1. Quivers and their representations: Basic definitions and examples.

1.1. Quivers.

A *quiver* Q (sometimes also called a directed graph) consists of vertices and oriented edges (arrows): loops and multiple arrows are allowed. An arrow goes from some vertex (its tail) to some vertex (its head), if we denote the tail of the arrow α by $t(\alpha)$, the head by $h(\alpha)$, we see that we deal with two set-theoretical maps

$$t, h\colon Q_1 \to Q_0,$$

where Q_0 denotes the set of vertices, Q_1 the set of arrows. Here is the formal definition of a *quiver* $Q = (Q_0, Q_1, t, h)$: there are given two sets Q_0, Q_1 and two maps $h, t\colon Q_1 \to Q_0$, the elements of Q_0 are called *vertices,* the elements of Q_1 are called *arrows,* and for every arrow $\alpha \in Q_1$, there is defined its *tail* $t(\alpha)$ and its *head* $h(\alpha)$. One depicts this in the usual way:

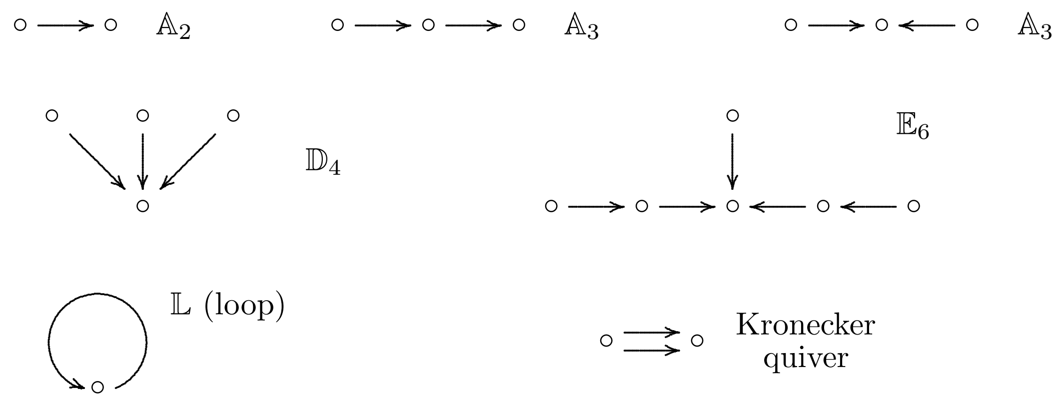
$$t(\alpha) \xrightarrow{\alpha} h(\alpha). \quad \text{or also} \quad \alpha\colon t(\alpha) \to h(\alpha)$$

(actually, often we will draw arrows from right to left, or also in any possible direction). Arrows α with $h(\alpha) = t(\alpha)$ are called *loops.*

Given a quiver Q, one may delete the orientation of the arrows and obtains in this way the *underlying graph* $\overline{Q}$, this is the triple consisting of the two sets Q_0, Q_1 and the functions which attaches to $\alpha \in Q_1$ the set $\{t(\alpha), h(\alpha)\}$ (this means that one does no longer distinguish which one of the vertices is the head and which one is the tail. The reverse process will be called *choosing an orientation.*

> The wording was chosen by Gabriel (1972): "quiver" means literally a box for holding arrows. Before Gabriel, quivers were called "diagram schemes" by Grothendieck.

Here is a collection of typical quivers, with the names which are now usually attached, often these names refer just to the underlying graph.



Of course, one may consider much more complicated quivers, say with 1000 vertices and 7000 arrows, but the representation theory already of quite small quivers usually turns out to be quite complicated. There are quivers with many edges which we will deal with, for example

$$\circ \!-\!\!-\!\!-\! \circ \!-\!\!-\!\!-\! \circ \,-\, \cdots \,-\, \circ \!-\!\!-\!\!-\! \circ \qquad \mathbb{A}_n$$

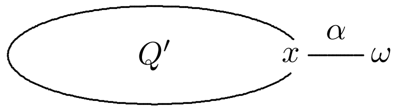
with n vertices, usually labeled $1, 2, \ldots, n$, and with $n-1$ arrows α_i with $\{t(\alpha_i), h(\alpha_i)\} = \{i, i+1\}$, but usually one is interested in rather small quivers, for example the Dynkin quivers $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, or the corresponding Euclidean quivers $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$.

If Q is a quiver, a *subquiver* Q' of Q is of the form $Q' = (Q'_0, Q'_1, t', h')$, with subsets $Q'_0 \subseteq Q_0$, $Q'_1 \subseteq Q_1$, such that $t(Q'_1) \subseteq Q'_0$ and $h(Q'_1) \subseteq Q'_0$, and such that t', h' are the restrictions of t, h, respectively.

