ON THE GALOIS COHOMOLOGY OF SPIN(14)

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`Preliminary Notes`

Note from May/June 2006

I am very grateful to Skip Garibaldi for comments. They led to several corrections and additions.

In the version from 1999 I had claimed without proof $\mathrm{ed}(\mathrm{Spin}_{13}) = 6$. I have now added a new section (Section 10) containing a proof.

Abstract

Let k be a field with $\operatorname{char} k \neq 2$. For $i = 6, 7$ we define invariants

$$h_i \colon H^1\big(k, \mathrm{Spin}(14)\big) \to H^i(k, \mathbf{Z}/2)/(-1)H^{i-1}(k, \mathbf{Z}/2).$$

Further we show that the natural map

$$H^1\big(k, (G_2 \times G_2) \rtimes \mu_8\big) \to H^1\big(k, \mathrm{Spin}(14)\big)$$

is surjective.

One concludes that the essential dimension of Spin(14) is equal to 7.

Similar considerations are done for Spin(12). We also present the list of essential dimensions of the split groups $\mathrm{Spin}(n)$ for $n \leq 14$.

Contents

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1. The Arason invariant

1.1. The invariants e_i, $i \leq 3$. Let

$$e_i\colon I^i(k)/I^{i+1}(k) \to H^i(k, \mathbf{Z}/2), \quad i = 0, \ldots, 3$$

be the first invariants on the graded Witt ring given by dimension, discriminant, the Hasse-Witt invariant, and Arason's invariant, cf. [1, 26].

1.2. The split groups of type D_n. We denote by $\mathrm{SO}(n,n)$ the automorphism group of the quadratic form

$$\sum_1^n (x_i^2 - y_i^2).$$

Furthermore, $\mathrm{Spin}(n,n)$ denotes the universal cover of $\mathrm{SO}(n,n)$ and

$$\mathrm{PSO}(n,n) = \mathrm{SO}(n,n)/\{\pm 1\} = \mathrm{Spin}(n,n)/\mu$$

denotes the corresponding adjoint group. Here μ is the center of $\mathrm{Spin}(n,n)$. One has $\mu = \mu_2 \times \mu_2$ if n is even and $\mu = \mu_4$ if n is odd.

If n is odd, every split group of type D_n is isomorphic to one of $\mathrm{Spin}(n,n)$, $\mathrm{SO}(n,n)$, $\mathrm{PSO}(n,n)$.

1.3. Galois cohomology of $\mathrm{SO}(n,n)$. The set $H^1\big(k, \mathrm{SO}(n,n)\big)$ consists of the isomorphism classes of $2n$-dimensional quadratic forms with trivial discriminant. We consider $H^1\big(k, \mathrm{SO}(n,n)\big)$ as a subset of $I^2(k) \subset W(k)$.

The image of

$$H^1\big(k, \mathrm{SO}(n,n)\big) \to H^1\big(k, \mathrm{PSO}(n,n)\big)$$

consists of the similarity classes of the quadratic forms in $H^1\big(k, \mathrm{SO}(n,n)\big)$. For $u \in H^1\big(k, \mathrm{Spin}(n,n)\big)$ let q_u be the corresponding quadratic form.

The image of

$$H^1\big(k, \mathrm{Spin}(n,n)\big) \to H^1\big(k, \mathrm{SO}(n,n)\big)$$

consists of those classes in $H^1\big(k, \mathrm{SO}(n,n)\big)$ with trivial Hasse-Witt invariant.

1.4. The invariant $\tilde{e}_3$ in $K_3^M/2$. Let $K_n^M k$ be Milnor's K-group [18].

By Merkurjev's theorem [2, 16, 31] the invariant e_2 is bijective. Furthermore, Milnor's homomorphism

$$s_3\colon K_3^M k/2 \to I^3(k)/I^4(k)$$

is bijective (cf. [11, 17, 18, 25]).

Putting things together yields natural maps

$$\tilde{e}_3\colon H^1\big(k, \mathrm{Spin}(n,n)\big) \to K_3^M k/2.$$

For $u \in H^1\big(\mathrm{Spin}(n,n)\big)$ the class $\tilde{e}_3(u)$ depends alone on q_u. For $u \in H^1\big(\mathrm{Spin}(8,8)\big)$ the corresponding quadratic form q_u is a 3-fold Pfister form (cf. [5, 15, 20, 26]); if $q_u = \langle\langle a,b,c \rangle\rangle$, then $\tilde{e}_3(u) = \{a,b,c\}$. Furthermore, the maps $\tilde{e}_3$ behave additively with respect to the natural inclusions

$$\mathrm{Spin}(n,n) \times \mathrm{Spin}(m,m) \to \mathrm{Spin}(n+m, n+m).$$

These properties determine the family of maps $\tilde{e}_3$ uniquely.

2. Reduced squares

It has been observed by Serre that for any $n \geq 2$ there is a natural map

$$P\colon K_n^M k/2 \to K_{2n}^M k/(2K_{2n}^M k + \{-1\}^{n-1} K_{n+1}^M k)$$

characterized by

$$P\Big(\sum_i x_i\Big) = \sum_{i<j} x_i x_j \bmod (2K_{2n}^M k + \{-1\}^{n-1} K_{n+1}^M k)$$

where x_i are symbols. (An element $x \in K_n^M k/2$ is called a symbol if it is of the form $x = \{a_1, \dots, a_n\}$ for some $a_i \in k^*$.)

To define the operation P one checks that the right hand side of this formula does not depend on the presentation of an element as a sum of symbols. This follows easily from the definition of Milnor's K-theory and the identity $\{a, a\} = \{a, -1\}$, cf. [18].

Let

$$\alpha_n\colon K_n^M k/2 \to H^n(F, \mathbf{Z}/2)$$

be the norm residue homomorphism [18]. Milnor's conjecture (cf. [30]) asserts that α_n is bijective. With Milnor's conjecture, the operations P give rise to corresponding maps

$$H^n(k, \mathbf{Z}/2) \to H^{2n}(k, \mathbf{Z}/2)/(-1)^{n-1} H^{n+1}(k, \mathbf{Z}/2).$$

Combining this with the fact that $(-1)H^{2n-1}(k, \mathbf{Z}/2)$ is in the kernel of the natural maps $H^{2n}(k, \mathbf{Z}/2) \to H^{2n}(k, \mathbf{Z}/4)$, one obtains operations

$$H^n(k, \mathbf{Z}/2) \to H^{2n}(k, \mathbf{Z}/4).$$

In the case $n = 2$ this operation is nothing else than the Pontryagin square, cf. [3, 4, 32, 33]. For $n > 2$ I don't know any explanation of the operations P by an operation defined on the cohomology of topological spaces.

