_i E$-module structure on $\Lambda^i E$ and thus we have homomorphisms

$$\rho_i \colon S_i E \to \operatorname{End}_F(\Lambda^i E)$$

The F-module $\Lambda^n E$ is invertible. Therefore we get a ring homomorphism

$$\nu_{E/F} = \rho_n \colon S_n E \to \operatorname{End}_F(\Lambda^n E) = F$$

called the *norm* of E/F. Indeed, for $x \in E$ one has

$$\nu_{E/F}([x]) = \left(\Lambda^n E \xrightarrow{\Lambda^n \mu_x} \Lambda^n E\right) = \det(\mu_x) = N_{E/F}(x)$$

where $\mu_x \colon E \to E$ is the multiplication map.

For a finitely generated locally free E-module M we put

$$\nu_{E/F}(M) = S_n M \otimes_{S_n E} F = (\nu_{E/F})_*(S_n M)$$

Suppose E/F is separable. Then $\nu_{E/F}(M)$ is a locally free F-module. If M is of rank d as E-module, then $\nu_{E/F}(M)$ is of rank d^n as F-module. If $e_1, \ldots, e_n$ are orthogonal idempotents of E (necessarily of rank 1: $e_i E = F$), then

$$\nu_{E/F}(M) = M_1 \otimes \cdots \otimes M_n \tag{5}$$

where $M_i = e_i M$.

Let $b \colon M \times M \to E$ be a non-degenerate bilinear symmetric form. Then the form

$$b^{\otimes n} \colon M^{\otimes n} \times M^{\otimes n} \to E^{\otimes n}$$

restricts to a form

$$S_n b \colon S_n M \times S_n M \to S_n E$$

which we tensor with $S_n E \to F$ to get a form

$$\nu_{E/F}(b) \colon \nu_{E/F}(M) \times \nu_{E/F}(M) \to F$$

If E/F is separable, then $\nu_{E/F}(b)$ is again a non-degenerate bilinear symmetric form. In the situation of (5) we have

$$\nu_{E/F}(b) = b_1 \otimes \cdots \otimes b_n$$

with $b_i = b|(M_i \times M_i)$.

In the next sections we study basic properties of the norm.

4. The extensions $C_{i_1,\dots,i_r}E$

Let $(i_1,\dots,i_r)$ be a sequence of integers $i_j \geq 0$ with $i_1+\cdots+i_r=n$. Such a sequence will be called a partition of n. (We allow $i_j=0$ for convenience.)

Let

$$S_{i_1,\dots,i_r}E = S_{i_1}E\otimes\cdots\otimes S_{i_r}E$$

Note that S_nE is a subring of $S_{i_1,\dots,i_r}E$.

We define

$$C_{i_1,\dots,i_r}E = (\nu_{E/F})_*(S_{i_1,\dots,i_r}E) = S_{i_1,\dots,i_r}E\otimes_{S_nE}F$$

Let $C^{i_1,\dots,i_r}X = \operatorname{Spec} C_{i_1,\dots,i_r}E$. Then there is a pull back diagram

$$\begin{array}{ccc} C^{i_1,\dots,i_r}X & \longrightarrow & S^{i_1}X\times\cdots\times S^{i_r}X \\ \Big\downarrow & & \Big\downarrow \\ Y & \xrightarrow{\nu} & S^nX \end{array}$$

If X/Y is separable, then $C^{i_1,\dots,i_r}X$ represents the decompositions of the fibers into subsets of size i_1, …, i_r.

If E/F is separable, then $C_{i_1,\dots,i_r}E$ is a separable F-algebra of rank

$$\binom{n}{i_1,\dots,i_r} = \binom{i_1+\cdots+i_r}{i_1}\binom{i_2+\cdots+i_r}{i_2}\cdots\binom{i_r}{i_r}$$

Example. If E/F is separable, then

$$\bar E = C_{1,\dots,1}E = E^{\otimes n}\otimes_{S_nE}F$$

is a separable F-algebra of rank $n!$. Moreover there is a natural $\mathcal{S}_n$-action on $\bar E$. If E/F is a field extension with maximal possible Galois group, then $\bar E$ is *the* Galois closure of E with the n embeddings $E\to\bar E$ given by the maps

$$E \mapsto E^{\otimes n}$$
$$x \mapsto 1\otimes\cdots\otimes 1\otimes x\otimes 1\otimes\cdots 1$$

5. The extensions C_iE

This section contains a first attempt to understand the algebras $C_{i_1,\dots,i_r}E$.

We study the case $r=2$. For $0\leq i\leq n$ let $C_iE = C_{i,n-i}E$.