For example, a quiver of type $\mathbb{A}_3$ has two subquivers of type $\mathbb{A}_2$.

A quiver Q is said to be *connected*, provided for any decomposition $Q_0 = Q'_0 \cup Q''_0$ of the set of vertices of Q with non-empty subsets Q'_0, Q''_0, there is an arrow α such that $h(\alpha) \in Q'_0, t(\alpha) \in Q''_0$ or $h(\alpha) \in Q''_0, t(\alpha) \in Q'_0$.

A connected quiver with n vertices and $n-1$ arrows is called a *tree quiver* (this just means that the underlying graph is a tree in the sense of graph theory). The tree quivers can be constructed inductive as follows: first of all, the quiver $\mathbb{A}_1$ (it consists of a single vertex and there is no arrow) is a tree quiver, and a quiver Q with $n \geq 2$ vertices is a tree quiver provided it is obtained from a tree quiver Q' with $n-1$ vertices by *attaching an arm of the form* $\mathbb{A}_2$ *at the vertex* x (this means that x is a vertex of Q' and one obtains Q_0 by adding to Q'_0 a vertex, say labeled ω, and that one obtains Q_1 by adding to Q'_1 an arrow α such that $\{h(\alpha), t(\alpha)\} = \{x, \omega\}$)



1.2. Representations of a quiver.

Let k be some field. Al the vector spaces to be considered are assumed to be k-spaces. For most considerations, the structure of k itself will not play a role, but we should stress that we always work with a **fixed** (commutative) field k.

A *representation* of the quiver Q is of the form $M = (M_x, M_\alpha)_{x,\alpha}$, where M_x is a vector space, for every vertex $x \in Q_0$, and $M_\alpha \colon M_{t(\alpha)} \to M_{h(\alpha)}$ is a linear map, for every $\alpha \in Q_1$; instead of M_α one often writes just α. Thus, representations of quivers are nothing else than collections of vector spaces and linear maps between these vector spaces. *We usually will assume that the vector spaces which we consider are finite-dimensional* (however most of the considerations carry over to the general case of dealing with vector spaces of arbitrary dimension).

Given a representation $M = (M_x, M_\alpha)_{x,\alpha}$, we call the sum of the dimensions of the vector spaces M_x with $x \in Q_0$ the *dimension* of M and denote it by $\dim M$. Later it will be convenient to denote the maps M_α with $\alpha \in Q_1$ just by α (clearly an abuse of notation, but quite convenient).

> Why do we use the letter M for a representation of a quiver? The representations of a quiver M may be considered as the "**m**odules" over the "path algebra" of Q, see section 4.

Of course, for any quiver there is defined the corresponding *zero representation* (or "trivial" representation) with all the vector spaces being zero (and all the maps being zero maps). The zero representation is usually just denoted by 0.

Representations M with all vector spaces M_x of dimension at most 1 are said to be *thin.*

> We will deal with thin representations in 1.6. Here is a typical representation which is not thin (and not isomorphic to a direct sum of thin modules). We deal with a quiver of type $\mathbb{D}_4$:
>
> $$\begin{tikzcd} \circ \arrow[dr] & \circ \arrow[d] & \circ \arrow[dl] \\ & \circ & \end{tikzcd} \qquad\qquad \begin{tikzcd} k0 \arrow[dr] & 0k \arrow[d] & \Delta \arrow[dl] \\ & k^2 & \end{tikzcd}$$
>
> with $\Delta = \{(x, x) \mid x \in k\}$ and all the maps being the corresponding inclusion maps. In section 3 we will see that this is an "indecomposable representation" (but we did not yet define what means "indecomposable").
>
> Also we may be interested in a vector space V with 4 subspaces $U_1,\ U_2,\ U_3,\ U$ such that $U_1 \subseteq U_2 \subseteq U_3$. Such a system can be considered as a representation of the following quiver of type $\mathbb{A}_5$
>
> $$\circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longleftarrow \circ \qquad \mathbb{A}_5$$
>
> namely as
>
> $$U_1 \longrightarrow U_2 \longrightarrow U_3 \longrightarrow V \longleftarrow U$$
>
> where again all the maps are the inclusion maps.

