## The descent algebra of the symmetric group

(Manfred Schocker)

The descent set of a permutation $\pi$ in the symmetric group $S_n$ is

$$\mathsf{Des}\,\pi = \{\, i \leq n-1 \,|\, i\pi > (i+1)\pi \,\}.$$

Due to a remarkable result of Solomon, the linear span $\mathcal{D}$ of the sums of descent classes

$$\Delta^D = \sum_{\mathsf{Des}\,\pi = D} \pi \qquad (D \subseteq \{1, \ldots, n-1\})$$

is a sub-ring of the integral group ring of $S_n$. Furthermore, there is a homomorphism of rings mapping $\mathcal{D}$ onto the ring of class functions of $S_n$, with kernel equal to the Jacobson radical of $\mathcal{D}$.

As a consequence, the irreducible representations of $\mathcal{D}$ are of dimension 1 and indexed by partitions of $n$ — and provide a strong connection to the representation theory of $S_n$. In fact, there is the (certainly vague) dictum that the Specht modules of $S_n$ have been shrunk to their linear nuclei here.

Results of Blessenohl and Laue describe the Cartan invariants and the descending Loewy series of $\mathcal{D}$, and allow to determine its quiver and representation type. Also, the $S_n$-representations arising from the primitive idempotents of $\mathcal{D}$ achieve a natural realization, namely as the multi-homogenous parts of the tensor algebra over a finite-dimensional vector space $V$ with respect to the Poincaré-Birkhoff-Witt basis, thanks to pioneering work of Garsia and Reutenauer. However, in spite of extensive study of these modules through the last years, there is still a haunting lack of comprehension.

Numerous related results have a different backdrop and link the descent algebra to combinatorial settings such as the theory of quasi-symmetric functions, problems related to permutation statistics, or card shuffling and associated random walks. Further connections to representation theory are provided by Jöllenbeck's *noncommutative character theory of the symmetric group*, as well as the huge body of research papers on the *algebra of noncommutative symmetric functions* introduced by Gelfand et al.

A satisfying overall explanation of the mediating role of the algebra $\mathcal{D}$ between the various fields is yet to be discovered. It might be hidden in the depths of its representation theory. I am going to display major results on the descent algebra and its representations, hopefully inviting parts of the audience to introduce their algebraic skills onto the subject.