First a general fact:

Lemma 3. *One has $S_iE\otimes S_jE = (S_iE\otimes 1)S_{i+j}E$.*

Proof. In this first draft, we leave the proof to the reader. □

Geometrically speaking, the lemma says that

$$S^iX\times S^jX \to S^iX\times S^{i+j}X$$

is a closed embedding.

Corollary 3. *The homomorphism*

$$\nu_i\colon S_iE \xrightarrow{\otimes 1} S_iE\otimes S_{n-i}E \to C_iE$$

is surjective.

Geometrically speaking, this means that

$$C^iX \to S^iX \times S^{n-i}X \xrightarrow{\text{projection}} S^iX$$

is a closed embedding.

Note that we have a canonical isomorphism $\tau\colon C_iE \to C_{n-i}E$ by switching factors of $S_iE \otimes S_{n-i}E$ (so that $\tau \circ \tau(\alpha) = \alpha$ when defined). If n is even, we get an involution $\tau\colon C_{n/2} \to C_{n/2}$.

Example. If E/F is separable, then C_iE is a separable F-algebra of rank $\binom{n}{i}$.

Example. If E/F is separable of degree 4, then $(C_2E)^\tau$ is the "cubic resolvent" of E/F.

Example. Consider the "most degenerate" extension of degree 4, $E = F[x, y, z]$, $x^2 = y^2 = z^2 = xy = xz = yz = 0$. In this case one finds that $S_2E \to C_2E$ is an isomorphism. Hence C_2E is of rank 10 in this case, in contrast to the case of separable quartic extensions where C_2E is of rank 6. See [1] for closely related examples.

Recall the homomorphisms $\rho_i\colon S_iE \to \mathrm{End}_F(\Lambda^iE)$. Composing ρ_{n-i} with duality

$$\mathrm{End}_F(\Lambda^{n-i}E) = \mathrm{End}_F\big(\Lambda^nE \otimes (\Lambda^iE)^*\big) = \mathrm{End}_F\big((\Lambda^iE)^*\big) = \mathrm{End}_F(\Lambda^iE)$$

we get a homomorphism

$$\bar\rho_{n-i}\colon S_{n-i}E \to \mathrm{End}_F(\Lambda^iE)$$

Lemma 4. *The homomorphisms ρ_i, $\bar\rho_{n-i}$ commute, i.e.,*

$$\rho_i(\alpha)\bar\rho_{n-i}(\beta) = \bar\rho_{n-i}(\beta)\rho_i(\alpha)$$

The resulting homomorphism

$$\rho_i \otimes \bar\rho_{n-i}\colon S_iE \otimes S_{n-i}E \to \mathrm{End}_F(\Lambda^iE)$$

factors through a homomorphism

$$\hat\rho_i\colon C_iE \to \mathrm{End}_F(\Lambda^iE)$$

Proof. In this first draft, we leave the proof to the reader. Note that we don't assume any separability condition here.

One may check this first in the split separable case, which gives the general separable case, and then use specialization to the degenerate case. (I don't like this way, rather I prefer a tensor game, which shouldn't be very technical.) $\square$

Let us define

$$\mathrm{trace}_i = \mathrm{trace}_{\mathrm{End}_F(\Lambda^iE)} \circ \hat\rho_i\colon C_iE \to F$$

In the separable case, $\hat\rho_i$ is the embedding of a maximal commutative subalgebra of $\mathrm{End}_F(\Lambda^iE)$. In this case trace_i is the usual trace map for C_iE/F.

Example. Consider the "most degenerate" extension of degree 4, see above. In this case one finds that the image of $\hat\rho_2$ has rank 4.

Exercise. Show that if E is a subalgebra of an Azumaya algebra A, then there is a natural homomorphism $C_iE \to \lambda^iA$ (see [3] for the definition of the exterior powers of Azumaya algebras). If E is locally a maximal commutative separable subalgebra of A, then C_iE locally a maximal commutative separable subalgebra of λ^iA. (I have not done this exercise.)

Proposition 1.

(1) $C_0E = C_nE = F$
(2) *The map* $\nu_1 \colon E \to C_1E$ *is an isomorphism.*
(3) *The map*
$$\# \colon S_{n-1}E \xrightarrow{\nu_{n-1}} C_{n-1}E \xrightarrow{\tau} C_1 \xrightarrow{\nu_1^{-1}} E$$
is the adjoint map $([x]^{\#}x = N(x))$.

Proof. In this first draft, we leave the proof to the reader. Note that we don't assume any separability condition here.

One may check this easily in the split separable case, which gives the general separable case.