When looking at representations of quivers, we often will replace a given representation by an "isomorphic" one, whenever this is suitable. Given representations M, M' of a quiver Q, an *isomorphism* $f = (f_x)_x \colon M \to M'$ is given by vector space isomorphisms $f_x \colon M_x \to M'_x$ such that for any arrow $\alpha \colon t(\alpha) \to h(\alpha)$ the following diagram commutes:

$$\begin{tikzcd} M_{t(\alpha)} \arrow[r, "f_{t(\alpha)}"] \arrow[d, "M_\alpha"'] & M'_{t(\alpha)} \arrow[d, "M'_\alpha"] \\ M_{h(\alpha)} \arrow[r, "f_{h(\alpha)}"] & M'_{h(\alpha)}. \end{tikzcd}$$

(We often will have to consider such diagrams, they are given by an arrow say $\alpha \colon x \to y$; the usual convention will be to draw the data concerning M vertically on the left, those concerning M' vertically on the right, and the maps f_x horizontally.)

Of course, the commuativity of the diagram above implies that also the diagram

$$\begin{array}{ccc} M'_{t(\alpha)} & \xrightarrow{f^{-1}_{t(\alpha)}} & M_{t(\alpha)} \\ {\scriptstyle M'_\alpha}\downarrow & & \downarrow{\scriptstyle M_\alpha} \\ M'_{h(\alpha)} & \xrightarrow{f^{-1}_{h(\alpha)}} & M_{h(\alpha)}. \end{array}$$

commutes; thus if $f = (f_x)_x \colon M \to M'$ is an isomorphism, also $f^{-1} = (f_x^{-1})_x \colon M' \to M$ is an isomorphism.

> **Slogan:** *Representation theory studies properties of representations which are invariant under isomorphisms.*
>
> In section 3, we will introduce the notion of a homomorphism $f \colon M \to M'$; isomorphisms are special homomorphisms.

For example:

Polishing. *Let Q be a quiver, $\alpha \colon x \to y$ an arrow of Q, but not a loop, and M a representation such that M_α is injective. Then M is isomorphic to a representation M' such that M'_α is the inclusion of a subspace* (namely the inclusion of the image of M_α into M_y).

Proof: Let M' be defined as follows: Let $M'_x = \operatorname{Im}(M_\alpha)$, and $M'_a = M_a$ for all vertices $a \neq x$ of Q. If $\beta \colon b \to x$ and $b \neq x$, let $M'_\beta = M_\alpha M_\beta \colon M_b \to M'_a$. If $\gamma \colon x \to c$ and $c \neq x$, let $M'_\gamma = M_\gamma (M_\alpha)^{-1} \colon M'_x \to M_c$. Finally, if $\delta \colon x \to x$, let $M'_\delta = M_\alpha M_\delta (M_\alpha)^{-1} \colon M'_x \to M'x$. Always note that $(M_\alpha)^{-1}$ is defined on M'_a and the definition for $\gamma = \alpha$ shows that $M'_\alpha = M_\gamma (M_\alpha)^{-1} \colon M'_x \to M_y$ is just the inclusion map. The representations M and M' are isomorphic, with an isomorphism $f : M \to M'$ given by $f_a = 1$ for $a \neq x$ and $f_x = M_\alpha \colon M_x \to M'_x$; in order to see that f is a homomorphism, let us exhibit two typical squares: on the left we consider an arrow $\beta \colon b \to x$, on the right an arrow $\gamma \colon x \to c$.

$$\begin{array}{ccc} M_b & = & M'_b \\ {\scriptstyle M_\beta}\downarrow & & \downarrow{\scriptstyle M_\alpha M_\beta} \\ M_x & \xrightarrow{M_\alpha} & M'_x \end{array} \qquad \begin{array}{ccc} M_x & \xrightarrow{M_\alpha} & M'_x \\ {\scriptstyle M_\gamma}\downarrow & & \downarrow{\scriptstyle M_\gamma (M_\alpha)^{-1}} \\ M_c & = & M'_c \end{array}$$

Warning: If M is a representation with several of the maps M_α being injective, one may try to replace successively all these maps by inclusion maps, but in general this will not be possible. For example, consider a cycle with all the maps being invertible. If there are n arrows, we may replace $n-1$ of them by corresponding identity maps, but trying to replace also the remaining map by an identity map may destroy a previous identity map.

Similarly, *if M is a representation of a quiver Q and $\alpha \colon x \to y$ is an arrow which is not a loop, such that M_α is surjective, then M is isomorphic to a representation M' such that M'_α is the canonical projection $M_y \to M_y / \operatorname{Ker}(M_\alpha)$.*

1.3. Direct decomposition.

Given a representation M of a quiver Q, a *direct sum decomposition* of M is of the following form: for every $x \in Q_0$, there is given a direct sum $M_x = M'_x \oplus M''_x$ and for every $\alpha\colon x \to y$, one has $M_\alpha(M'_x) \subseteq M'_y$ and $M_\alpha(M''_x) \subseteq M''_y$. One may denote the restriction of M_α to M'_x by $M'_\alpha\colon M'_x \to M'_y$, and similarly, the restriction of M_α to M''_x by $M''_\alpha\colon M''_x \to M''_y$. One obtains in this way representations $M' = (M'_x, M'_\alpha)_{x,\alpha}$ and $M'' = (M''_x, M''_\alpha)_{x,\alpha}$ and one writes $M = M' \oplus M''$.