3. Lambda operations

Let $\widehat{W}(k)$ be the Grothendieck (-Witt) ring of quadratic forms over k. One defines λ-operations

$$\lambda^i\colon \widehat{W}(k) \to \widehat{W}(k)$$

in the usual fashion (see for instance [13]):

For a quadratic form $\varphi\colon V \to k$ let $\lambda^i\varphi\colon \bigwedge^i V \to k$ be its i-th exterior power. One has $\lambda^0\varphi = \langle 1\rangle$ and $\lambda^1\varphi = \varphi$. The form λ^2 is also given by the Killing form on the Lie algebra $\mathrm{so}(\varphi)$ (at least if $\bar{\mathbf{Q}} \subset k$).

One forms the formal power series

$$\lambda_t\varphi = \sum_{i\geq 0} t^i \lambda^i \varphi.$$

Then

$$\lambda_t(\varphi \perp \psi) = \lambda_t\varphi \otimes \lambda_t\psi.$$

The series λ_t extends to $\widehat{W}(k)$ by

$$\lambda_t(\varphi - \psi) = \lambda_t\varphi \otimes (\lambda_t\psi)^{-1}$$

and the operations λ^i on $\widehat{W}(k)$ are defined by

$$\lambda_t(x) = \sum_{i\geq 0} t^i \lambda^i(x)$$

for $x \in \widehat{W}(k)$.

We are mainly interested in λ^2. Note that

$$\begin{aligned}
\lambda^0(x) &= 1,\\
\lambda^1(x) &= x,\\
y^2 &= \dim y + 2\lambda^2(y),\\
\lambda^2(x+y) &= \lambda^2(x) + xy + \lambda^2(y),\\
\lambda^2(x-y) &= \lambda^2(x) - y(x-y) - \lambda^2(y),\\
\lambda^2(x-y) &= \lambda^2(x) - xy + \dim y + \lambda^2(y),\\
\lambda^2(\langle a\rangle x) &= \lambda^2(x)
\end{aligned}$$

for x, $y \in \widehat{W}(k)$ and $a \in k^*$.

Let $\widehat{I}(k) \subset \widehat{W}(k)$ be the fundamental ideal of zero dimensional virtual quadratic forms. The projection $\widehat{W}(k) \to W(k)$ induces identifications $\widehat{I}^n(k) = I^n(k)$ for $n > 0$. $\widehat{I}^n(k)$ is additively generated by elements of the form

$$\langle\langle a_1,\dots,a_n\rangle\rangle - \langle\langle 1\rangle\rangle^n = \langle\langle a_1,\dots,a_{n-1}\rangle\rangle - \langle a_n\rangle\langle\langle a_1,\dots,a_{n-1}\rangle\rangle.$$

Lemma 3.1. *Let φ be an n-fold Pfister form and $x = \varphi - \langle\langle 1\rangle\rangle^n$. Then*

$$\lambda^2(x) = \langle\langle -1\rangle\rangle^{n-1}x.$$

Proof. Write $\varphi = \psi\langle\langle a\rangle\rangle$ where ψ is an $(n-1)$-fold Pfister form and where $a \in k^*$. Then $x = \psi - \langle a\rangle\psi$ and one finds

$$\begin{aligned}
\lambda^2(x) &= \lambda^2(\psi - \langle a\rangle\psi)\\
&= \lambda^2(\psi) - \langle a\rangle\psi x - \lambda^2(\psi)\\
&= -\langle a\rangle\langle\langle -1\rangle\rangle^{n-1}x\\
&= \langle\langle -1\rangle\rangle^{n-1}\langle -a\rangle x = \langle\langle -1\rangle\rangle^{n-1}x
\end{aligned}$$

Here one uses $\psi^2 = \langle\langle -1\rangle\rangle^{n-1}\psi$, $\langle -a\rangle x = -\langle a\rangle x$ if $\dim x = 0$, and $\langle -a\rangle\langle\langle a\rangle\rangle = \langle\langle a\rangle\rangle$. □

Corollary 3.2. *Let φ be an n-fold Pfister form. Then*

$$\begin{aligned}
\lambda^2(\varphi) &\simeq \varphi'\langle\langle -1\rangle\rangle^{n-1},\\
\lambda^2(\varphi') &\simeq \varphi'(\langle\langle -1\rangle\rangle^{n-1})'. \qquad \square
\end{aligned}$$

We define operations

$$\begin{aligned}
P' \colon I^n(k) &\to I^{2n}(k),\\
P'(x) &= \lambda^2(x) - \langle\langle -1\rangle\rangle^{n-1}x.
\end{aligned}$$

It follows from Lemma 3.1 and $\lambda^2(x+y) = \lambda^2(x) + xy + \lambda^2(y)$ that indeed $P'(x) \in I^{2n}(k)$.

These operations lift the operations P to the Witt ring.

4. Multiplicative transfer

Let L/F be separable field extension. In addition to the restriction map

$$r_{L/F}\colon W(F)\to W(L),\quad [\varphi]\mapsto[\varphi_L]$$

and the corestriction map

$$c_{L/F}\colon W(L)\to W(F),\quad [\psi]\mapsto[\mathrm{trace}_{L/F}\,\varphi]$$

one may define a multiplicative transfer map

$$N_{L/F}\colon W(L)\to W(F).$$

This map is analogous to the multiplicative transfer in cohomology, cf. [6, 12, 29].

We are interested in the case $[L:F]=2$. Let σ denote the generator of the Galois group. Then for a quadratic form $\psi\colon W\to L$ the form $N_{L/F}(\psi)$ is given by the restriction of $\psi\otimes{}^{\sigma}\psi\colon W\otimes{}^{\sigma}W\to L$ to the subspace of invariants $(W\otimes{}^{\sigma}W)^{\sigma}$.

