

On Vector Product Algebras

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This text contains some remarks on vector product algebras and the graphical techniques. It is partially contained in the diploma thesis of D. Boos and S. Maurer.

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1. Vector Product Algebras

A vector product algebra consists of a vector space V together with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on V , and a linear map $V \otimes V \rightarrow V$, $x \otimes y \mapsto x \times y$ such that

$$\langle x \times y, z \rangle = \langle x, y \times z \rangle, \quad x \times x = 0, \quad (x \times y) \times x = \langle x, x \rangle y - \langle x, y \rangle x$$

In a vector product algebra one has $x \times (y \times x) = (x \times y) \times x$. Moreover $\langle x \times y, z \rangle$ is alternating.

V is called associative if

$$(x \times y) \times z = \langle x, z \rangle y - \langle y, z \rangle x$$

V is called commutative if the product is trivial:

$$x \times y = 0$$

In this case $\langle x, x \rangle y = \langle x, y \rangle x$.

2. The Fundamental Relation in Vector Product Algebras

For a vector product algebra one introduces the following tensors $R_n: V^{\otimes n} \rightarrow V$.

$$R_1(x) = x$$

$$R_2(x, y) = x \times y,$$

$$R_3(x, y, z) = (x \times y) \times z - y \langle x, z \rangle + x \langle y, z \rangle$$

$$R_4(x_1, x_2, x_3, t) = R_3(x_1, x_2, x_3) \times t - \sum_{i=1}^3 x_i \langle x_{i+1} \times x_{i+2}, t \rangle + \sum_{i=1}^3 x_{i+1} \times x_{i+2} \langle x_i, t \rangle$$

with i taken mod 3.

One has the following fundamental relation for vector product algebras. It holds over any ring F and for possibly degenerate bilinear forms $\langle \cdot, \cdot \rangle$.

(2.1) Main Lemma. $2 \cdot R_4 \equiv 0$.

Proof. Put

$$\Delta(u, v, w) = (u \times v) \times w + u \times (v \times w)$$

and check the equality

$$(*) \quad 2((x \times y) \times z) \times t = \Delta(x \times y, z, t) - \Delta(x, y, z \times t) + x \times \Delta(y, z, t) - \Delta(x, y \times z, t) + \Delta(x, y, z) \times t$$

In fact, you will find the cancelling terms:

$$(x \times y) \times (z \times t), \quad x \times (y \times (z \times t)), \quad x \times ((y \times z) \times t), \quad (x \times (y \times z)) \times t$$

On the other hand, the anti-commutativity and the polarization of the second axiom give

$$\Delta(u, v, w) = 2w\langle u, v \rangle - u\langle v, w \rangle - v\langle u, w \rangle$$

Inserting this in (*) leads to the claim.

3. Relations for the Dimension in a Vector Product Algebra

We next consider the norms N_n of the tensors R_n . Let

$$Q_n(x_1, \dots, x_{n+1}) = \langle R_n(x_1, \dots, x_n), x_{n+1} \rangle$$

and put

$$N_n = N_{V^{\otimes(n+1)}}(Q_n)$$

In other words, if e_i is an orthonormal basis of V (over some algebraic closure of V), then

$$N_n = \sum_{i_1, \dots, i_{n+1}} Q_n(e_{i_1}, \dots, e_{i_{n+1}})^2 = \sum_{i_1, \dots, i_n} N(R_n(e_{i_1}, \dots, e_{i_n}))$$

Let $d = \dim V \in F$ be the dimension of V considered as an element of the ground field.

Since $2R_4 = 0$ it follows immediately from the next Proposition that

$$4d(d-1)(d-3)(d-7) = 0$$

in any vector product algebra. Similarly for the associative and commutative case.

(2.2) Proposition.

$$\begin{aligned}
N_1 &= d \\
N_2 &= d(d-1) \\
N_3 &= d(d-1)(d-3) \\
N_4 &= d(d-1)(d-3)(d-7)
\end{aligned}$$

Proof. The claim for N_1 is obvious. Next note

$$(2.2a) \quad \sum_i e_i \times (v \times e_i) = \sum_i \langle e_i, e_i \rangle v - \sum_i \langle e_i, v \rangle e_i = dv - v = (d-1)v$$

Hence

$$\sum_i N(x \times e_i) = (d-1)N(x)$$

and

$$N_2 = \sum_{i,j} \langle e_i \times e_j, e_i \times e_j \rangle = \sum_{i,j} \langle e_i, e_j \times (e_i \times e_j) \rangle = (d-1) \sum_i \langle e_i, e_i \rangle = d(d-1)$$