The representation theory of quivers is concerned with the following question: given a representation M of some quiver Q, is it possible to decompose the representation? If there is no non-trivial decomposition and M is non-zero, then M is said to be indecomposable: To repeat: M is *indecomposable* if and only if $M \neq 0$ and for any decomposition $M = M' \oplus M''$, either $M' = 0$ or $M'' = 0$.

There is the following question: describe all the indecomposable representations of a given quiver. For some (quite small) quivers, this will be possible (and indeed for all the examples exhibited above), but in general it seems to be impossible (there is a notion of "wildness": nearly all the large quiver are wild and one does not expect that there is a decent way to classify all the indecomposable representations of any wild quiver).

> **Slogan:** *Representation theory studies the isomorphism classes of indecomposable representations.*

Let us consider the quiver $\mathbb{A}_2$, we label the vertices 1 and 2 so that the unique arrow is $\alpha\colon 2 \to 1$. The representations of Q are of the form $M = (M_2, M_1, M_\alpha)$, where M_1, M_2 are vector spaces and $M_\alpha\colon M_2 \to M_1$ is a linear map, we will denote M just by writing $M = (M_\alpha\colon M_2 \to M_1)$. There are three indecomposable representations of V which are easy to describe:

$$(0 \to k), \quad (k \to 0), \quad (1_k\colon k \to k).$$

(and later it will turn out that these are the only indecomposable representations up to isomorphism. Why are these representations indecomposable? This should be clear for the first two representations, thus let us look at the third one: write it as $M = (M_\alpha\colon M_2 \to M_1)$ with $M_1 = M_2 = k$ and M_α the identity map. What is important is only that $M_\alpha \neq 0$. Assume we have given a direct decomposition $M = M' \oplus M''$, thus $M_2 = M'_2 \oplus M''_2$, $M_1 = M'_1 \oplus M''_1$, such that $M_\alpha(M'_2) \subseteq M'_1$ and $M_\alpha(M''_2) \subseteq M''_1$. Since $M_2 = k$ is one-dimensional, we must have $M'_2 = 0$ or $M''_2 = 0$. Without loss of generality, we can assume that $M''_2 = 0$, thus $M'_2 = M_2$. Now M_α is non-zero and maps M'_2 into M'_1, therefore also $M'_1 \neq 0$. Since $M_1 = M'_1 \oplus M''_1$ is one-dimensional and $M'_1 \neq 0$, it follows that $M''_1 = 0$. Thus $M'' = 0$.

Given a representation M and for every $x \in Q_0$ a subspace M'_x of M_x with $M_\alpha(M'_x) \subseteq M'_y$, for every arrow $\alpha\colon x \to y$, then we may denote the restriction of M_α to M'_x by M'_α, for $\alpha\colon x \to y$, and we obtain in this way a representation $M' = (M'_x, M'_\alpha)$ of Q which is called a *subrepresentation* of M.

Direct decomposition $M = M' \oplus M''$ are given by subrepresentations M', M'' of M such that $M_x = M'_x \oplus M''_x$ for all x.

Definition. We say that a representation N of a quiver Q with a vertex y is *y-sincere,* provided for any direct decomposition $N = N' \oplus N''$ with $N''_y = 0$ we have $N'' = 0$.

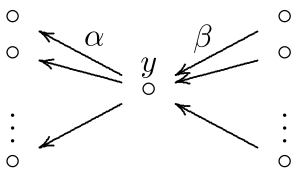
Proposition. *Let Q be obtained from a quiver Q' by attaching an arm of type $\mathbb{A}_2$ at the vertex x. Let M be an indecomposable representation of Q with support not contained in Q'. Then the restriction M' of M to Q' is x-sincere.*

Prof: Let $N = N' \oplus N''$ with $N''_x = 0$. Then we obtain a direct decomposition of $M = M' \oplus M''$ by taking $M'_\omega = M_\omega$, $M''_\omega = 0$ and such that the restriction of M' to Q' is N', the restriction of M'' to Q' is N''. Since $M' \neq 0$, and M is indecomposable, we conclude that $M'' = 0$, thus $N'' = 0$.

