4. The path algebra of a quiver.

4.1. Paths.

For definitions see section 2.1 (In particular: path; head, tail, length of a path; concatenation; oriented cycle).

Lemma. *Let Q be a quiver. If there is a path of length at least $|Q_0|$, then there are cyclic paths, and thus infinitely many paths.*

Proof: Assume that there exists a path of length greater or equal to $|Q_0|$. Then there exists a path of length $|Q_0|$, say $\alpha_n \cdots \alpha_1$. Consider the vertices $x_i = t(\alpha_i)$ for $1 \leq i \leq n$ and $x_{n+1} = h(\alpha_n)$. Then these are $n+1$ vertices, thus there has to exists $i < j$ with $x_i = x_j$. Let $w = \alpha_{j-1} \cdots \alpha_i$, this is a path with head and tail $x_i = x_j$, thus a cyclic path. But then w^m is a path for any natural number m. The path w has length $j - i \geq 1$, thus w^m has length $m(j-i)$. This shows that these paths are pairwise different.

Corollary. *Let Q be a quiver. The number of paths is finite if and only if Q is finite and there are no oriented cycles.*

Proof: The number of paths of length at most 1 is $|Q_0| + |Q_1|$, thus an infinite quiver has infinitely many paths. Also, any oriented cycle w gives rise to infinitely many paths, namely the paths w^m with m a natural number.

Conversely, assume that Q is a finite quiver. The number of paths of length 0 is $|Q_0|$, the number of paths of length s is at most $|Q_1|^s$. Thus, if there are infinitely many paths, there has to exist paths of arbitrarily large length. According to the lemma, this implies that there are oriented cycles.

4.1. The path algebra of a quiver.

Definition: Let kQ be the vector space with basis the set of all paths in Q, and with the following *multiplication:* if w, w' are paths, let ww' be the concatenation of w and w' provided the tail of w is the head of w', and the zero vector otherwise, and extend this multiplication bilinearly to kQ.

Note that kQ is an associative k-algebra. Proof of the associativity: Let w, w', w'' be paths. Then both $(ww')w''$ and $w(w'w'')$ are the concatenation of w on the left, w' in the middle and w'' on the right, in case both conditions $t(w) = h(w')$ and $t(w') = h(w'')$ are satisfied, and otherwise the zero element (since $(ww')0 = 0$, $0(w'w'') = 0$, according to bilinearity.

Since the multiplication is defined on a basis and extended bilinearly, we clearly deal with a k-algebra.

The elements e_x with $x \in Q_0$ are pairwise orthogonal idempotents.

Below we also will see that any e_x is a primitive idempotent.

If Q_0 is finite, then kQ has a unit element, namely $\sum_{x \in Q_0} e_x$. Proof: Let $e = \sum_{x \in Q_0} e_x$. We have to show that $ew = w = we$ for any path w (then we also have $er = r = re$ for any linear combination r of paths, thus for any element $r \in kQ$). Let

w be a path with tail x and head y, then $e_y w = w$ and $e_z w = 0$ for all $z \neq y$, thus $ew = e_y w + \sum_{z \neq y} e_z w = w$. Similarly, $we_x = w$ and $we_z = 0$ for $z \neq x$.

More generally, we can say that for an arbitrary quiver Q the path algebra always has sufficiently many idempotents. Recall that a ring R is said to have *sufficiently many idempotents* provided there is a set of pairwise orthogonal idempotents e_i in R indexed by a set I such that for any element $r \in R$, there is a finite subset $I' \subseteq I$ such that $\left(\sum_{i \in I'} e_i\right) r = r = r \left(\sum_{i \in I'} e_i\right)$. In our case $R = kQ$, we take $I = Q_0$.

Warning. *A path algebra has usually many additional idempotents.* Example: Let $\alpha \colon x \to y$ be an arrow which is not a loop. Then $e_x + \alpha$ is an idempotent. Namely:

$$(e_x + \alpha)^2 = e_x^2 + e_x\alpha + \alpha e_x + \alpha^2 = e_x + 0 + \alpha + 0.$$

Finite-dimensionality. *The algebra kQ is finite-dimensional if and only if there are only finitely many paths in Q, thus if and only if Q is a finite quiver without oriented cycles.*

The ideal kQ_+. Let kQ_+ be the subspace of kQ with basis all paths of length at least 1. This is clearly an ideal of kQ.

