The Calabi-Yau dimension of tame symmetric algebras

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(joint work with Andrzej Skowroński)

Let K be a field and $\mathcal{T}$ a K-linear triangulated category which has Serre duality, that is there is a triangle functor S such that $D\,\mathrm{Hom}_{\mathcal{T}}(X,-) \cong \mathrm{Hom}_{\mathcal{T}}(-,SX)$ for each object X in $\mathcal{T}$. By a definition of Kontsevich [8], $\mathcal{T}$ is Calabi-Yau if S is isomorphic to some power of the shift of $\mathcal{T}$, if so, then the CY-dimension is the minimal $d \geq 0$ such that S is isomorphic to $[d]$ (see also [7]).

Let A be a finite-dimensional selfinjective algebra over K and assume that K is algebraically closed. The stable category $\underline{\mathrm{mod}}(A)$ is triangulated, with shift given by Ω^{-1}, which is the inverse of the functor which sends a module M to the kernel $\Omega(M)$ of a minimal projective cover. The stable category $\underline{\mathrm{mod}}(A)$ has Serre duality given by $\Omega\circ\nu$. Here ν is the Nakayama functor $\nu = D\,\mathrm{Hom}_A(-,A)$. Then $\underline{\mathrm{mod}}(A)$ is Calabi-Yau of CY-dimension d if $d \geq 0$ is minimal such $\Omega\circ\nu$ is isomorphic to Ω^{-d}. When A is symmetric, $\Omega\circ\nu \cong \Omega^{-d}$ if and only if Ω^{d+1} is isomorphic to the identity on $\underline{\mathrm{mod}}(A)$.

Suppose A is symmetric. If $\underline{\mathrm{mod}}(A)$ has finite CY-dimension then it is necessary that all simple A-modules are Ω-periodic. Recently we completed classifying tame symmetric algebras which have only Ω-periodic simple modules [4]. These are precisely the algebras whose connected components are, up to Morita equivalence,

(1) symmetric algebras of Dynkin type;
(2) symmetric algebras of tubular type;
(3) algebras of quaternion type;
(4) socle deformations of algebras in (1) or (2).

The algebras in (1) and (2) are of the form $\hat{B}/(\varphi)$ where $\hat{B}$ is the repetitive algebra of B and φ is an appropriate root of the Nakayama automorphism $\nu_{\hat{B}}$. Here the algebras B are tilted of Dynkin type (in (1)), or of tubular type (in (2)); for details and further references see [3]. An algebra A is of quaternion type if it is connected, tame and symmetric with non-singular Cartan matrix, and such that all indecomposable non-projective A-modules are periodic of period ≤ 4. Any such algebra belongs, up to Morita equivalence, to a small list, explicitly given by quivers and relations [2].

These algebras have been known and studied extensively over the last years. In [4] we show that these are all tame symmetric algebras with only periodic simple modules. Moreover, we have:

Theorem 1. *Assume A is tame and symmetric. Then $\underline{\mathrm{mod}}(A)$ has finite CY-dimension if and only if A is one of the algebras in this list.*

This is proved in [3]. In each case, we determine the Calabi-Yau dimension of $\underline{\mathrm{mod}}(A)$ explicitly. For algebras as in (1) the CY-dimension is given by a formula involving the Coxeter number of the associated Dynkin diagram, and it turns out that all integers can occur. These are the symmetric algebras of finite type. As a contrast, for selfinjective algebras of finite type, the stable category need not have finite CY-dimension. For the algebras in (2) the CY-dimensions are precisely the prime numbers $2, 3, 5, 7$ and 11.

If A is an algebra of quaternion type, then the stable module category has CY-dimension 3. As the main part of the proof, we show, using [6], that all derived equivalence classes of such algebras, except for a few (which are of tubular type), contain an algebra which has a periodic bimodule resolution of period 4. As a consequence we can complete the classification of algebras of quaternion type. Namely it follows that for the algebras in the list given in [2], all indecomposable non-projective modules have Ω-period at most 4.

Furthermore, we study arbitrary selfinjective algebras A such that $\underline{\mathrm{mod}}(A)$ has CY-dimension 2. For this to happen it is necessary that every simple A-module S satisfies $\nu(S) \cong \Omega^{-3}(S)$; and algebras with this property were studied in [1]. The main result is:

Theorem 2 ([1]). *Let A be a connected finite-dimensional selfinjective algebra. Then the following are equivalent:*

(a) *Every simple A-module S satisfies $\nu(S) \cong \Omega^{-3}(S)$;*
(b) *A is either generalized preprojective, i. e. A is Morita equivalent to $P(\Delta)$ with Δ either Dynkin of type ADE, or of type L, or A is Morita equivalent to a certain deformation $P^f(\Delta)$.*

Moreover, any such algebra has a periodic bimodule resolution.

The *preprojective algebra* $P(\Delta)$ for Δ a Dynkin graph has quiver Q_Δ obtained from Δ by replacing each edge by a pair of vertices, one in each direction, denoted by a and $\bar{a}$, setting $\bar{\bar{a}} = a$. Then $P(\Delta) = KQ_\Delta/I$ where I is the ideal of the path algebra generated by all relations of the form

$$\sum_{a, ia=v} a\bar{a} \quad (v \text{ a vertex of } Q_\Delta).$$

The algebra $P(L_n)$, which we call *generalized preprojective*, is defined similarly. Its quiver is obtained from Q_{A_n} by by attaching a loop, ε say, to one of the end vertices. We set $\bar{\varepsilon} = \varepsilon$ and define $P(L_n)$ by the same relations as the preprojective algebras; see also [7]. The algebras $P^f(\Delta)$ are deformations of $P(\Delta)$ where only the relation at the branch vertex (or at the loop) is deformed. The precise definition is given in [1]. This theorem is proved by exploiting subadditive functions, as studied in [5].

The stable category of an algebra $P(\Delta)$ in the Dynkin case is known to have CY-dimension 2; and for $\Delta = L_n$ this also holds. Our theorem implies that the stable categories of the deformed algebras $P^f(\Delta)$ have finite CY-dimension. We do not know at present whether they also have CY-dimension 2.

References

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