1.4. The simple representations $S(x)$.

Let x be a vertex of Q. The representation $S(x)$ of Q is defined by $S(x)_x = k$, $S(x)_y = 0$ for $y \neq 0$, and $S(x)_\alpha = 0$ for all arrows α (note that the latter condition concerns only loops $\alpha\colon x \to x$).

Proposition. *If y is a vertex of the quiver Q, and M a representation of Q, define subspaces K_y, I_y of M_y as follows: K_y is the intersection of the kernels of the maps M_α, where α is an arrow with tail $t(\alpha) = y$ and I_y is the sum of the images of the maps M_β where β is an arrow with head $h(\beta) = y$. Then $S(y)$ is a direct summand of M if and only if $K_y \not\subseteq I_x$.*



Better:

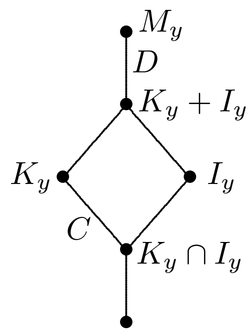
Splitting off copies of $S(y)$. *Let C, D be subspaces of M_y such that*

$$(K_y \cap I_y) \oplus C = K_y \quad \textit{and} \quad (K_y + I_y) \oplus D = M_y.$$

Let $M'_y = I_y \oplus D$ and $M'_x = M_x$ for all $x \neq y$. Let $M''_y = C$ and $M''_x = 0$ for all $x \neq y$. Then M', M'' are subrepresentations of M, and $M = M' \oplus M''$. The representation M' has no direct summand of the form $S(x)$, whereas M'' is a direct sum of copies of $S(x)$.

If x is a vertex, a representation without a direct summands $S(x)$ may be called *y-reduced*, thus here we deal with the *y-reduction.*

Proof: Here is the lattice of the relevant subspace of M_y:



We will need that $(K_y \cap I_y) \oplus C = K_y$ implies that $I_y \oplus C = K_y + I_y$.

In order to see that M' is a subrepresentation of M, we need to look only at arrows β ending in y, since $M'_x = M_x$ for $x \neq y$. But by construction M_y contains I_y, thus the image of any map $M\beta$ with $h(\beta) = y$. In order to see that M'' is a subrepresentation of M, we only have to note that C is contained in the kernel of any map M_α with $t(\alpha) = y$. Actually, this also shows that M'' is a direct sum of copies of $S(y)$. Namely, take a basis $\mathcal{B}$ of C and observe that any $b \in \mathcal{B}$ yields a copy of $S(y)$.

Since $I_y \oplus C = K_y + I_y$, it follows that $M_y = I_y \oplus C \oplus D = M'_y \oplus M''_y$, and therefore $M = M' \oplus M''$. Looking at M', we see that the sum of the images of the maps M'_β with $h(\beta) = y$ is precisely I_y, whereas the intersection of the kernels of the maps M'_α with $t(\alpha) = y$ is $K_y \cap I_y$ and thus a subset of I_y.

Of course, in general, if M has a direct summand isomorphic to $S(y)$, there is an element $b \in M_y$ which belongs to K_x and not to I_y, thus $K_y \not\subseteq I_y$. Conversely, the splitting-off assertion shows: If $K_y \not\subseteq I_y$, then $K_y \cap I_y$ is a proper subspace of K_y and therefore $C \neq 0$. The splitting-off assertion shows that we split off the the direct sum of c copies of $S(x)$, where c is the dimension of C, thus the dimension of $K_y/(K_y \cap I_y)$.

Corollaries. *Let y be a vertex of Q and M an indecomposable representation of Q which is not isomorphic to $S(y)$.*

(a) *Always, $K_y \subseteq I_y$.*

(b) *If y is a source, then $K_y = 0$.*

(c) *If y is a sink, then $I_y = M_y$.*

Namely, if y is a source, then $I_y = 0$, and $K_y \subseteq I_y = 0$. And if y is a sink, then $K_y = M_y$ and then $M_y = K_y \subseteq I_y$.

Let us consider again the quiver Q of type $\mathbb{A}_2$ with the arrow $\alpha\colon 2 \to 1$. Let M be an indecomposable representation. If M is not isomorphic to $S(1)$, then M_α has to be surjective, according to (c). If M is not isomorphic to $S(2)$, then M_α has to be injective, according to (b). Thus if M is neither isomorphic to $S(1)$ not to $S(2)$, then M_α is both injective and surjective, thus a vector space isomorphism. It follows that M is isomorphic to a direct sum of say n copies of $(1_k\colon k \to k)$ (here n may be a non-negative integer or some cardinality. Namely, choose a basis $\mathcal{B}$ of M_2, this yields a vector space isomorphism

$\Phi\colon k^n \to M_2$ and a commutative diagram

$$\begin{tikzcd} k^n \arrow[r, "\Phi"] \arrow[d, "1"'] & M_2 \arrow[d, "M_\alpha"] \\ k^n \arrow[r, "M_\alpha\Phi"] & M_1 \end{tikzcd}$$

which is an isomorphism of representations of Q. Note that the left vertical map is the direct sum of n copies of $(1_k\colon k \to k)$. We see:

Let Q be the quiver of type $\mathbb{A}_2$. Any representation of Q is a direct sum of copies of $S(1), S(2)$ and $(1_k\colon k \to k)$, thus of thin representations.