Also, let kQ_0 be the subspace of kQ with basis the paths of length 0. This is a subalgebra, it is a direct sum of copies of k (one for each vertex x), with component wise multiplication (or, we may reformulate this by saying that kQ_0 is the path algebra of the quiver $(Q_0, \emptyset)$ with the same vertices as Q, but no arrows.

Now $kQ = kQ_0 \oplus kQ_+$, or better $kQ_0 \ltimes kQ_+$, since this is a semi-direct product (kQ_0 is a subalgebra, kQ_+ an ideal).

The powers of kQ_+ can be described easily: *$(kQ_+)^m$ is the subspace with basis the set of paths of length at least m,* for all natural numbers m.

There are the following consequences:

(a) *If there is no path of length m, then $(kQ_+)^m = 0$.*

(b) *If Q is a finite quiver without oriented cycles, say with n vertices, then kQ_+ is a nilpotent ideal: $(kQ_+)^n = 0$.*

It follows that if Q is a finite quiver without oriented cycles, then kQ_+ is the radical of kQ (it is a nilpotent ideal, with semisimple factor ring).

Warning. In general, kQ_+ is not the (Jacobson or nil) radical of kQ. For example, in case $Q = \mathbb{L}$, the algebra $k\mathbb{L}$ is the polynomial ring in one variable: its radical is 0, whereas kQ_+ is a maximal ideal.

If Q is a quiver, one calls a vertex x a *source* provided no arrow ends in x, and a *sink* provided no arrow starts in x. *All vertices of Q are sinks or sources, if and only if there* *are no paths of length* 2 *if and only if* $(kQ_+)^2 = 0$. Typical examples of quivers with all the vertices sinks or sources are the subspace quivers $\mathbb{S}_n$.

Description of kQ by generators and relations. Looking at the construction, we see that kQ is generated as a k-algebra by the paths of length at most 1 in Q. Also, we see that the following relations are satisfied:

- If x is a vertex, then $e_x^2 = e_x$,
- If $x \neq y$ are vertices, then $e_x e_y = 0$,
- If $\alpha\colon x \to y$ is an arrow, then $e_y \alpha = \alpha = \alpha e_x$.

Actually, it is not difficult so see that these are all the relations needed in order to define kQ by generators and relations.

4.3. Examples of path algebras.

(a) **The loop quiver $\mathbb{L}$.** We have $k\mathbb{L} = k[T]$, the polynomial ring in one variable with coefficients in k.

(b) **The n-loops quiver.** Let Q be the quiver with one vertex and $n \geq 2$ loops. Then kQ is the free (non-commutative!) algebra in n generators.

(c) **The linearly oriented quiver Q of type** $\mathbb{A}_n$. Here, $kQ = T_n(k)$, the ring of upper triangular $(n \times n)$-matrices.

$$\overset{1}{\circ} \xleftarrow{\ \alpha_2\ } \overset{2}{\circ} \xleftarrow{\ \alpha_3\ } \overset{3}{\circ} \longleftarrow \cdots\cdots \longleftarrow \overset{n-1}{\circ} \xleftarrow{\ \alpha_n\ } \overset{n}{\circ}$$

An isomorphism $\eta\colon kQ \to T_n(k)$ is defined as follows:

$$\eta(e_i) = E_{ii}, \quad \eta(\alpha_i) = E_{i-1,i}.$$

(By definition, $E_{i,j}$ is the $(n \times n)$-matrix with one coefficient 1, namely in the intersection of the i-th row and the j-th column, all other coefficients being zero.) Note: paths of length at least 1 are of the form $\alpha_i\alpha_{i+1}\cdots\alpha_j$, with $2 \leq i \leq j \leq n$, and

$$\eta(\alpha_i\alpha_{i+1}\cdots\alpha_j) = E_{i-1,i}E_{i,i+1}\cdots E_{j-1,j} = E_{i-1,j}$$

for the longest path (there is such a path) we see:

$$\eta(\alpha_2\alpha_3\cdots\alpha_j) = E_{1,n}.$$

(d) **The n-subspace quiver $\mathbb{S}_n$.** The path algebra $k\mathbb{S}_n$ of the n-subspace quiver

$$\begin{tikzcd}
\overset{1}{\circ} \arrow[drr, "\gamma_1"'] & \overset{2}{\circ} \arrow[dr, "\gamma_2"] & \cdots\cdots & \overset{n}{\circ} \arrow[dl, "\gamma_n"] \\
 & & \underset{0}{\circ} &
\end{tikzcd}$$

is the subalgebra of $T_{n+1}(k)$ of matrices with non-zero coefficients only on the diagonal and in the first row:

$$\begin{bmatrix} * & * & \cdots & * \\ & * & & \\ & & \ddots & \\ & & & * \end{bmatrix}$$

An isomorphism is defined as follows:

$$e_0 \mapsto E_{11}, \quad e_i \mapsto E_{i+1,i+1}, \quad \alpha_i \mapsto E_{1,i+1}.$$

for $1 \leq i \leq n$.