As for (2) in the general case: We know from Corollary 3 that ν_1 is surjective. On the other hand, one has
$$\hat{\rho}_1 \circ \nu_1(x) = \mu_x$$
for $x \in E$. This shows the injectivity of ν_1. □

Lemma 5. *The trace maps* $T_{i,j} \colon S_iE \otimes S_jE \to S_{i+j}E$ *induce trace maps*
$$\bar{T}_{i,j} \colon C_iE \otimes C_jE \to C_{i+j}E$$
which are $C_{i+j}E$*-linear. One has*
$$\bar{T}_{i,n-i}(\alpha \otimes \beta) = \mathrm{trace}_i\big(\alpha\tau(\beta)\big)$$

Proof. No proof. □

We formulate a basic formula.

Proposition 2. *Let* x, $y \in E$. *Then*
$$\nu_k([x+y]) = \sum_{i=0}^{k} T_{i,k-i}\big(\nu_i([x]) \otimes \nu_{k-i}([y])\big)$$
In particular,
$$N_{E/F}(x+y) = \sum_{i=0}^{n} \mathrm{trace}_i\Big(\nu_i([x])\tau\big(\nu_{n-i}([y])\big)\Big)$$

Proof. In this first draft, we leave the proof to the reader. Note that we don't assume any separability condition here.

□

It follows that the $\mathrm{trace}_i\big(\nu_i([x])\big)$ are the coefficients of the characteristic polynomial of x.

6. The tensorial norm

Let F be a commutative ring and let E be a locally free ring extension of F of rank n. We assume that E/F is separable (=etale).

Let $M_1, \dots, M_r$ be finitely generated locally free E-modules and let $(i_1, \dots, i_r)$ be a partition of n.

We put
$$S_{i_1,\dots,i_r}(M_1, \dots, M_r) = S_{i_1}M_1 \otimes \cdots \otimes S_{i_r}M_r$$

and

$$\begin{aligned} C_{i_1,\dots,i_r}(M_1,\dots,M_r) &= (\nu_{E/F})_*\big(S_{i_1,\dots,i_r}(M_1,\dots M_r)\big) \\ &= S_{i_1,\dots,i_r}(M_1,\dots,M_r)\otimes_{S_{i_1,\dots,i_r}E} C_{i_1,\dots,i_r}E \end{aligned}$$

Lemma 6. *Let $f_r\colon C_{i_1,\dots,i_r}E \to C_{r,i_1-r,\dots,i_r}E$ be the natural homomorphisms, induced from $S_{i_1}E \to S_rE\otimes S_{i_1-r}E$. Then*

$$C_{i_1,\dots,i_r}(M\oplus N, M_2,\dots M_r) = \bigoplus_{r=0}^{i_1} f_r^*\big(C_{r,i_1-r,\dots,i_r}(M,N,M_2,\dots M_r)\big)$$

Proof. Easy. □

Corollary 4. *There exist a unique multiplicative polynomial map*

$$N_{E/F}\colon K_0^{\oplus}E \to K_0^{\oplus}F$$

of degree n such that

$$N_{E/F}([M]) = [\nu_{E/F}(M)]$$

Proof. Use Lemma 6 and Corollary 2... □

7. The norm for the Grothendieck-Witt ring

We denote by $GW(F)$ the Grothendieck-Witt ring of F of symmetric bilinear forms (let us say in this draft that F is a field).

Let E/F be a separable field extension of degree n. Using an obvious extension of section 6, one gets

Corollary 5. *There exist a unique multiplicative polynomial map*

$$N_{E/F}\colon GW(E)\to GW(F)$$

of degree n such that

$$N_{E/F}([b]) = [\nu_{E/F}(b)]$$

I guess one should also define polynomial maps

$$\nu_i\colon GW(E)\to GW(C_iE)$$

of degree i. For instance, for $n=4$ this would give a quadratic map

$$GW(E)\xrightarrow{\nu_2} GW(C_2E)\xrightarrow{\text{trace}} GW\big((C_2E)^\tau\big)$$

to the Grothendieck-Witt ring of the cubic resolvent. . .

8. Computations

I haven't done much besides for the case of a quadratic field extensions in characteristic not 2. See the text "A Pfister form invariant for etale algebras".

References

[1] D. Ferrand, *Un foncteur norme*, Bull. Soc. Math. France **126** (1998), no. 1, 1–49.
[2] S. Joukhovitski, *K-theory of the Weil transfer functor*, K-Theory **20** (2000), no. 1, 1–21, Special issues dedicated to Daniel Quillen on the occasion of his sixtieth birthday, Part I.
[3] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, American Mathematical Society, Providence, RI, 1998, With a preface in French by J. Tits.

Department of Mathematics, The Ohio State University, 231 W 18th Avenue, Columbus, OH 43210, USA
E-mail address: rost@math.ohio-state.edu
URL: http://www.math.ohio-state.edu/~rost