Stable cohomology over local rings

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(joint work with Oana Veliche)

In the mid-1980s Pierre Vogel introduced a cohomology theory that associates to each pair (M,N) of modules over an associative ring A groups $\widehat{\mathrm{Ext}}^n_A(M,N)$ defined for every $n\in\mathbb{Z}$, which vanish when either M or N has finite projective dimension. The first published account is in [5], and different constructions were independently found by Benson and Carlson [2] and by Mislin [8]. Kropholler's survey [6, §4] contains background and details. Known as *stable cohomology*, this theory contains as a special case Tate's cohomology theory for modules over a finite group G (namely, $\widehat{\mathrm{Ext}}^n_{\mathbb{Z}G}(\mathbb{Z},N)=\widehat{\mathrm{H}}(G,N)$, where $\mathbb{Z}G$ is the group ring), as well as its extension by Buchweitz [3] to two-sided noetherian Gorenstein rings.

Little is known about the meaning or the properties of stable cohomology outside of the original context of group representations. One reason for that may be the fact that the stable groups, and the multiplicative structures they support, are not readily amenable to computations through classical techniques.

We develop new approaches to their computation and present applications to commutative algebra. For the rest of this text, R denotes a commutative local ring with residue field k. Historical precedents indicate that considerable ring theoretic information on R is reflected in the homological behavior of k, so we focus on the stable cohomology of that module.

The classical Auslander-Buchsbaum-Serre theorem characterizes regular local rings as the local rings of finite global dimension. In particular, when R is regular all functors $\widehat{\mathrm{Ext}}^n_R(-,-)$ are trivial. We prove a strong converse:

1. If $\widehat{\mathrm{Ext}}^n_R(k,k)=0$ for a single $n\in\mathbb{Z}$, then R is regular.

When R is Gorenstein and M is finitely generated, $\widehat{\mathrm{Ext}}^n_R(M,N)$ can be computed from a complete resolution of M, which is a complex of finite free R-modules. It follows that if N is finitely generated as well, then so is $\widehat{\mathrm{Ext}}^n_R(M,N)$ for each $n\in\mathbb{Z}$. No characterization of Gorenstein rings is known in terms of the numbers $\operatorname{rank}_k \mathrm{Ext}^n_R(k,k)$, so the next result comes as a surprise:

2. If $\operatorname{rank}_k \widehat{\mathrm{Ext}}^n_R(k,k)<\infty$ for a single $n\in\mathbb{Z}$, then R is Gorenstein.

The statements above concern R-module structures, but their proofs use the fact that $\mathcal{E}=\mathrm{Ext}_R(k,k)$ and $\mathcal{S}=\widehat{\mathrm{Ext}}_R(k,k)$ are graded k-algebras, linked by a canonical homomorphism $\iota\colon\mathcal{E}\to\mathcal{S}$. The structure of $\mathcal{E}$ has been the subject of numerous investigations. The structure of $\mathcal{S}$ is a major topic of the talk.

When R is regular, **1.** yields $\mathcal{S}=0$. Martsinkovsky [7] proved that for singular rings ι is injective. We reprove this as part of the next result, where Σ denotes the translation functor and $\mathcal{E}$ acts canonically on $\mathcal{I}=\mathrm{Hom}_k(\mathcal{E},k)$. This theorem leads to an effective procedure for checking the finiteness condition in **2.**

3. If R is singular, then there is an exact sequence

$$0 \longrightarrow \mathcal{E} \xrightarrow{\iota} \mathcal{S} \longrightarrow \coprod_{i=d-1}^{\infty} (\Sigma^{-i}\mathcal{I})^{\mu^{i+1}} \longrightarrow 0$$

of graded left $\mathcal{E}$-modules, where $d = \operatorname{depth} R$ and $\mu^i = \operatorname{rank}_k \operatorname{Ext}_R^i(k, R)$.