(e) **The Kronecker quiver** $\mathbb{K}$**.** This is the quiver:

$$\begin{tikzcd} \overset{1}{\circ} & \overset{2}{\circ} \arrow[l, bend right, "\alpha"'] \arrow[l, bend left, "\beta"] \end{tikzcd}$$

It is of interest, since the representations of $\mathbb{K}$ are pairs of linear maps $\alpha, \beta \colon M_2 \to M_1$, in matrix language, one deals with *matrix pencils.* The path algebra can be written as the (2×2)-matrices

$$\begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}$$

Note that in general, given two rings R, S and a bimodule ${}_RM_S$, the set of matrices of the form

$$\begin{bmatrix} r & m \\ 0 & s \end{bmatrix}$$

with $r \in R,\ m \in M,\ s \in S$, with the usual matrix addition and matrix multiplication, a ring: for such upper triangular (2×2)-matrices, we need the addition in R, in M and in S separately, the multiplication in R and in S, as well as multiplications $R \times M \to M$ and $M \times S \to M$, and the bimodule axioms are just the correct axioms in order to obtain a ring. This ring is denoted by

$$\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}.$$

There is a fancy way to realize $k\mathbb{K}$, namely to consider the subspace R of $M_2(k[t])$ with k-basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix},$$

this obvious is a subring and of course of the form $\begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}$.
But this is fancy only on first sight. Namely it turns out that the inclusion of $R \to M_2(k[t])$ is a categorical epimorphism of rings (be aware that categorical epimorphisms of rings do not have to be surjective!) and provides a full embedding of the category of $M_2(k[t])$-modules into the category of $k\mathbb{K}$-modules. This kind of embeddings are of great interest.

4.4. Representations of quivers, modules over the path algebra.

Reminder: Given a ring R with identity $1 = 1_R$, an *R-module* M is by definition an abelian group M with a given biadditive map $R \times M \to M$, called the scalar multiplication, the image of (r, m) under this map is usually denoted just by rm, such that the following two rules are satisfied:

- $r(r'm) = (rr')m$ for all $r, r' \in R$, and all $m \in M$.
- $1_R m = m$ for all $m \in M$.

One can show that the last condition is equivalent to the condition $RM = M$; here RM denotes the abelian subgroup of M generated by the set of elements of M of the form rm with $r \in R, m \in M$.

Theorem. *Let Q be a quiver with finitely many vertices and k a field. The category of representations of Q over k is equivalent to the category of kQ-modules.*

The following functors are equivalences which are inverse to each other:

Given a representation $(M_x, M_\alpha)_{x,\alpha}$ of the quiver Q, let $M = \bigoplus_{x \in Q_0} M_x$ be the corresponding kQ module, with operation by the paths when ever possible: thus the path $(y|\alpha_1, \ldots, \alpha_m|x)$ sends $a \in M_x$ to $\alpha_1 \cdots \alpha_m(a) \in M_y$, and the elements in M_z with $z \neq x$ to zero.

Conversely, given a kQ-module M, let $M_x = e_x M$ and for $\alpha \colon x \to y$ let $M_\alpha \colon M_x \to M_y$ be the multiplication with α (note that $\alpha = e_y \alpha e_x$).

It is straightforward (but tedious to verify) that this works well. (See for example the text books by Auslander-Reiten-Smalø (Theorem III.1.5, p.57) or Assem-Simson-Skowronski.)

What about morphisms? Of course, if we start with a homomorphism

$$(f_x)_x \colon (M_x, M_\alpha)_{x,\alpha} \to (M'_x, M'_\alpha)_{x,\alpha},$$

we just form $f = \bigoplus_x f_x \colon \bigoplus_x M_x \to \bigoplus_x M'_x$.