Suppose $L=F(\sqrt{a})$. One has the following rules

$$\begin{aligned}
\dim_F\bigl(N_{L/F}(\psi)\bigr)&=(\dim_L\psi)^2,\\
N_{L/F}(\langle\alpha\rangle)&=\langle N_{L/F}(\alpha)\rangle,\\
N_{L/F}(x+y)&=N_{L/F}(x)+c_{L/F}(x\sigma(y))+N_{L/F}(y),\\
N_{L/F}(x-y)&=N_{L/F}(x)-c_{L/F}(x\sigma(y))+N_{L/F}(y),\\
\lambda^2\bigl(c_{L/F}(x)\bigr)&=c_{L/F}\bigl(\lambda^2(x)\bigr)+aN_{L/F}(x),\\
N_{L/F}(\langle\langle\alpha\rangle\rangle)&=\langle\langle a\rangle\rangle+\begin{cases}\langle\langle\mathrm{trace}\,\alpha,-aN_{L/F}(\alpha)\rangle\rangle & \text{if trace}\,\alpha\neq0,\\ 0 & \text{if trace}\,\alpha=0,\end{cases}\\
N_{L/F}(\langle\langle\alpha_1,\dots,\alpha_n\rangle\rangle)&=\langle\langle a\rangle\rangle^n+\begin{cases}\prod_i\langle\langle\mathrm{trace}\,\alpha_i,-aN_{L/F}(\alpha_i)\rangle\rangle & \text{if trace}\,\alpha_i\neq0,\\ 0 & \text{else.}\end{cases}
\end{aligned}$$

In particular, if -1 is a square in F, then

$$N_{L/F}\bigl(I^n(L)\bigr)\subset I^{2n}(F)$$

for $n\geq2$.

5. The invariants h_6 and h_7

For this section it is assumed for simplicity that $\sqrt{-1}\in k$.

We define

$$\begin{gathered}h_6\colon H^1\bigl(k,\mathrm{Spin}(7,7)\bigr)\to H^6(k,\mathbf{Z}/2),\\ h_6(u)=\alpha_6\circ P\circ\tilde{e}_3(u).\end{gathered}$$

The invariant $h_6(u)$ depends only on q_u.

By the remarks of Section 3 one can lift this invariant to $I^6(k)$.

In some cases the invariant h_6 can be described explicitly. For a Pfister form φ one denotes by φ' its pure subform (one has $\varphi=\langle1\rangle\perp\varphi'$). Let a_i, b_i, $c\in k^*$, $i=1$, 2, 3, and put

$$q=c(\langle\langle a_1,a_2,a_3\rangle\rangle'\perp-\langle\langle b_1,b_2,b_3\rangle\rangle')\tag{1}$$

Then $q=q_u$ for some $u\in H^1\bigl(k,\mathrm{Spin}(7,7)\bigr)$ and for any such u one finds

$$h_6(u)=(a_1,a_2,a_3,b_1,b_2,b_3).\tag{2}$$

Lemma 5.1. *Let $u\in H^1\bigl(k,\mathrm{Spin}(7,7)\bigr)$. If q_u is isotropic, then $h_6(u)=0$.*

Proof. If q_u is isotropic, the q_u has a representation (1) with $a_1 = b_1$, see [19, Satz 14, Zusatz] or [24]. The claim follows from (2). $\square$

Proposition 5.2. *Let $u \in H^1\big(k, \mathrm{Spin}(7,7)\big)$ and let c be a nonzero value of q_u. The element*
$$h_6(u) \cup (c) \in H^7(k, \mathbf{Z}/2)$$
does not depend on the choice of c.

Proof (Variant 1). Write $q = q_u$. If q is isotropic, then $h_6(u) = 0$ by Lemma 5.1. We may therefore assume that q is anisotropic. Let $c = q(v)$ and $c' = q(v')$ be two values of q with v, v' linearly independent. Then c/c' is a norm from the quadratic extension L splitting the 2-dimensional subform $q|(vk + v'k)$. Say $c/c' = N_{L/k}(\lambda)$. Then
$$\begin{aligned} h_6(u) \cup (c) - h_6(u) \cup (c') &= h_6(u) \cup (c/c') \\ &= h_6(u) \cup N_{L/k}\big((\lambda)\big) \\ &= N_{L/k}\big(h_6(u_L) \cup (\lambda)\big) \\ &= N_{L/k}\big(0 \cup (\lambda)\big) = 0 \end{aligned}$$
since q_L is isotropic and by Lemma 5.1. $\square$

Proof (Variant 2). Write $q = q_u$ as $q\colon V \to k$. Then any $x = [v] \in \mathbf{P}V$ determines an element
$$q(x) \in \kappa(x)/\big(\kappa(x)^*\big)^2.$$
Let $\xi \in \mathbf{P}V$ be the generic point and consider
$$\omega = h_6(u) \cup \big(q(\xi)\big) \in H^7\big(k(\mathbf{P}V), \mathbf{Z}/2\big).$$

The element ω is unramified on $\mathbf{P}V$, except possibly at the divisor
$$Z = \{q = 0\} \subset \mathbf{P}V$$
Here the residue is a multiple of (in fact, equal to)
$$h_6(u)_{k(Z)} \in H^6\big(k(Z), \mathbf{Z}/2\big)$$
But the quadratic form $q_{k(Z)}$ is isotropic, whence $h_6(u)_{k(Z)} = 0$ by Lemma 5.1. Hence ω is unramified everywhere on $\mathbf{P}V$ and therefore $\omega = (\omega_0)_{k(\mathbf{P}V)}$ for some $\omega_0 \in H^7(k, \mathbf{Z}/2)$. The claim follows by specialization. $\square$

Proposition 5.2 gives rise to an invariant
$$h_7\colon H^1\big(k, \mathrm{Spin}(7,7)\big) \to H^7(k, \mathbf{Z}/2),$$
$$h_7(u) = h_6(u) \cup \big(q_u(v)\big)$$
where $q_u(v)$ is any nonzero value of q_u.

As for h_6, the invariant $h_7(u)$ depends only on q_u. If $q_u = q$ with q as in (1), then
$$h_7(u) = (a_1, a_2, a_3, b_1, b_2, b_3, c).$$
This computation shows that the invariant h_7 is non-trivial.

In the next two statements (Proposition 5.3, Lemma 5.4) we assume that k contains the algebraic closure of $\mathbf{Q}$. This assumption is made to be sure that we can neglect some universal constants arising in decompositions of Killing forms and of $\lambda^2(q)$. I have not tried to figure out the best possible conditions.

Proposition 5.3. *Assume $\bar{\mathbf{Q}} \subset k$. Any value of h_6 and of h_7 is a symbol.*

Proof. It suffices to consider h_6. Let $u \in H^1\big(k, \mathrm{Spin}(7,7)\big)$ and write $q = q_u$. Then $h_6(u)$ is represented by 92-dimensional form

$$\lambda^2 q \perp \langle 1 \rangle.$$

The form $\lambda^2 q$ is also given by the Killing form on $\mathrm{so}(q)$.

We may assume that u is induced from an element $x \in H^1\big(k, (G_2 \times G_2) \rtimes \mu_8\big)$, see Corollary 7.3. Let $\mathfrak{g} \subset \mathrm{so}(q)$ be the Lie algebra of type $G_2 + G_2$ corresponding to x. Its Killing form is the trace of the Killing form of a Lie algebra of type G_2 over some quadratic extension. In view of the next Lemma, this form is hyperbolic.