Moreover

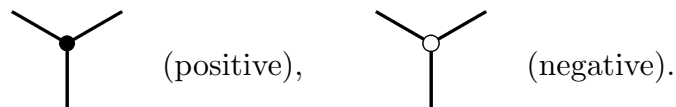
$$\begin{aligned}
N_3 &= \sum_{i,j,k} N(R_3(e_i, e_j, e_k)) \\
&= \sum_{i,j,k} N((e_i \times e_j) \times e_k - e_j \langle e_i, e_k \rangle + e_i \langle e_k, e_j \rangle) \\
&= \sum_{i,j,k} \left[N((e_i \times e_j) \times e_k) + N(e_j) \langle e_i, e_k \rangle^2 + N(e_i) \langle e_k, e_j \rangle^2 \right. \\
&\quad \left. - 2 \langle (e_i \times e_j) \times e_k, e_j \rangle \langle e_i, e_k \rangle + 2 \langle (e_i \times e_j) \times e_k, e_i \rangle \langle e_j, e_k \rangle \right. \\
&\quad \left. - 2 \langle e_i, e_k \rangle \langle e_j, e_k \rangle \langle e_k, e_i \rangle \right] \\
&= d(d-1)^2 + d^2 + d^2 - 2N_2 - 2N_2 - 2d \\
&= d(d-1)^2 + 2d(d-1) - 4d(d-1) = d(d-1)(d-3)
\end{aligned}$$

Finally, by re-indexing and using $\langle e_i, e_i \times e_j \rangle = \langle e_i, e_k \times e_i \rangle = 0$ one finds:

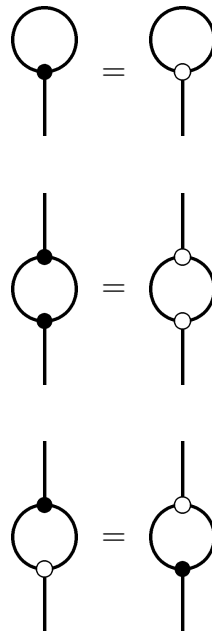
$$\begin{aligned}
N_4 &= \sum_{i,j,k,l} N(R_4(e_i, e_j, e_k, e_l)) \\
&= \sum_{i,j,k,l} N(R_3(e_i, e_j, e_k) \times e_l - e_i \langle e_j \times e_k, e_l \rangle - e_j \langle e_k \times e_i, e_l \rangle - e_k \langle e_i \times e_j, e_l \rangle \\
&\quad + e_j \times e_k \langle e_i, e_l \rangle + e_k \times e_i \langle e_j, e_l \rangle + e_i \times e_j \langle e_k, e_l \rangle) \\
&= \sum_{i,j,k,l} \left[N(R_3(e_i, e_j, e_k) \times e_l) + 3 \cdot N(e_i) \langle e_j \times e_k, e_l \rangle^2 + 3 \cdot N(e_j \times e_k) \langle e_i, e_l \rangle^2 \right. \\
&\quad - 3 \cdot 2 \langle R_3(e_i, e_j, e_k) \times e_l, e_i \rangle \langle e_j \times e_k, e_l \rangle \\
&\quad + 3 \cdot 2 \langle R_3(e_i, e_j, e_k) \times e_l, e_j \times e_k \rangle \langle e_i, e_l \rangle \\
&\quad + 3 \cdot 2 \langle e_i, e_j \rangle \langle e_j \times e_k, e_l \rangle \langle e_k \times e_i, e_l \rangle \\
&\quad + 3 \cdot 2 \langle e_i, e_j \times e_k \rangle \langle e_j \times e_k, e_l \rangle \langle e_i, e_l \rangle \\
&\quad \left. + 3 \cdot 2 \langle e_j \times e_k, e_k \times e_i \rangle \langle e_i, e_l \rangle \langle e_j, e_l \rangle \right] \\
&= \sum_{i,j,k} (d-1) \cdot N(R_3(e_i, e_j, e_k)) + 3 \cdot \sum_{j,k} d \cdot N(e_j \times e_k) + 3 \cdot \sum_{j,k} d \cdot N(e_j \times e_k) \\
&\quad + 3 \cdot 2 \sum_{i,j,k} \langle R_3(e_i, e_j, e_k) \times e_i, e_j \times e_k \rangle \\
&\quad - 3 \cdot 2 \sum_{i,j,k} \langle R_3(e_i, e_j, e_k) \times (e_j \times e_k), e_i \rangle \\
&\quad + 3 \cdot 2 \sum_{i,k} \langle e_i \times e_k, e_k \times e_i \rangle \\
&\quad - 3 \cdot 2 \sum_{j,k} \langle e_j \times e_k, e_j \times e_k \rangle \\
&\quad + 3 \cdot 2 \sum_{i,k} \langle e_i \times e_k, e_k \times e_i \rangle \\
&= (d-1)N_3 + 6d \cdot N_2 - 12N_3 - 18N_2 = (d-1-12+6)N_3 \\
&= d(d-1)(d-3)(d-7).
\end{aligned}$$