Conversely, assume that there are given two kQ-modules M, M' and a module homomorphism $f \colon M \to M'$. The important fact is that $f(e_x M) \subseteq e_x M'$ for any $x \in Q_0$ (this is due to the fact that f commutes with scalar multiplication, here with the multiplication with the scalar $e_x \in kQ$. Thus, denote by f_x the restriction of f to M_x (with values in M'_x), then we really have $f = \bigoplus_x f_x \colon \bigoplus_x M_x \to \bigoplus_x M'_x$.

SLOGAN: The representations of a quiver Q are just the kQ-modules.

If $M = (M_x, M_\alpha)_{x,\alpha}$ is a representation of Q, then the information provided by the M_x and the M_α is quite different: the vector spaces M_x are subspaces "of the module M", we may think of M as $M = \bigoplus_x M_x$, whereas the maps M_α provide the action of kQ on M.

If we denote the category of representations of Q over k by $\operatorname{Rep}(Q,k)$, and the module category of a ring R by $\operatorname{Mod} R$, then the Theorem can be noted as follows:

$$\operatorname{Rep}(Q,k) \simeq \operatorname{Mod} kQ$$

Also, if we denote the category of finite-dimensional representations of Q over k by $\operatorname{rep}(Q,k)$, and the category of finite-dimensional kQ-modules by $\operatorname{mod} kQ$, then we similarly have:

$$\operatorname{rep}(Q,k) \simeq \operatorname{mod} kQ$$

The categories $\operatorname{Rep}(Q,k)$ and $\operatorname{Mod} kQ$ are not only equivalent, but (nearly) isomorphic. Recall that an *equivalence* of categories $\mathcal{C}$ and $\mathcal{D}$ requires the existence of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ such that the composition GF is naturally equivalent to the identity of $\mathcal{C}$, and the composition FG is naturally equivalent to the identity of $\mathcal{D}$, whereas for an *isomorphism* one requires that GF and $\mathcal{C}$ GF **are** the respective identity functors. Let us look at our functors. The functor $F\colon \operatorname{Rep}(Q,k) \to \operatorname{Mod} kQ$ attaches to the given vector spaces M_x indexed by Q_0 the direct sum $M = \bigoplus_x M_x$, this is an **external** direct sum, the functor G sends a module M to the set of spaces e_xM indexed by Q_0, note that the spaces M_x are subspaces of M and $M = \bigoplus_x M_x$, but this is now an **internal** direct sum. The composition FG of applying first G, then F would be the identity, if we would use the internal direct sum, not the external direct sum when applying the functor F. Also, when we look at the composition GF, we are faced with the question whether $e_y(\bigoplus_x M_x)$ can be considered as being equal to M_y, or only (canonically) isomorphic to M_y.

4.5. Finite-dimensional k-algebras in general.

This is a report (essentially without proofs) which outlines in which way the representation theory of quivers can be used in order to study the module category of a finite-dimensional k-algebra.

Any finite-dimensional k-algebras Λ (associative, with 1) has a maximal nilpotent ideal J (called its *radical*) and Λ/J is a semisimple k-algebra: it is the product of finitely many matrix rings over division k-algebras.

Proposition 1. *Let Λ be a finite-dimensional k-algebra with radical J such that $\Lambda/J = k \times \cdots \times k$ (n copies of k) and such that $J^r = 0$* (such an r exists, since J is nilpotent). *Then Λ is isomorphic as a k-algebra to kQ/I, where Q is a quiver with n vertices, and I is an ideal with $(kQ_+)^r \subseteq I \subseteq (kQ_+)^2$. The quiver Q is uniquely determined by Λ* (and called the *quiver* of Λ).

Conversely, if Q is a quiver with n vertices, and I is an ideal with $(kQ_+)^r \subseteq I \subseteq (kQ_+)^2$, then $\Lambda = kQ/I$ is a finite-dimensional k-algebra with radical kQ_+/I and Λ modulo its radical is of the form $k \times \cdots \times k$ with n copies of k.

Idea of proof: Start with a finite-dimensional k-algebra Λ. We need to find the quiver of Q. The theorem mentions already how many vertices we need. We want to construct an algebra homomorphism $\eta\colon kQ \to \Lambda$, and we want to have the elements $\eta(e_x)$ from the start. These elements have to be orthogonal idempotents in Λ. Now $\Lambda/J = k \times \cdots \times k$ (with n copies of k) has precisely n primitive idempotents, namely the elements $\overline{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i-th position. It is well-known that a complete set of primitive pairwise orthogonal idempotents can be lifted modulo any nilpotent ideal, thus there is a complete set of primitive pairwise orthogonal idempotents $e_1, \dots, e_n$, with $e_i + J = \overline{e}_i$. These are the elements we are looking for. Thus we take as vertices of Q the numbers $1, 2, \dots, n$, and we will start to define η by setting $\eta(e_i) = e_i$ for $1 \le i \le n$ (the first e_i is the path of length 0 corresponding to the vertex i, the second e_i is an idempotent in Λ).