Therefore the 92-dimensional form $\lambda^2 q \perp \langle 1 \rangle$ contains a 28-dimensional hyperbolic subform. Thus $h_6(u)$ is represented by a $92 - 28 = 64$-dimensional quadratic form, which therefore must be a multiple of a 6-fold Pfister form. □

This proof indicates that one may represent $h_6(u)$ by a form on the spinor representation S, cf. below. In fact there is a natural way to represent $h_6(u)$ as $N_{L/k}(\psi)$ on S, where ψ/L is the 3-fold Pfister form corresponding to a reduction $x \in H^1\big(k, (G_2 \times G_2) \rtimes \mu_8\big)$ of u, cf. [24].

Lemma 5.4. *Assume $\bar{\mathbf{Q}} \subset k$. Let $\mathfrak{g}$ be a Lie algebra of type G_2 and let φ be the associated 3-fold Pfister form. Then the Killing form on $\mathfrak{g}$ is hyperbolic.*

Proof. Let V be the 7-dimensional representation of $\mathfrak{g}$. Then

$$\mathfrak{g} \perp V = \bigwedge^2 V.$$

Let further ψ denote the Killing form on $\mathfrak{g}$ and let φ be the associated 3-fold Pfister form. Then

$$\psi \perp \varphi' = \lambda^2(\varphi') = \varphi' \langle\!\langle -1, -1 \rangle\!\rangle'$$

by Corollary 3.2. The claim follows. □

Our considerations in the construction of the invariants h_6, h_7 may be also applied to the group $\mathrm{SO}(6)$. This leads to invariants

$$H^1\big(k, \mathrm{SO}(6)\big) \to H^i(k, \mathbf{Z}/2)$$

for $i = 4$, 5, given by

$$c(\langle\!\langle a_1, a_2 \rangle\!\rangle' \perp \langle\!\langle b_1, b_2 \rangle\!\rangle') \mapsto (a_1, a_2, b_1, b_2),$$
$$c(\langle\!\langle a_1, a_2 \rangle\!\rangle' \perp \langle\!\langle b_1, b_2 \rangle\!\rangle') \mapsto (a_1, a_2, b_1, b_2, c).$$

The latter coincides with the invariant

$$\langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle \mapsto (a_1, a_2, a_3, a_4, a_5)$$

defined by Serre.

6. A reduction lemma

Let G be an algebraic group over k and let $i \colon H \subset G$ be a subgroup. For $x \in H^1(k, G)$ we denote by P_x a corresponding G-torsor.

Lemma 6.1. *Let $x \in H^1(k, G)$. Then x is in the image of*

$$i_* \colon H^1(k, H) \to H^1(k, G)$$

if and only if the variety P_x/H has a k-rational point.

Proof. Indeed, if $x = i_*(y)$, then $P_x \simeq P_y \times_H G$ and P_x/H has the k-rational point given by $[P_y, 1] \bmod H$.

Conversely, if $z \in P_x/H$ is k-rational, then the fiber of z under $P \to P_x/H$ is an H-torsor Q with $Q \times_H G \simeq P_x$. $\square$

This simple lemma is the basis of many structure theorems on quadratic forms and algebras. It applies usually when there is a "small" representation of G, i.e., a representation $G \to \mathrm{GL}(V)$ with $\dim V < \dim G$.

A fairly simple example is given by $G = \mathrm{O}(n)$ and $H = \mathrm{O}(n-1) \times \mu_2$: Let $x \in H^1\big(k, \mathrm{O}(n)\big)$; if $q_x \colon V \to k$ is the corresponding quadratic form, then P_x/H is naturally isomorphic to $U = \mathbf{P}V \setminus \{q_x = 0\}$. Since U has a rational point, it follows that x has a reduction to H.

Her majesty E_8 does not have a small representation.

7. 14-dimensional spinors

Let $\mathrm{Spin}(7,7) \to \mathrm{GL}(S)$ be one of the spinor representations ($\dim S = 64$) and let $\mathrm{PSO}(7,7) \to \mathrm{PGL}(S)$ be the induced homomorphism. We denote $G = \mathrm{PSO}(7,7)$ and define $H \subset G$ as the image of

$$(G_2 \times G_2) \rtimes \mathbf{Z}/2 \to \mathrm{PSO}(7,7)$$

given by

$$(g,h)\epsilon^n \mapsto \begin{pmatrix} \rho(g) & 0 \\ 0 & \rho(h) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n$$

where $\rho \colon G_2 \to \mathrm{Spin}(7)$ is the standard representation.

We need the following fact, see [7, 9, 21, 24].

Proposition 7.1. *The action of G on $\mathbf{P}S$ has an open and dense orbit U. If k is algebraically closed, then the isotropy group H_u of $u \in U$ is conjugate to H. In particular, $U = G/H$.*

Now let $x \in H^1(k, G)$. Then $X_x = P_x \times_G \mathbf{P}S$ is a Brauer-Severi variety whose Brauer class coincides with the Tits class $t(x) \in H^2(k, \mu_4)$ of x. Further, the variety $U_x = P_x \times_G U = P_x/H$ is a dense open subscheme of X_x. It follows that P_x/H has k-rational points if and only if $t(x) = 0$ (to be sure, let us assume that k is infinite). Lemma 6.1 shows

Corollary 7.2. *An element $x \in H^1(k, G)$ has an H-reduction if and only if $t(x) = 0$.* $\square$

Let $\widetilde{H}$ be the preimage of H under $\mathrm{Spin}(7,7) \to \mathrm{PSO}(7,7)$. One finds (see [24])

$$\widetilde{H} = (G_2 \times G_2) \rtimes \mu_8$$

where $\mu_8 \subset \mathrm{Spin}(7,7)$ is the normalizer of $G_2 \times G_2$.

Corollary 7.3. *The homomorphism*

$$H^1\big(k, (G_2 \times G_2) \rtimes \mu_8\big) \to H^1\big(k, \mathrm{Spin}(7,7)\big)$$

is surjective.

