On the derived category of coherent sheaves on an irreducible projective curve of arithmetic genus one

Igor Burban

(joint work with Bernd Kreußler)

In my talk based on a joint work with B. Kreußler I am going to discuss various properties of the bounded derived category of coherent sheaves on a singular Weierstrass cubic curve.

Singular Weierstrass curves are irreducible one-dimensional Calabi-Yau manifolds and the study of their derived category is important from the point of view of the homological mirror symmetry [6], applications to F-theory [5] and to the theory of the Yang-Baxter equation [7].

A classification of the indecomposable objects of the derived category of coherent sheaves on a smooth elliptic curve follows from the classification of vector bundles of Atiyah [1]. The main difference in the case of a singular Weierstrass curve is that the homological dimension of the category of coherent sheaves is infinite, and hence there are indecomposable complexes with an arbitrary number of non-zero cohomologies, see [3]. In the singular case there are indecomposable vector bundles and torsion free sheaves which are not semi-stable and there are indecomposable sheaves which are neither torsion sheaves nor torsion free sheaves. The indecomposable objects of the derived category of a nodal cubic curve were described in [3], in [4] the Fourier-Mukai transform on Weierstrass curves was studied. One of the goals of my talk is to compare common features and to point out main differences between the derived category of a smooth and a singular Weierstrass cubic curve.

Let $\mathbf{E}$ be a a Calabi-Yau curve, i.e. a curve with trivial canonical bundle $\omega_{\mathbf{E}} = \mathcal{O}$ and let $\mathcal{E}$ be a spherical object of $D^b(\mathrm{Coh}_{\mathbf{E}})$, i.e. a perfect complex such that

$$\mathrm{Hom}(\mathcal{E}, \mathcal{E}[i]) = \begin{cases} \mathbf{k} & \text{if } \ i = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

It was shown by Polishchuk [7] that one can associate to a pair $(\mathbf{E}, \mathcal{E})$ a solution of the classical Yang-Baxter equation over the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n)$. Motivated by this application the following two conjectures were posed [7]:

(1) Let $\mathbf{E}$ be a Calabi-Yau curve, $\mathcal{E}$ be a spherical object, then there exists $\mathbb{F} \in \mathrm{Aut}(D^b(\mathrm{Coh}_{\mathbf{E}}))$ such that $\mathcal{E} = \mathbb{F}(\mathcal{O}_{\mathbf{E}})$;
(2) The group $\mathrm{Aut}(D^b(\mathrm{Coh}_{\mathbf{E}}))$ is generated by $\mathrm{Aut}(\mathbf{E})$, $\mathrm{Pic}^0(\mathbf{E})$ and tubular mutations.

I use the technique of Harder-Narasimhan filtrations in triangulated categories [2] to prove these conjectures in the case of Weierstrass cubic curves.

References

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