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Twisting and Tilting

We consider the principal block $\mathcal{O}_o$ of the Bernstein-Gelfand category $\mathcal{O}$ for a semisimple complex Lie algebra $\mathfrak{g}$ ([BGG]). This is a highest weight category with standard objects $\Delta(x)$ indexed by the elements of the Weyl group. For each element w of the Weyl group, Arkhipov defined a right exact endofunctor of the category of $\mathfrak{g}$-modules which restricts to an endofunctor of $\mathcal{O}_o$. A very nice property of these functors is that they satisfy braid relations. If w is a simple reflection then $T_w\Delta(x)$ has the same composition factors as $\Delta(sx)$, but the modules are not isomorphic in general. This is why we call the modules $T_w\Delta(x)$ "twisted" standard objects (or twisted Verma modules in [AL]) and the functors "twisting functors".
We prove that twisting functors and the functors given by tensoring with finite dimensional modules commute (moreover, they commute with Jantzen's translation functors, [Ja]). This is a very useful property to describe twisted dual standard objects. Of course, twisting functors do not commute with the duality. It turns out that we get the right adjoint functor of T_w by conjugating $T_{w^{-1}}$ with the duality.
These properties are used to prove that the derived functors $\mathcal{L}T_w$ define an auto-equivalence of the bounded derived category $\mathrm{D}^b(\mathcal{O}_o)$. Therefore, we get an action of the braid group via derived equivalences.
We get the following applications:

- A rather easy looking reformulation for the Kazdhan-Lusztig conjecture in terms of twisted simple objects (for details see [AS]).
- A definition of "twisted tilting modules" (see [St]).
- Each tilting module has a "w-twisted standard flag" (see the special cases [Ir], [M] and in general [MS]).

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