One measure of the singularity of R is provided by a non-negative number, $\operatorname{codepth} R = \operatorname{edim} R - \operatorname{depth} R$, where $\operatorname{edim} R$ denotes the minimal number of generators of $\mathfrak{m}$ and $\operatorname{depth} R$ the depth of the ring. One has $\operatorname{codepth} R = 0$ precisely when R is regular. The condition $\operatorname{codepth} R \leq 1$ characterizes hypersurface rings. Their stable cohomology algebra, determined by Buchweitz [3], satisfies:

4. When R is a hypersurface, $\mathcal{S} = \mathcal{E}[\vartheta^{-1}]$, where $\vartheta \in \mathcal{E}^2$ is a central non-zero-divisor and $\mathcal{E}/(\vartheta)$ is an exterior algebra on $\operatorname{edim} R$ generators of degree 1.

Except for the special case of group algebras of finite abelian groups, little is known about the structure of $\mathcal{S}$ for local rings R having $\operatorname{codepth} R \geq 2$.

Our results on the subject involve the number

$$\operatorname{depth} \mathcal{E} = \inf\{n \in \mathbb{Z} \mid \operatorname{Ext}^n_{\mathcal{E}}(k, \mathcal{E}) \neq 0\}.$$

Clearly, one always has $\operatorname{depth} \mathcal{E} \geq 0$. When R is regular, the k-algebra $\mathcal{E}$ is finite dimensional, so $\operatorname{depth} \mathcal{E} = 0$. The converse also holds, but this time for a non-trivial reason. Indeed, a fundamental structure theorem, due to Milnor and Moore, André, and Sjödin, shows that $\mathcal{E}$ is the universal enveloping algebra of a graded Lie algebra π_R. If R is singular, then $\pi_R^2 \neq 0$, so the Poincaré-Birkhoff-Witt theorem implies $\operatorname{depth} \mathcal{E} \geq 1$; see [1] for details on π_R. Félix *et al.* [4] pioneered the use of $\operatorname{depth} \mathcal{E}$ in the study of the structure of $\mathcal{E}$. We show that this invariant provides also a lot of information on the structure of the k-algebra $\mathcal{S}$.

To describe the structure of $\mathcal{S}$ we use the subset

$$\mathcal{N} = \{\tau \in \mathcal{S} \mid \mathcal{E}^{\geqslant i}\tau = 0 \text{ for some } i \geq 0\}.$$

For instance, if $\operatorname{codim} R = 1$, then **4.** shows that $\operatorname{depth} \mathcal{E} = 1$ and $\mathcal{N} = 0$. From the next result a completely different picture emerges 'in general'.

5. If R is a Gorenstein ring and one of the following conditions holds:

(a) $\operatorname{depth} \mathcal{E} \geq 2$; or

(b) $\operatorname{codepth} R \geq 2$, and $\mathcal{E}^{\geqslant 1}$ contains a central non-zero-divisor,

then $\mathcal{N}$ is a two-sided ideal of $\mathcal{S}$, such that

$$\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{N} \quad \text{and} \quad \mathcal{N}^2 = 0.$$

The theorem applies in many cases. For example, we prove that (a) holds when R is Gorenstein and $\operatorname{codim} R = 3$; when R has minimal multiplicity; when R is a localization of a graded Gorenstein Koszul algebra; or when R is a tensor product of singular Gorenstein algebras over a field. Condition (b) is known to apply to all complete intersection rings R with $\operatorname{codepth} R \geq 2$.

However, there exist examples of Gorenstein rings for which depth $\mathcal{E} = 1$ and $\mathcal{E}^{\geqslant 1}$ does not contain non-zero central elements. The structure of their stable cohomology algebra is not known at present.

Our results on the structure of the stable cohomology algebra $\mathcal{S} = \widehat{\mathrm{Ext}}_R(k,k)$ for a Gorenstein ring R are similar to—and partly motivated by—results of Benson and Carlson [2] on the structure of the Tate cohomology algebra $\widehat{\mathrm{H}}(G,k)$ for a finite group G. The similarity is rather unexpected, as the cohomology algebra $\mathrm{H}(G,k)$ is always noetherian, while the absolute cohomology algebra $\mathcal{E} = \mathrm{Ext}_R(k,k)$ is noetherian precisely when R is complete intersection.

The structure of the algebra $\mathcal{S}$ when R is not Gorenstein is the subject of work in progress. We have found out that in some cases $\mathcal{S}$ can be described in terms of $\iota(\mathcal{E})$ and $\mathcal{N}$, as in **5.**, but that fundamentally new phenomena also occur.

References

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