Next, consider J/J^2 and multiply this bimodule from the left by e_i, from the right by e_j, we obtain a k-vector space $e_i(J/J^2)e_j$, its dimension yields the number of arrows $j \to i$. Actually, let us choose elements $a_1, \dots, a_t$ in e_iJe_j which form modulo J^2 a basis of $e_i(J/J^2)e_j$. By definition, there are precisely t arrows $j \to i$ in Q, label them $\alpha_1, \dots, \alpha_t$. We continue to define η by setting $\eta(\alpha_1) = a_1, \ \dots, \ \eta(\alpha_t) = a_t + J^2$.

We have defined Q, thus there is the corresponding path algebra kQ. We have described in which way we want to define $\eta(w)$ for all the paths of length at most 1 and we extend the definition to all of kQ, so that η is multiplicative and k-linear. Since the paths of length at most 1 are generators of the algebra kQ, we have to verify that the relations which define kQ are satisfied for the elements $e_1, \dots, e_n$ and the chosen elements in J/J^2. However, this is clear: the elements $e_1, \dots, e_n$ are orthogonal idempotents, and all the elements $a \in e_iJe_j$ satisfy $e_iae_j = a$. This shows that we obtain a k-algebra homomorphism

$$\eta\colon kQ \to \Lambda.$$

It remains to be shown that η is surjective (this means, we have to show that the chosen elements in Λ generate Λ). And we have to see that the kernel I of η satisfies

$$(kQ_+)^r \subseteq I \subseteq (kQ_+)^2.$$

Application. *Let Λ be a finite-dimensional k-algebra with radical J such that $\Lambda/J = k \times \cdots \times k$. Let Q be the quiver of Λ. Then* $\operatorname{Mod}\Lambda$ *is a full exact subcategory of* $\operatorname{Mod} kQ$ *and* $\operatorname{mod}\Lambda$ *is a full exact subcategory of* $\operatorname{mod} kQ$.

Proof: This is just a special case of the following general result: *If I is an ideal of the ring R, then* $\operatorname{Mod} R/I$ *is a full exact subcategory of* $\operatorname{Mod} R$*,* it consists just of those

R-modules M which are annihilated by I (this means that $ra = 0$ for all $r \in I$ and all $a \in M$).

In our case, dealing with the algebra Λ, we write $\Lambda = kQ/I$ where I is an ideal von kQ. [Actually, since we know that we can assume that $(kQ_+)^r \subseteq I \subseteq (kQ_+)^2$, we have more information about the embedding $\operatorname{Mod}\Lambda \subseteq \operatorname{Mod} kQ$: for example, the simple modules $S(x)$ with $x \in Q_0$ are annihilated by I, thus they are in the subcategory.]

Proposition 2. *If Λ is a finite-dimensional k algebra, then there exists a finite-dimensional k-algebra Λ' (unique up the algebra isomorphisms) such that the module categories of Λ and Λ' are equivalent and all simple factor algebras of Λ' are division ring.*

The algebra Λ' can be constructed as follows: Let $e_1, \dots, e_n$ be a complete set of pairwise inequivalent, but pairwise orthogonal primitive idempotents, and let $e = \sum e_i$. Then take $\Lambda' = e\Lambda e$. The algebra Λ is called a *basic* algebra, the algebras Λ and Λ' are said to be *Morita equivalent*.

Summery, in case k is algebraically closed. Let Λ be a finite-dimensional k-algebra, where k is an algebraically closed field. According to Proposition 2, there is a basic k-algebra Λ' which is Morita-equivalent to Λ. Since k is algebraically closed, the only finite-dimensional k-algebra which is a division ring, is k itself. Let J' be the radical of Λ', let $(J')^r = 0$ It follows that $\Lambda'/J' = k \times \cdots \times k$, thus there is a quiver Q and an ideal I with $(kQ_+)^r \subseteq I \subseteq (kQ_+)^2$ such that Λ' and kQ/I are isomorphic. Altogether, we see:

- The categories $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda'$ are equivalent (this is a Morita equivalence),
- the categories $\operatorname{mod}\Lambda'$ and $\operatorname{mod} kQ/I$ are equivalent (or even isomorphic; this is trivial, since the algebras Λ' and kQ/I are isomorphic,
- the category $\operatorname{mod} kQ/I$ is a full exact subcategory of $\operatorname{mod} kQ$,

thus there is a full exact embedding of $\operatorname{mod}\Lambda$ into $\operatorname{mod} kQ$.

