REALIZATION OF SIMPLE LIE ALGEBRAS VIA HALL ALGEBRAS

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Let A be a finite-dimensional hereditary algebra over a finite field k, and consider the free abelian group $\mathcal{H}(A)$ with basis the isoclasses of finite A-modules. In [4] Ringel introduced a multiplication on $\mathcal{H}(A)$ to make it an associative ring with identity. The structure constants were given by the numbers of filtrations of modules with factors isomorphic to modules that are multiplied. The ring $\mathcal{H}(A)$ obtained in this way is called the *Hall algebra* of A. Assume in addition that A is representation-finite, i.e., that the underlying graph Δ of the quiver of A is a Dynkin graph. Then it was shown that the structure constants are given by polynomials in the cardinality q of k, called the *Hall polynomials*. This makes it possible to define a *generic Hall algebra* $\mathcal{H}(A, \mathbb{Z}[T])$, which is a free $\mathbb{Z}[T]$-module with the same basis as $\mathcal{H}(A)$ and the multiplication is given by the Hall polynomials. The specialization $\mathcal{H}(A)_1$ of $\mathcal{H}(A, \mathbb{Z}[T])$ for $T = 1$, called the *degenerate Hall algebra* of A, yields a Lie subalgebra $\overline{L}(A)_1$, which is the free abelian subgroup of the degenerate Hall algebra with basis the set of isoclasses of indecomposable A-modules. In [5] he has shown that $\overline{L}(A)_1^{\mathbb{C}} := \overline{L}(A)_1 \otimes_{\mathbb{Z}} \mathbb{C}$ and the positive part $\mathfrak{n}_+(\Delta)$ of the simple complex Lie algebra $\mathfrak{g}(\Delta)$ of type Δ are isomorphic as Lie algebras. Namely, the positive part $\mathfrak{n}_+(\Delta)$ of $\mathfrak{g}(\Delta)$ for a Dynkin graph Δ was recovered by the representation theory of A. After that, Peng and Xiao [2] constructed the simple Lie algebra $\mathfrak{g}(\Delta)$ of type Δ itself for each Dynkin graph Δ, mainly but partly using a Hall algebra. In their construction, the positive part and the negative part were given by the root category $\mathcal{R}$ of the path-algebra of a quiver with the underlying graph Δ ($\mathcal{R}$ can be regarded as a double of the category of finite-dimensional A-modules), and the Cartan subalgebra was given by the Grothendieck group of $\mathcal{R}$. The Hall multiplication was used to define the Lie bracket only inside $\mathcal{R}$, and when the bracket should not be closed in $\mathcal{R}$ the definition was changed. Their realization of simple Lie algebras is useful, but the definition of the bracket is a patchwork of several multiplication tables and seems to be rather artificial.

In the lecture, for each integer $n \geq 1$ we first give simple and explicit realization of the simple Lie algebra $\mathfrak{g}(A_n)$ of type A_n, i.e., the special linear algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$ and of the general linear algebra $\mathfrak{gl}_{n+1}(\mathbb{C})$ only by the Hall multiplication as a quotient of the degenerate composition Lie algebra (resp. of the degenerate Lie algebra) with coefficients in $\mathbb{C}$ of the category of nilpotent representations of the cyclic quiver with $n+1$ vertices. Note that until now this categoy was used to realize the positive part of the affine Lie algebra of type $\tilde{A}_n (= A_n^{(1)})$ as in [7] or [1]. The realization of these immediately yields realization of their universal enveloping algebras via Hall algebras. Next, we will give a general way to realize all types of simple Lie algebras via Hall algebras of finite-dimensional tame hereditary algebras (in contrast with the first part, in which we use an infinite-dimensional algebra by the speciality of a cyclic orientation on $\tilde{A}_n$), which were well studied by Ringel [6] and Peng and Xiao [3], and give explicit realization for the types A_n (a different presentation) and $D_n (n \geq 4)$.

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