4. Graph Considerations for Vector Product Algebras

We consider 3-valent graphs with cyclically oriented vertices. We describe the orientation at a vertex by replacing it by an oriented disk. The orientation of a disk is indicated by black or white coloring:



First note that rotation around the vertical symmetry axis give the following identities:



The following rules (R1) and (R2) are the graph versions of the axioms for vector product algebras.

(R1) $\begin{matrix} \diagup \\ \circ \\ \diagdown \\ | \end{matrix} = - \begin{matrix} \diagup \\ \bullet \\ \diagdown \\ | \end{matrix}$

(R2) $\begin{matrix}) \\ \text{---} \\ (\end{matrix} + \begin{matrix} \text{---} \\ \cup \\ \text{---} \end{matrix} = 2 \cdot \begin{matrix} \diagup \\ \diagdown \\ \diagdown \\ \diagup \end{matrix} - \begin{matrix} \diagdown \\ \diagup \\ \diagup \\ \diagdown \end{matrix} \quad \left(- \begin{matrix} \text{---} \\ \cap \\ \text{---} \end{matrix} \right)$

Here we use the following **convention**: If in a plane graph no orientation is indicated we understood the positive orientation (black coloring). The rule (R1) makes it possible to give this orientation to the pictures in this section.

5. Graphical proof of $d(d-1)(d-3)(d-7) = 0$

We assume now that 2 is invertible in the ground ring and show that the two rules (R1) and (R2) imply

$$d(d-1)(d-3)(d-7) = 0$$

where

$$d = \bigcirc$$

Since 2 is invertible, one has

$$(1.0) \quad \bigcirc \text{---} = 0$$

by (0.1) and (R1).

Next consider the following consequence of (R2):

$$\begin{array}{c} | \\ | \\ \bigcirc \text{---} \\ | \\ | \end{array} + \begin{array}{c} | \\ | \\ \bigcirc \\ | \\ | \end{array} = 2 \cdot \begin{array}{c} | \\ | \\ \bigcirc \text{---} \\ | \\ | \end{array} - \begin{array}{c} | \\ | \\ \bigcirc \text{---} \\ | \\ | \end{array} - \begin{array}{c} | \\ | \\ \text{---} \\ | \\ | \end{array}$$

By (1.0) this gives

$$(1.1) \quad \begin{array}{c} | \\ | \\ \bigcirc \\ | \\ | \end{array} = -(d-1) \cdot \begin{array}{c} | \\ | \\ | \\ | \end{array}$$

This yields immediately

$$(1.2) \quad \begin{array}{c} | \\ | \\ \bigcirc \\ | \\ | \end{array} = -(d-1) \cdot \bigcirc = -d(d-1)$$

We next prove Springer's formula. (R2) gives

$$\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ | \end{array} + \begin{array}{c} \diagdown \diagup \\ \circ \\ | \end{array} = 2 \cdot \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \circ \\ | \end{array} - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ | \end{array} - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \circ \\ | \end{array}$$

Inserting (1.0) and (1.1) shows

$$\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ | \end{array} - (d-1) \cdot \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ | \end{array} = -2 \cdot \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ | \end{array} - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ | \end{array} - 0$$

Hence

$$(1.3) \quad \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ | \end{array} = (d-4) \cdot \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ | \end{array}$$

Moreover one finds

$$(1.4) \quad \begin{array}{|c|} \hline \diagdown \diagup \\ \diagup \diagdown \\ \hline \end{array} = (d-4)^2 \cdot \begin{array}{|c|} \hline \circ \\ \hline \end{array} = -d(d-1)(d-4)^2$$

by applying (1.3) to both of the triangles and using (1.2).