Proof. This follows from a diagram chase in

$$\begin{array}{ccccccc}
H^1(k,\mu_4) & \longrightarrow & H^1\big(k,(G_2\times G_2)\rtimes\mu_8\big) & \longrightarrow & H^1\big(k,(G_2\times G_2)\rtimes \mathbf{Z}/2\big) & \longrightarrow & H^2(k,\mu_4) \\
\| & & \downarrow & & \downarrow & & \| \\
H^1(k,\mu_4) & \longrightarrow & H^1\big(k,\mathrm{Spin}(7,7)\big) & \longrightarrow & H^1\big(k,\mathrm{PSO}(7,7)\big) & \xrightarrow{t} & H^2(k,\mu_4)
\end{array}$$

□

It can be shown that there exist a field k and $x \in H^1\big(k,\mathrm{Spin}(7,7)\big)$ such that x has no reduction to the subgroup $(G_2\times G_2)\times\mu_4$. This means that the appearing forms of $G_2\times G_2$ are necessarily of type $R_{\ell/k}(G_2)$ with ℓ/k a quadratic *field* extension. Examples have been provided in [8] using residue arguments and in [10] using computations of the K-theory of certain homogeneous varieties.

8. The essential dimension of Spin(14)

We denote by $\mathrm{ed}(G)$ the essential dimension of G, see [22].

Proposition 8.1. *$\mathrm{ed}\big(\mathrm{Spin}(14)\big) = 7$.*

Proof. $\mathrm{ed}\big(\mathrm{Spin}(14)\big) \geq 7$ follows from the non-triviality of the invariant h_7.

It remains to show $\mathrm{ed}\big(\mathrm{Spin}(14)\big) \leq 7$. By Corollary 7.3 it suffices show $\mathrm{ed}(\widetilde{H}) \leq 7$. To describe any $\widetilde{H}$-torsor one needs one parameter to describe a class $(a) \in H^1(k,\mu_8) = k^*/(k^*)^8$ and $3\cdot 2$ parameters to describe an octonion algebra

$$O(a_1+\sqrt{a}b_1, a_2+\sqrt{a}b_2, a_3+\sqrt{a}b_3)$$

over $k(\sqrt{a})$. □

9. On the cohomology of Spin(12)

We briefly sketch a proof of $\mathrm{ed}\big(\mathrm{Spin}(6,6)\big) = 6$.

We define $H \subset \mathrm{SO}(6,6)$ as the image of

$$\mathrm{SL}(6)\rtimes\mathbf{Z}/2 \to \mathrm{SO}(6,6)$$

given by

$$g\epsilon^n \mapsto \begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}^n.$$

Here we understand coordinates (x,y) with respect to the quadratic form $\sum_i x_i y_i$. The preimage $\widetilde{H}$ of H in $\mathrm{Spin}(6,6)$ is

$$\widetilde{H} = \mathrm{SL}(6)\rtimes\mu_4.$$

By the mentioned theorem of Pfister ([19, Satz 14, Zusatz] or [24]), any $\mathrm{Spin}(6,6)$-torsor admits an $\widetilde{H}$-reduction. Since any hermitian form can be diagonalized, the map

$$H^1\big(k,\mathrm{SO}(6)\times\mu_4\big) \to H^1(k,\widetilde{H})$$

is surjective. Hence

Corollary 9.1. *$\mathrm{ed}\big(\mathrm{Spin}(6,6)\big) \leq 6$.*

We define invariants in $H^5(\mathbf{Z}/2)$, $H^6(\mathbf{Z}/2)$ by a variant of the previous method. It is based on the following facts:

Lemma 9.2. *Let $a \in k^*$. Then the kernel of*

$$W(k) \to W(k), \quad x \mapsto \langle\langle a\rangle\rangle x$$

is generated by 2-dimensional forms of the form $\langle\langle N_{\ell/k}(\alpha)\rangle\rangle$ with $\alpha \in \ell^*$, $\ell = k(\sqrt{a})$.

Proof. Well known... □

Lemma 9.3. *Let a, $b \in k^*$ and let x, $y \in W(k)$. If*

$$\langle\langle a\rangle\rangle x = \langle\langle b\rangle\rangle y,$$

then there exist $z \in W(k)$ with

$$\langle\langle a\rangle\rangle x = \langle\langle a\rangle\rangle z = \langle\langle b\rangle\rangle z = \langle\langle b\rangle\rangle y.$$

Moreover, any such z may be written as a sum of 2-dimensional forms of the form $\langle\langle N_{\ell/k}(\alpha)\rangle\rangle$ with $\alpha \in \ell^*$, $\ell = k(\sqrt{ab})$.

Proof. Let φ be a quadratic form representing x, let $K = k(\sqrt{b})$, and suppose that $\langle\langle a\rangle\rangle\varphi_K$ is split.

Since $\langle\langle a\rangle\rangle\varphi_K$ is isotropic, one has $\langle\langle a\rangle\rangle\varphi = \langle\langle a\rangle\rangle(c\langle\langle d\rangle\rangle + \varphi')$ such that $\langle\langle a, d\rangle\rangle_K$ is isotropic. To see this, let $\varphi = \langle a_1, \dots, a_n\rangle$ and let

$$q\colon V = L^n \to k$$

$$q(\lambda_1, \dots, \lambda_n) = \sum_i a_i N_{L/k}(\lambda_i)$$

with $L = k(\sqrt{a})$. Note that $q = \langle\langle a\rangle\rangle\varphi$ and that $q(\lambda v) = N_{L/k}(\lambda)q(v)$ for $\lambda \in L$. If q_K is isotropic, there exists a 2-dimensional L-submodule W of V such that $q|W$ is isotropic over K. Next note that $q|W = c\langle\langle a, d\rangle\rangle$ for some c, d.

We may assume $\varphi = c\langle\langle d\rangle\rangle \perp \varphi'$. There exists e such that

$$\langle\langle a, d\rangle\rangle = \langle\langle a, e\rangle\rangle = \langle\langle b, e\rangle\rangle$$

Then $\langle\langle a\rangle\rangle\varphi = c\langle\langle a, e\rangle\rangle + \langle\langle a\rangle\rangle\varphi'$. The claim follows by induction on $\dim\varphi$ and Lemma 9.2. □

Let $I_2(k) \subset I(k)$ be the subset of elements which are split over *some* quadratic extension. One defines an operation

$$Q\colon I_2(k) \to I_2(k),$$

$$Q(\langle\langle a\rangle\rangle x) = \langle\langle a\rangle\rangle\lambda^2(x).$$

This map is well defined by Lemma 9.2 and Lemma 9.3.