1.5. The indecomposable representations of quivers of type $\mathbb{A}$.

Let us first consider the quivers of type $\mathbb{A}_3$.

(1) The quiver Q of type $\mathbb{A}_3$ with linear orientation. This is the following quiver

$$\overset{1}{\circ} \xleftarrow{\alpha} \overset{2}{\circ} \xleftarrow{\beta} \overset{3}{\circ}$$

Splitting off copies of $S(1)$ we can assume that we deal with a representation M with M_α surjective; splitting off copies of $S(3)$ we can assume that M_β is injective. After polishing. we can assume that M_β is the inclusion of a subspace U of $V = M_2$, and that there is a subspace U' of V such that M_α is the canonical projection $V \to V/U'$. Thus we deal with a vector space V with two subspaces U, U' and consider the corresponding representation of Q:

$$V/U \longleftarrow V \longleftarrow U'$$

One knows that there is a basis $\mathcal{B}$ of V such that both subspaces U, U' are generated by subsets of $\mathcal{B}$. But this means that we can decompose M into a direct sum of copies of the following representations

$$\begin{array}{cccc} 0 \longleftarrow k \xleftarrow{1} k & 0 \longleftarrow k \longleftarrow 0 & k \xleftarrow{1} k \xleftarrow{1} k & k \xleftarrow{1} k \longleftarrow 0 \\ b \in U \cap U' & b \in U \setminus U' & b \in U' \setminus U & b \notin U \cup U' \end{array}$$

(always, we specify which elements $b \in \mathcal{B}$ give rise to the representation in question). In particular, we see: *Any indecomposable representation of Q is thin.*

(2) The 2-subspace quiver. This is the following quiver

$$\overset{1}{\circ} \xrightarrow{\alpha} \overset{2}{\circ} \xleftarrow{\beta} \overset{3}{\circ}$$

Splitting off copies of $S(1)$ and of $S(3)$, we can assume that we deal with a representation M with both M_α, M_β injective. After polishing. we can assume that M_α and M_β are the inclusions subspaces U, U' of $V = M_2$, respectively. Thus we deal with a vector space V with two subspaces U, U'.

$$U \longrightarrow V \longleftarrow U'$$

One knows that there is a basis $\mathcal{B}$ of V such that both subspaces U, U' are generated by subsets of $\mathcal{B}$. But this means that we can decompose M into a direct sum of copies of the following representations:

$$\begin{array}{cccc}
k \xrightarrow{1} k \xleftarrow{1} k & k \xrightarrow{1} k \longleftarrow 0 & 0 \longrightarrow k \xleftarrow{1} k & 0 \longrightarrow k \longleftarrow 0 \\[1em]
b \in U \cap U' & b \in U \setminus U' & b \in U' \setminus U & b \notin U \cup U'
\end{array}$$

(again, we specify which elements $b \in \mathcal{B}$ give rise to the representation in question). Also here, we see: *Any indecomposable representation of Q is thin.*

(3) The 2-factor-space quiver. This is the quiver

$$\overset{1}{\circ} \xleftarrow{\alpha} \overset{2}{\circ} \xrightarrow{\beta} \overset{3}{\circ}$$

One uses vector space duality in order to relate the representations of the 2-factor-space quiver Q and the representations of the 2-subspace quiver Q' (at least when dealing with finite-dimensional representations):

$$\begin{array}{ccc}
\text{representation of } Q & \qquad\qquad & \text{representation of } Q' \\[1em]
M_1 \xleftarrow{M_\alpha} M_2 \xrightarrow{M_\beta} M_3 & & M_1^* \xrightarrow{M_\alpha^*} M_2^* \xleftarrow{M_\beta^*} M_3^*
\end{array}$$

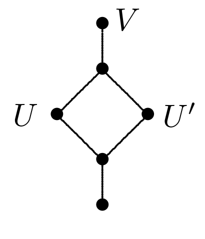
Or else, one shows that any polished representation of Q without direct summands $S(1), S(3)$ is of the form

$$V/U \longleftarrow V \longrightarrow V/U'$$

where U, U' are subspaces of a vector space V. Thus, again, we see: *any indecomposable representation is thin.*