4.6. The indecomposable projective kQ-modules $P(x)$.

Let x be a vertex of the quiver Q. Let $P(x)$ be the vector space with basis the set of all paths w with tail x. By definition, $P(x)$ is a subspace of kQ, but it is even a submodule, thus a left ideal. And we have:

$$kQ = \bigoplus_x P(x).$$

Proposition. *The evaluation map $f \mapsto f_x(e_x)$ yields a natural isomorphism*

$$\eta_M \colon \operatorname{Hom}(P(x), M) \to M_x$$

for all kQ-modules M.

Proof: Let $f\colon P(x) \to M$ be a homomorphism, then $f_x(e_x) = f_x(e_x^2) = e_x f_x(e_x)$, thus $f_x(e_x)$ is an element of $M_x = e_x M$, thus we really get a (set-theoretical) map $\eta = \eta_M \colon \operatorname{Hom}(P(x), M) \to M_x$. And clearly η is k-linear. We have to show that η is surjective and that its kernel is zero.

In order to show that the map η is surjective, let $a \in M_x$. For every path w with tail x and head y, the path w lies in $P(x)_y$, we have to define $f_y(w) \in M_y$. Thus, let $w = \alpha_1 \cdots \alpha_n$; we take (and have to take)

$$f_y(w) = f_y(\alpha_1 \cdots \alpha_n) = \alpha_1 \cdots \alpha_n(a).$$

In this way, f is defined on all paths in $P(x)$ and we extend it k-linearly in order to obtain $f\colon P(x) \to M$. Actually, it is easy to verify that we obtain not just a map, but a homomorphism $f\colon P(x) \to M$, and by definition, $f_x(e_x) = a$.

Now let us consider the kernel, thus let $f\colon P(x) \to M$ be a homomorphism such that $f_x(e_x) = 0$. But then for any path w with tail x and head y, we have $f_y(w) = f_y(we_x) = wf_x(e_x) = 0$, thus $f = 0$.

What means the naturality? If there is given a homomorphism $g\colon M \to M'$ of quiver representations, then the following square must commute:

$$\begin{array}{ccc}
\operatorname{Hom}(P(x), M) & \xrightarrow{\eta_M} & M_x \\
{\scriptstyle \operatorname{Hom}(P(x),g)}\downarrow & & \downarrow{\scriptstyle g_x} \\
\operatorname{Hom}(P(x), M') & \xrightarrow{\eta_{M'}} & M'_x
\end{array}$$

Start with $f \in \operatorname{Hom}(P(x), M)$, to the right we get $\eta_N(f) = f_x(e_x)$, under g_x we get $g_x f_x(e_x)$. On the other hand, $\operatorname{Hom}(P(x), g)(f) = gf$, and $\eta_{M'}(gf) = (gf)(e_x) = g_x f_x(e_x)$.

Corollary. *If $p\colon M' \to M$ is a surjective homomorphism of quiver representations, then, for every homomorphism $f\colon P(x) \to M$, there is a homomorphism $f'\colon P(x) \to M'$ such that $pf' = f$.* Thus $P(x)$ is a projective module.

Proof: Since p is surjective, $p_x\colon M'_x \to M_x$ is a surjective linear map. Now assume there is given $f\colon P(x) \to M$. Then $f_x(e_x) \in M_x$, thus there is $a \in M'_x$ such that $p_x(a) = f_x(e_x)$. According to the Proposition, there is $f'\colon P(x) \to M'$ with $f'(e_x) = a$ (the surjectivity of $\eta_{M'}$). But then

$$\eta_M(f) = f_x(e_x) = p_x(a) = p_x f'_x(a) = (pf')_x(a = \eta_M(pf').$$

The injectivity of η_M asserts that $f = pf'$.

Of course, if Q has only finitely many vertices, then $R = kQ$ is a ring with 1, and it is well-known, that the module ${}_RR$ (the ring considered as a left module over itself) is projective, as well as that direct summands of projective modules are projective. Thus, since $kQ = \bigoplus_x P(x)$ is a direct sum of left ideals, thus left modules, we see that all the modules $P(x)$ are projective left modules.