We assume that -1 is a square. Let $u \in H^1\big(k, \mathrm{Spin}(6,6)\big)$. Then

$$q_u = a\langle\langle b\rangle\rangle(\langle\langle c, d\rangle\rangle' - \langle\langle e, f\rangle\rangle')$$

and

$$Q(q_u) = \langle\langle b, c, d, e, f\rangle\rangle$$

Hence we an invariant

$$k_5\colon H^1\big(k, \mathrm{Spin}(6,6)\big) \to H^5(k, \mathbf{Z}/2).$$

If q_u is isotropic, then $q_u = a\langle\langle b, c', d'\rangle\rangle \perp \langle 1, -1\rangle$. This shows $Q(q_u) = 0$. By the same argument as in the proof of Proposition 5.2 we get an invariant

$$k_6\colon H^1\big(k, \mathrm{Spin}(6,6)\big) \to H^6(k, \mathbf{Z}/2),$$

$$k_6(u) = k_5(u) \cup \big(q_u(v)\big)$$

where $q_u(v)$ is any nonzero value of q_u.

If $q_u = a\langle\langle b\rangle\rangle(\langle\langle c,d\rangle\rangle' - \langle\langle e,f\rangle\rangle')$, then

$$k_6(u) = (a,b,c,d,e,f).$$

This shows that k_6 is nontrivial.

Corollary 9.4. $\mathrm{ed}\big(\mathrm{Spin}(6,6)\big) \geq 6$.

10. On the cohomology of Spin(13)

Let

$$\begin{gathered} q\colon V = k^{13} \to k \\ q(x_1,\dots,x_{13}) = (x_1^2+x_2^2+x_3^2) - (x_4^2+x_5^2+x_6^2) \\ - (x_7^2+x_8^2+x_9^2) + (x_{10}^2+x_{11}^2+x_{12}^2) - x_{13}^2 \end{gathered}$$

An element of $H^1(k,\mathrm{SO}(q))$ is given by a 13-dimensional quadratic form q' with

$$q' \perp \langle 1\rangle \in H^1(k,\mathrm{SO}(7,7)) \subset I^2 \subset W(k)$$

Let G be the subgroup of $\mathrm{SO}(q)$ generated by (matrix notation with respect to $k^{13} = k^3 \times k^3 \times k^3 \times k^3 \times k$)

$$U(g,h) = \begin{pmatrix} g & 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 & 0 \\ 0 & 0 & h & 0 & 0 \\ 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$V(\alpha,\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & \alpha\beta \end{pmatrix}, \quad W(\eta) = \begin{pmatrix} 0 & 0 & \eta & 0 & 0 \\ 0 & 0 & 0 & \eta & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with g, $h \in \mathrm{SO}(3)$, α, $\beta \in \mu_2$ and $\eta \in \mu_4$. One has

$$G = \big(\mathrm{SO}(3)\times\mu_2\big)^2 \rtimes \mu_4 \subset \mathrm{SO}(q)$$

with μ_4 acting via the projection $\mu_4 \to \mu_2 = \mathbf{Z}/2$ by permutation of the factors.

We consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \widetilde{G} & \xrightarrow{\pi_G} & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \tilde{j}} & & \downarrow{\scriptstyle j} & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{Spin}(q) & \xrightarrow{\pi} & \mathrm{SO}(q) & \longrightarrow & 1 \end{array} \tag{3}$$

where $\widetilde{G} \subset \mathrm{Spin}(q)$ is the preimage of G under the projection $\pi\colon \mathrm{Spin}(q) \to \mathrm{SO}(q)$ and $\tilde{j}$, j are the inclusions.

We describe the image

$$J = j_*\big(H^1(k,G)\big) \subset H^1(k,\mathrm{SO}(q))$$

Lemma 10.1. *The set J consists exactly of the (isomorphism classes of) quadratic forms q' of the type*

$$q' = \tilde{q} \perp \langle -\det(\tilde{q})\rangle \tag{4}$$

with

$$\tilde{q} = (T_{K/k})_*\big(\langle s\rangle\langle 1, -\lambda\rangle\langle -\mu_1, -\mu_2, \mu_1\mu_2\rangle\big) \tag{5}$$

with $K = k[s]/(s^2 - b)$ for some $b \in k^\times$ and λ, μ_1, $\mu_2 \in K^\times$.

Proof. Note that $G \subset \mathrm{SO}(q)$ leaves the subspace $V' = k^{12} \times \{0\} \subset V$ invariant. Let

$$\ell\colon G \to \mathrm{O}(q|V')$$
$$\ell(g) = j(g)|V'$$

Then

$$j(g) = \big(\ell(g), \det(\ell(g))\big) \in \mathrm{O}(q|V') \times \mathrm{O}(1) \subset \mathrm{O}(q)$$

This yields the decomposition (4).

It remains to show that $\ell_*\big(H^1(k,G)\big) \subset H^1(\mathrm{O}(q|V'))$ consists of the forms $\tilde{q}$ as in (5).

Elements of $H^1(k,G)$ are the isomorphism classes of triples (K', φ, φ_1), where $K' = k[t]/(t^4 - b)$ is a Galois μ_4-algebra and where φ, φ_1 are quadratic forms over the quadratic subextension $K = k[s] \subset K'$, $s = t^2$ with φ of rank 3 and determinant 1 and with φ_1 of rank 1. Let

$$H = \big(\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(1)\big) \cap \mathrm{SO}(3) \simeq \mu_2 \times \mu_2$$

and

$$G' = (H \times \mu_2)^2 \rtimes \mu_4 \subset G$$

Since quadratic forms (over K) can be diagonalized, it follows that $H^1(k,G') \to H^1(k,G)$ is surjective.

The claim follows from Corollary 10.3 below. □

Lemma 10.2. *Let*

$$G'' = (\mu_2)^2 \rtimes \mu_4$$

generated by μ_4 and elements α, β with the relations

$$\alpha^2 = \beta^2 = (\alpha\beta)^2 = 1, \quad \zeta\alpha\zeta^{-1} = \beta, \quad \zeta\beta\zeta^{-1} = \alpha$$

for a generator ζ of μ_4.

Let

$$q_0\colon k^2 \to k$$
$$q_0(x,y) = x^2 - y^2$$

and let

$$\varphi\colon G'' \to \mathrm{O}(q_0)$$

be the homomorphism with

$$\varphi(\alpha) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi(\beta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varphi(\eta) = \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}$$

with α, $\beta \in \mu_2$ and $\eta \in \mu_4$.

Let $\xi \in H^1(k,G'')$ and write the corresponding Galois G''-algebra as

$$E_\xi = k[t,s,x,y]/(t^4 - b, s - t^2, x^2 - u - sv, y^2 - u + sv)$$

with $b,\ u,\ v \in k$, $b \neq 0$, $u^2 - bv^2 \neq 0$. Here the action of G'' is given by

$$\begin{aligned}
&\zeta(t) = \zeta t, \quad \zeta(s) = -s, \quad \zeta(x) = y, \quad \zeta(y) = x\\
&\alpha(t) = t, \quad \alpha(s) = s, \quad \alpha(x) = -x, \quad \alpha(y) = y\\
&\beta(t) = t, \quad \beta(s) = s, \quad \beta(x) = x, \quad \beta(y) = -y
\end{aligned}$$

Then the associated quadratic form $q_\xi = \varphi_*(\xi) \in H^1(k, \mathrm{O}(q_0))$ is given by

$$q_\xi = (T_{K/k})_*\big(\langle s\rangle\langle u + sv\rangle\big)$$

with $K = k[s] \subset E_\xi$.