Now comes the final step. Rule (R2) gives

$$\begin{array}{|c|} \hline \diagdown \diagup \\ \diagup \diagdown \\ \hline \end{array} + \begin{array}{|c|} \hline \diagdown \diagup \\ \diagup \diagdown \\ \hline \end{array} = 2 \cdot \begin{array}{|c|} \hline \diagdown \diagup \\ \diagup \diagdown \\ \hline \end{array} - \begin{array}{|c|} \hline \diagdown \diagup \\ \diagup \diagdown \\ \hline \end{array} - \begin{array}{|c|} \hline \diagdown \diagup \\ \diagup \diagdown \\ \hline \end{array}$$

The two leftmost and the two rightmost graphs are the same. So, after dividing by 2 we have

$$(1.5) \quad \begin{array}{|c|} \hline \diagdown \diagup \\ \diagup \diagdown \\ \hline \end{array} = \begin{array}{|c|} \hline \diagdown \diagup \\ \diagup \diagdown \\ \hline \end{array} - \begin{array}{|c|} \hline \diagdown \diagup \\ \diagup \diagdown \\ \hline \end{array}$$

For the middle term of (1.5) one finds

$$(1.6) \quad \begin{array}{|c|} \hline \diagdown \diagup \\ \diagup \diagdown \\ \hline \end{array} = \begin{array}{|c|} \hline \diagdown \diagup \\ \diagup \diagdown \\ \hline \end{array} = (d-4) \cdot \begin{array}{|c|} \hline \circ \\ \hline \end{array} = -d(d-1)(d-4)$$

Here the first equality follows, since both pictures are just different projections of the same graphs but with sign changes at two vertices. Rule (R1) give then equality. The other two equalities follow from (1.3) and (1.2).

To compute the rightmost graph in (1.5) one applies formula (1.1) twice and finds

$$(1.7) \quad \begin{array}{|c|} \hline \diagdown \diagup \\ \diagup \diagdown \\ \hline \end{array} = (d-1)^2 \cdot \begin{array}{|c|} \hline \circ \\ \hline \end{array} = d(d-1)^2$$

The formulas (1.4), (1.6), and (1.7) give the desired relation. □

Here are some of the translations of the graph formulas above to the algebraic formulas in [Rost, M., On the Dimension of a Composition Algebra, *Doc. Math.* **1** No. 10 (1996) 209–214]:

$$(R1) \leftrightarrow (2.4a)$$

$$(R2) \leftrightarrow (2.5a,b)$$

$$(1.1) \leftrightarrow (3.1)$$

$$(1.2) \leftrightarrow (3.2)$$

$$(1.3) \leftrightarrow (3.3)$$

Exercise. Draw the pictures which derive the relations $d(d-1)(d-3) = 0$ and $d(d-1) = 0$ for “associative” and “commutative” vector product algebras (the “toy models” in loc. cit.)

6. Graphical illustration of $2R_4 = 0$

The relation (R2) gives

The diagram shows two 4-vertex graphs. The first graph has a central vertex with four edges extending outwards. The second graph is a rotation of the first. They are separated by a plus sign, followed by an equals sign.

(1.8)

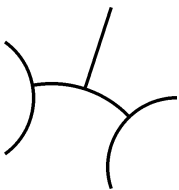
The diagram shows a 4-vertex graph with a coefficient '2' to its left. This is followed by a minus sign and three other 4-vertex graphs, each with a minus sign to its left.

Hence

The diagram shows a 4-vertex graph on the left, followed by an equals sign, a minus sign, another 4-vertex graph, a plus sign, and the text 'sum of graphs with less vertices'.

The two graphs differ just by sign and a rotation of order 5. Therefore, iterating this

relation 5 times yields

$$2 \cdot \text{graph} = \text{sum of graphs with less vertices}$$


This shows that the morphism

$$V^{\otimes 4} \rightarrow V, \quad x_1 \otimes x_2 \otimes x_3 \otimes x_4 \mapsto ((x_1 \times x_2) \times x_3) \times x_4$$

can be expressed as a sum of tensors involving only 1 \times -product. The precise formula for this is

$$2R_4 = 0$$

Together with (1.0) and (1.1) it follows from (1.8):

(1.1) Proposition. *Any closed connected graph with at most $2k - 2$ vertices can be expressed by a polynomial in d of degree $\leq k$. In particular, $\text{End}(\emptyset)$ is generated by d .*

If I remember well one can use (1.8) directly to show at least the following qualitative result:

(1.2) Proposition. *One has $P(d) = 0$ in $\text{End}(\emptyset)$ for some polynomial $P \in \mathbb{Z}[d]$ of degree 4.*

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