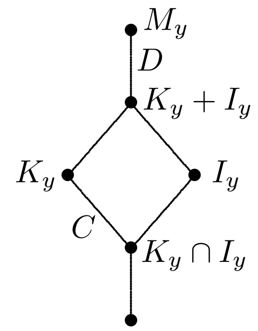
Altogether we have shown: *If Q is a quiver of type $\mathbb{A}_3$, then any indecomposable representation is thin.*

> Let us stress that the indecomposable non-simple representations of all quivers of type $\mathbb{A}_3$ have been obtained by looking at a vector space V with two subspaces U, U', thus by looking at the following subspace lattice of a vector space V:



Slogan: *Representation theory of quivers is just (a higher form of) linear algebra.*

Recall: The splitting-off of copies of $S(y)$ as considered in section 1.4 also relies on the same subspace lattice, namely we were dealing with:



Proposition. *Let Q be a quiver of type $\mathbb{A}_n$. Then any indecomposable representation of Q is thin.*

Proof: As we know already, the assertion is true for $n \leq 3$. Thus, consider now some $n \geq 4$. By induction, we may assume that any indecomposable representation of a quiver of type $\mathbb{A}_{n-1}$ is thin.

Let M be an indecomposable representation of Q with underlying graph

$$\overset{1}{\circ} \text{ --- } \overset{2}{\circ} \text{ --- } \cdots \text{ --- } \overset{n-1}{\circ} \text{ --- } \overset{n}{\circ}$$

and assume that both $M_1 \neq 0$ and $M_n \neq 0$. The restriction M' of M to the full subquiver Q' with vertices $1, \ldots, n-1$ is a direct sum of thin representations which are $(n-1)$-sincere, according to Proposition 1.3. For example, if $n = 6$, then M' is the direct sum of copies of the following 5 representations of Q' (where the edges have to be replaced by corresponding arrows, and all the maps $k \to k$ are identity maps):

$$\begin{array}{ccccccccc}
k & \text{---} & k & \text{---} & k & \text{---} & k & \text{---} & k \\
0 & \text{---} & k & \text{---} & k & \text{---} & k & \text{---} & k \\
0 & \text{---} & 0 & \text{---} & k & \text{---} & k & \text{---} & k \\
0 & \text{---} & 0 & \text{---} & 0 & \text{---} & k & \text{---} & k \\
0 & \text{---} & 0 & \text{---} & 0 & \text{---} & 0 & \text{---} & k
\end{array}$$

We claim that M' is *increasing from left to right*: this should mean that for any arrow α with $t(\alpha), h(\alpha) \in [1, n-1]$, the map $M'_\alpha (= M_\alpha)$ is a monomorphism provided $t(\alpha) < h(\alpha)$, and an epimorphism otherwise. Similarly, consider the restriction M'' of M to the full subquiver with vertices $2, \ldots, n$. The representation M'' is a direct sum of thin representations which are 2-sincere, according to Proposition 1.3. Thus M'' is *decreasing from left to right*: for any arrow α with $t(\alpha), h(\alpha) \in [2, n]$, the map $M''_\alpha (= M_\alpha)$ is an epimorphism, if $t(\alpha) < h(\alpha)$, otherwise a monomorphism.

It follows that all the maps M_α with $t(\alpha), h(\alpha) \in [2, n-1]$ are bijective, thus up to isomorphism, we can assume that these maps are identity maps. But then it is sufficient to look at the representation

$$M_1 \;\text{——}\; \begin{matrix} M_2 \\ = M_{n-1} \end{matrix} \;\text{——}\; M_n$$

(where the edges have to be replaced by the appropriate arrows), thus at a representation of a quiver of type $\mathbb{A}_3$. We know that this representation is a direct sum of thin representations, thus also M itself is a direct sum of thin representations. But since by assumption M is indecomposable, we conclude that M is thin.

Remark. The proof provides a normal form for all the indecomposable representations of Q. Namely, *any indecomposable representation of the quiver Q of type $\mathbb{A}$ is isomorphic to a representation M using as vector spaces only 0 and k and as non-zero maps only the identity map $1\colon k \to k$.* Thus M is determined by the pair of numbers $i \le j$, such that the support quiver $Q(M)$ consists of the vertices x with $i \le x \le j$ and all the arrows in-between. In particular, the classification of the indecomposable representations uses only combinatorial data.