Proof. One has (more or less by definition)

$$q_u = (q_0 \otimes_k E)|(k^2 \otimes_k E)^{G''}$$

with G'' acting on k^2 via $\mathrm{O}(q_0)$ and on E as Galois algebra, respectively.

The claim follows from the following explicit computation (for a related consideration see Garibaldi's Lens notes from May 2006, Example 16.5):

One finds that $(k^2 \otimes_k E)^{G''}$ is the free k-module with basis

$$X = (xt, -yt), \qquad Y = (xt^3, yt^3) = (xts, yts)$$

For $c,\ d \in k$ one has with $\lambda = x^2 = u + sv$ and $\bar\lambda = y^2 = u - sv$

$$\begin{aligned}
q_0(cX + dY) &= \big(xt(c + ds)\big)^2 - \big(yt(-c + ds)\big)^2\\
&= \lambda s(c + ds)^2 + \bar\lambda(-s)(c - ds)^2\\
&= T_{K/k}\big(\lambda s(c + ds)^2\big)
\end{aligned}$$

□

Corollary 10.3. *Let $n,\ m \geq 0$, let $U = (\mu_2)^n$ and let*

$$\Phi\colon U \to \mathrm{O}(1)^m \subset \mathrm{O}(m)$$

be some homomorphism. Let

$$G'' = U^2 \rtimes \mu_4$$

with μ_4 acting via the projection $\mu_4 \to \mu_2 = \mathbf{Z}/2$ by permutation of the factors and let

$$\varphi\colon G'' \to \mathrm{O}(m, m)$$

$$\varphi(u_1, u_2) = \begin{pmatrix} \Phi(u_1) & 0 \\ 0 & \Phi(u_2) \end{pmatrix}$$

$$\varphi(\zeta) = \begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix}$$

for $(u_1, u_2) \in U^2$ and a generator ζ of μ_4.

Let $\xi \in H^1(k, G'')$ and write the corresponding Galois G''-algebra as

$$E_\xi = k[t, s, x_i, y_i; i = 1, \ldots, n]/(t^4 - b, s - t^2, x_i^2 - u_i - sv_i, y_i^2 - u_i + sv_i)$$

with $b,\ u_i,\ v_i \in k$, $b \neq 0$, $u_i^2 - bv_i^2 \neq 0$ (with obvious G'' action, see Lemma 10.2).

Then the associated quadratic form $q_\xi = \varphi_*(\xi) \in H^1(k, \mathrm{O}(m, m))$ is given by

$$q_\xi = (T_{K/k})_*\big(\langle s\rangle\langle \mu_1, \ldots, \mu_m\rangle\big)$$

with $K = k[s] \subset E_\xi$ and with

$$\mu_j = \prod_{i=1}^{n} \lambda_i^{\Phi_{ij}} \in K, \quad j = 1, \dots, m$$

where

$$\lambda_i = u_i + s v_i$$

and where $\Phi_{ij} = 0$, 1 is defined by

$$\Phi(\alpha_1, \dots, \alpha_n) = \Big(\prod_{i=1}^{n} \alpha_i^{\Phi_{ij}}\Big)_{j=1,\dots,m}$$

Proof. One easily reduces to the case $m = 1$, $n = 1$ and $\Phi = \mathrm{id}$, which is treated in Lemma 10.2. □

Proposition 10.4. *The natural map $\tilde{j}_*\colon H^1(\widetilde{G}) \to H^1(\mathrm{Spin}(q))$ is surjective.*

Proof. Let $u \in H^1(k, \mathrm{Spin}(q))$ and let $q_u \in H^1(k, \mathrm{SO}(q))$ be the associated quadratic form. Then

$$q_u \perp \langle 1 \rangle \in I^3$$

By the results on 14-dimensional forms in I^3 one has

$$q_u \perp \langle 1 \rangle = (T_{K/k})_*(\langle s \rangle \varphi')$$

with $K = k[s]/(s^2 - b)$ for some $b \in k^\times$ and with φ a 3-fold Pfister form over K (and with $\varphi = \langle 1 \rangle \perp \varphi'$). Since the left hand side represents 1, there exists a value $-\lambda$ of φ' with $T_{K/k}(-s\lambda) = 1$. As for any (invertible) value $-\lambda$ of φ', one has $\varphi = \langle\langle \lambda, \mu_1, \mu_2 \rangle\rangle$ for some μ_1, $\mu_2 \in K^\times$. Note that

$$(T_{K/k})_*(\langle -s\lambda \rangle) = \langle 1, -N_{K/k}(\lambda) \rangle$$

Thus

$$q_u = (T_{K/k})_*\big(\langle s \rangle \langle\langle \lambda \rangle\rangle \langle\langle \mu_1, \mu_2 \rangle\rangle'\big) \perp \langle -N_{K/k}(\lambda) \rangle$$

By Lemma 10.1 it follows that $q_u \in J$. A diagram chase (see diagram (3)) involving the coboundary maps $H^1(k, G)$, $H^1(k, \mathrm{SO}(q)) \to H^2(k, \mu_2)$ shows that there exists $\tilde{u} \in H^1(k, \widetilde{G})$ such that $\tilde{j}(\tilde{u})$, $u \in H^1(k, \mathrm{Spin}(q))$ have the same image in $H^1(k, \mathrm{SO}(q))$. Another diagram chase shows that we can arrange $\tilde{j}(\tilde{u}) = u$. □

We next compute $\widetilde{G} \subset \mathrm{Spin}(q) \subset C(q)$ inside the Clifford algebra. Let e_1, …, e_{13} be the standard base of V.

Let ζ be a primitive 4-th root of unity.

For v, $w \in V$ with $q(v) = 1$, $q(w) = -1$ and $v \perp w$ let

$$\omega(v, w) = \frac{1 + \zeta w v}{\sqrt{2}}$$

Then $\omega(v, w)\omega(w, v) = 1$ and therefore $\omega(v, w) \in \mathrm{Spin}(q)$. Moreover $\omega(v, w)^2 = \zeta w v$ and $\omega(v, w)^4 = -1$. Furthermore $\omega(v, w) v = v \omega(v, w)^{-1}$ and $\omega(v, w) w = w \omega(v, w)^{-1}$. Also $\omega(v, w) v \omega(v, w)^{-1} = \zeta w$ and $\omega(v, w) w \omega(v, w)^{-1} = \zeta v$.

