Cluster tilting II

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(joint work with Robert J. Marsh and Idun Reiten)

This talk was part two in a series, where the first part was given by Robert Marsh. This part is mainly based on results of [2], a paper motivated by the interplay between the recent development of the theory of cluster algebras defined by Fomin and Zelevinsky in [4] (see [5] for an introduction) and the subsequent theory of cluster categories and cluster-tilted algebras [3, 1]. Our main results can be considered to be interpretations within cluster categories of the essential concepts in the theory of cluster algebras.

1. Matrix mutation

Given a skew-symmetric integer $n \times n$-matrix $B = (b_{ij})$, and an index $k \in \{1, \ldots, n\}$, let a *mutation in direction* k denote the following operation

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k = i \text{ or } k = j, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

One can associate with B a quiver Q_B with n vertices and with b_{ij} arrows from i to j if $b_{ij} > 0$. It is clear that Q_B will have no loops and no oriented cycles of length two. In fact, the skew-symmetric integer matrices are in one-one correspondence with quivers with these properties. So mutation induces an operation on such quivers.

2. Matrix mutation via quiver representations

Let $H = KQ$ be a hereditary algebra which is the path algebra of a quiver Q for some algebraically closed field K. Given a cluster-tilted algebra $\Gamma = \mathrm{End}_{\mathcal{C}_H}(T)^{\mathrm{op}}$, with $T = T_1 \amalg \cdots \amalg T_n$ a direct sum of n nonisomorphic indecomposable objects T_i in $\mathcal{C}_H$, there is a unique indecomposable object $T_i^* \not\simeq T_i$ in $\mathcal{C}_H$, such that we get a tilting object T' by replacing T_i by T_i^*. Our main result is to obtain a formula for passing from the quiver of $\Gamma = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ to the quiver of $\Gamma' = \mathrm{End}_{\mathcal{C}}(T')^{\mathrm{op}}$, not involving any information on relations. In fact, we show that this formula coincides with the formula for matrix mutation in direction i.

3. Cluster algebras

This has a nice interpretation in the case of cluster algebras. A cluster algebra (without coefficients) is defined via a choice of a free generating set $\underline{x} = \{x_1, \ldots, x_n\}$ in the field $\mathcal{F}$ of rational polynomials over $\mathbb{Q}$ and a skew-symmetrizable integer matrix B indexed by the elements of $\underline{x}$. The pair $(\underline{x}, B)$, called a seed, determines the cluster algebra as a subring of $\mathcal{F}$. More specifically, for each $i = 1, \ldots, n$, a new seed $\mu_i(\underline{x}, B) = (\underline{x'}, B')$ is obtained by replacing x_i in $\underline{x}$ by ${x_i}' \in \mathcal{F}$, where ${x_i}'$ is obtained by a so called *exchange multiplication rule* and B' is obtained from

B by applying so called *matrix mutation* at row/column i. Mutation in any direction is also defined for the new seed, and by iterating this process one obtains a countable number of seeds. For a seed $(\underline{x}, B)$, the set $\underline{x}$ is called a *cluster*, and the elements in $\underline{x}$ are called *cluster variables*. The desired subring of $\mathcal{F}$ is by definition generated by the cluster variables. Given a finite quiver Q with no oriented cycles, one can define on the one hand a cluster algebra $\mathcal{A}$, and on the other hand the cluster category $\mathcal{C}$ of kQ.

It was shown in [3] that in case Q is a Dynkin quiver, the cluster variables of $\mathcal{A}$ correspond to the indecomposable objects of $\mathcal{C}$, and that this correspondence induces a correspondence between the clusters and the tilting objects in $\mathcal{C}$. This was also conjectured to generalize to arbitrary quivers, except that in this case the exceptional objects should correspond to the cluster variables. Combining this with the results of [2], one obtains for finite type a precise interpretation of cluster algebras in terms of tilting theory in cluster categories. In [2] there is also an interpretation beyond finite type.

References

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[4] S. Fomin and A. Zelevinsky. *Cluster algebras I*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[5] A. Zelevinsky. *Cluster algebras: notes for 2004 IMCC (Chonju, Korea, August 2004).* Preprint `arXiv: math.RT/0407414`, (2004)