> Actually, the assertion concerning the normal form is an easy consequence, once we have established that any indecomposable representation is thin. We just have to use the process of polishing inductively, starting at one end. It is easy to see that all thin representations of tree quivers can be polished in this way, as we will outline in the next section.

Corollary. *Let V be a vector space with two filtrations*

$$\begin{aligned} U_1 \subseteq U_2 \subseteq \cdots \subseteq U_p \subseteq V, \\ U'_1 \subseteq U'_2 \subseteq \cdots \subseteq U'_q \subseteq V. \end{aligned}$$

Then there is a basis $\mathcal{B}$ of V such that any of the subspaces U_i, U'_j is generated by a subset of $\mathcal{B}$.

Proof: Consider the quiver Q of type $\mathbb{A}_n$ with $n = p+q+1$ with vertices labeled $1, \dots, p, 1', \dots, q'$ and 0 and the following orientation

$$\underset{\circ}{1} \longrightarrow \underset{\circ}{2} \;\text{—}\; \cdots \;\rightarrow \underset{\circ}{p} \longrightarrow \underset{\circ}{0} \longleftarrow \underset{\circ}{q'} \leftarrow \;\cdots\; \text{—}\; \underset{\circ}{2'} \longleftarrow \underset{\circ}{1'}$$

The two filtrations yield the following representation of Q (all maps are the inclusion maps):

$$U_1 \longrightarrow U_2 \;\text{—}\; \cdots \;\rightarrow U_p \longrightarrow V \longleftarrow U'_q \leftarrow \;\cdots\; \text{—}\; U'_2 \longleftarrow U'_1$$

If we write this representation as a direct sum of indecomposable representations, thus of thin representations, and choose in any direct summand N a non-zero element $b \in N_0$, we obtain the required basis.

Slogan: *One can use the representation theory of quivers in order to solve vector space problems.*

Theorem 1. *Let Q be a finite connected quiver. Then all indecomposable modules are thin if and only if Q is of type $\mathbb{A}_n$.*

Proof: We have seen above that the indecomposable representations of a quiver of type $\mathbb{A}_n$ are thin. Conversely, assume now Q is a connected quiver and all its indecomposable representations are thin. We look at some special cases of Q.

The Kronecker quiver Q. It has two vertices, labeled $1, 2$ and two arrows $\alpha, \beta\colon 2 \to 1$. Define M as follows: $M_1 = M_2 = k^2$, M_α the identity matrix, $M_\beta = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. One easily checks that M is indecomposable.

More general: Cycles. Say assume there are pairwise different vertices $x(1), \dots, x(n)$ with arrows $\alpha(i)$ such that $\{h(\alpha(i)), t(\alpha(i))\} = \{x(i), x(i+1)\}$ for all $1 \le i \le n$ (and $x(n+1) = x(1)$; in the Kronecker case, one also requires $\alpha(1) \ne \alpha(2)$). As in the Kronecker case, take $M_{x(i)} = k^2$, and take for all but one arrows $\alpha(i)$ the identity matrix, and $M_{\alpha(n)} = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Again, we get an indecomposable representation.

The quivers of type D_4. We have mentioned already the case of the subspace orientation. In general, we have to distinguish the 4 different orientations. The construction is quite similar in all cases. Namely, we start with the subspaces $k0, 0k, \Delta = \{(x, x) \mid x \in k\}$ of k^2 and take the following representations:

$$\begin{tikzcd} k0 \arrow[dr] & 0k \arrow[d] & \Delta \arrow[dl] \\ & k^2 & \end{tikzcd} \qquad \begin{tikzcd} k0 \arrow[dr] & 0k \arrow[d] & k^2/\Delta \\ & k^2 \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} k0 \arrow[dr] & k^2/0k & k^2/\Delta \\ & k^2 \arrow[u] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} k^2/k0 & k^2/0k & k^2/\Delta \\ & k^2 \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

The maps which we use are either the inclusion maps or the canonical projections. One may check directly that these representations are indecomposable. In section 1.7 we will see that the corresponding endomorphism rings are all equal to k.

If Q is not of type $\mathbb{A}_n$, then Q has a subquiver which is either a cycle or of type $\mathbb{D}_4$.

A quiver is said to be *representation-finite* (or to be of *finite representation type,* provided the number of isomorphism classes of indecomposable representations is finite. We have shown above that *any quiver of type $\mathbb{A}$ is representation-finite.*

> The connected quivers of finite representation type have been determined by Gabriel, they are the quivers whose underlying graph is a "Dynkin diagram". A typical example of a quiver which is not representation finite is the loop quiver $\mathbb{L}$; as we will point out in the next section, this is a consequence of the Jordan normal form of linear endomorphisms which usually is established in a Linear Algebra course.