Consider the element

$$\omega = \omega(e_1, e_7)\omega(e_2, e_8)\omega(e_3, e_9)\omega(e_{10}, e_4)\omega(e_{11}, e_5)\omega(e_{12}, e_6) \in \mathrm{Spin}(q)$$

Its image in $\mathrm{SO}(q)$ is $\pi(\omega) = W(\zeta)$. Moreover

$$\omega^4 = 1$$

Next let

$$\tilde{\alpha} = e_4e_5e_6e_{13}, \quad \tilde{\beta} = -\zeta e_{10}e_{11}e_{12}e_{13}$$

Both elements are in $\mathrm{Spin}(q)$ and $\pi(\tilde{\alpha}) = V(-1,1)$ and $\pi(\tilde{\beta}) = V(1,-1)$. Moreover

$$\begin{aligned}
\tilde{\alpha}^2 &= 1\\
\tilde{\beta}^2 &= 1\\
\tilde{\alpha}\tilde{\beta} &= -\tilde{\beta}\tilde{\alpha}\\
(\tilde{\alpha}\tilde{\beta})^2 &= -1\\
\omega\tilde{\alpha}\omega^{-1} &= \tilde{\beta}\\
\omega\tilde{\beta}\omega^{-1} &= -\tilde{\alpha}\\
\omega\tilde{\alpha}\tilde{\beta}\omega^{-1} &= \tilde{\alpha}\tilde{\beta}\\
\tilde{\alpha}\omega\tilde{\alpha}^{-1} &= \tilde{\alpha}\tilde{\beta}\omega\\
\tilde{\alpha}\omega^2\tilde{\alpha}^{-1} &= -\omega^2
\end{aligned}$$

Let H be the subgroup generated by ω and $\tilde{\alpha}$. Then $\tilde{\beta} \in H$ and

$$H = (\mu_4 \times \mu_4) \rtimes \mu_2$$

with the μ_2 generated by $\tilde{\alpha}$ and $\mu_4 \times \mu_4$ generated by ω and $\tilde{\alpha}\omega\tilde{\alpha}^{-1}$.

Note further that the diagonal embedding $\mathrm{SO}(3) \to \mathrm{SO}(3,3)$ lifts to $\mathrm{Spin}(3,3)$. Thus the connected component of G lifts (uniquely) to $\mathrm{Spin}(q)$. This yields:

Lemma 10.5. *One has*

$$\widetilde{G} \simeq \big(\mathrm{SO}(3)\big)^2 \rtimes_\varphi H$$

where H acts by permutation of the factors via $\varphi\colon H \to \mathbf{Z}/2$, $\varphi(\tilde{\alpha}) = 0$, $\varphi(\omega) = 1$.

(I was surprised about the simple structure of H. There ought to be a better approach to the subgroup $\widetilde{G}$ of $\mathrm{Spin}(13)$ than just by a computation starting from G.)

Proposition 10.6. $\mathrm{ed}(\widetilde{G}) \leq 6$

Proof. Elements of $H^1(k,H)$ are given by Galois H-algebras which can be written as

$$L = k[z,x,y]/(z^2 - a, x^4 - u - sv, y^4 - u + sv)$$

with a, u, $v \in k$, $a \neq 0$, $u^2 - av^2 \neq 0$. For the generic case we may assume $v \neq 0$ and replace s by sv and a by av^2. Then $v = 1$. Therefore H-torsors are parameterized by a and u and we have $\mathrm{ed}(H) \leq 2$.

Thus an element of $H^1(k,\widetilde{G})$ is given by a Galois H-algebra

$$L = k[z,x,y]/(z^2 - a, x^4 - u - s, y^4 - u + s)$$

and a quadratic form of rank 3 and determinant 1 over $K = k[t] \subset L$ with $t = (xy)^2$ and $t^2 = u^2 - a$. Thus $\mathrm{ed}(\widetilde{G}) \leq \mathrm{ed}(H) + 2 \cdot 2$. □

Corollary 10.7. $\mathrm{ed}(\mathrm{Spin}(q)) \leq 6$

Proof. This is clear from Proposition 10.4 and Proposition 10.6. □

11. The essential dimension of split Spin(n) for $n \leq 14$

Let Spin_n denote a split form of $\mathrm{Spin}(n)$.

Theorem.

$$
\begin{aligned}
\mathrm{ed}(\mathrm{Spin}_n) &= 0 \quad \textit{for } n \leq 6,\\
\mathrm{ed}(\mathrm{Spin}_7) &= 4,\\
\mathrm{ed}(\mathrm{Spin}_8) &= 5,\\
\mathrm{ed}(\mathrm{Spin}_9) &= 5,\\
\mathrm{ed}(\mathrm{Spin}_{10}) &= 4,\\
\mathrm{ed}(\mathrm{Spin}_{11}) &= 5,\\
\mathrm{ed}(\mathrm{Spin}_{12}) &= 6,\\
\mathrm{ed}(\mathrm{Spin}_{13}) &= 6,\\
\mathrm{ed}(\mathrm{Spin}_{14}) &= 7.
\end{aligned}
$$

Proof. (Sketch) The cases $n = 12, 14$ have been just considered. It is not difficult to extend our considerations to the case $n = 11$.

As for $n = 13$: By corollary 10.7 one has $\mathrm{ed}(\mathrm{Spin}_{13}) \leq 6$. The invariant h_6 restricted to Spin_{13} is nontrivial, for example for

$$q \perp \langle 1\rangle = b_1\big(\langle\langle a_1, a_2, a_3\rangle\rangle' - \langle\langle b_1, b_2, b_3\rangle\rangle'\big)$$

Hence $\mathrm{ed}(\mathrm{Spin}_{13}) \geq 6$.

For $n = 7, 10$ one uses that any Spin_n-torsor admits a reduction to $G_2 \times \mu_2$ resp. to $G_2 \times \mu_4$. For $n = 8, 9$ one may use the fact that

$$\mathrm{Spin}_8 \to \mathrm{Spin}_9 \to F_4$$

induce surjections on H^1 at the prime 2 and Serre's $H^5(\mathbf{Z}/2)$-invariant for F_4, cf. [27, III. Annexe, § 3.4] or [28, III. Appendix 2, 3.4] and [14, § 40], [23]. For $n \leq 6$ note that any n-dimensional quadratic form with trivial e_1-, e_2-invariants is split. □

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