Resolutions over Koszul algebras and a question of D. Happel

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(joint work with Ragnar-Olaf Buchweitz, Dag Madsen and Øyvind Solberg)

Suppose that Λ is a finite dimensional K-algebra where K is a field. We denote the n-th Hochschild cohomology group of Λ by $\mathrm{HH}^n(\Lambda)$. In [5], Dieter Happel asked: If $\mathrm{HH}^n(\Lambda) = 0$ for sufficiently large n, then is the global dimension of Λ finite? Using the quantum exterior algebra in two variables as an example, we give a negative answer to the question [2]. It should be noted that L. Avramov and S. Iyengar [1] show that if Λ is a commutative finite dimensional K-algebra, then Happel's question has an affirmative answer.

We now give the example. Let $\Lambda_q = K\langle x, y\rangle/(x^2, y^2, xy + qyx)$ where $K\langle x, y\rangle$ is the free associative algebra in two variables and $q \in K$. We have the following facts. For all $q \in K$, the dimension of Λ_q is 4 and the global dimension of Λ_q is infinite. If $q \neq 0$ then Λ_q is a self-injective Koszul algebra with Koszul dual being the quantum affine plane $K\langle x, y\rangle/(yx - qxy)$. We note that a minimal projective resolution of K, as a right Λ_q-module, is

$$\cdots \to \Lambda_q^4 \xrightarrow{\begin{pmatrix} x & y & 0 & 0 \\ 0 & qx & y & 0 \\ 0 & 0 & q^2x & 0 \end{pmatrix}} \Lambda_q^3 \xrightarrow{\begin{pmatrix} x & y & 0 \\ 0 & qx & y \end{pmatrix}} \Lambda_q^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \Lambda_q \to K \to 0.$$

Using this resolution, we are able to find a minimal projective Λ_q^e-resolution of Λ_q, where Λ_q^e is the enveloping algebra $\Lambda_q^{op} \otimes_K \Lambda_q$. We use this resolution to calculate the Hochschild cohomology groups.

We prove the following result.

Theorem. *Let $\Lambda_q = K\langle x, y\rangle/(x^2, y^2, xy + qyx)$ such that q is not a root of unity in K. Then the Hochschild cohomology ring, $\mathrm{HH}^*(\Lambda_q)$ is isomorphic to $K[z]/(z^2) \times_K \wedge^*(u_0, u_1)$ as graded algebras where z has degree one, u_0 and u_1 are of degree 1 and $\wedge^*(u_0, u_1)$ denotes the exterior algebra in two variables.*

From this theorem, we see that $\dim_K(\mathrm{HH}^0(\Lambda_q)) = 2 = \dim_K(\mathrm{HH}^1(\Lambda_q))$, $\dim_K(\mathrm{HH}^2(\Lambda_q)) = 1$, and $\mathrm{HH}^n(\Lambda_q) = 0$ for $n \geq 3$. Thus, each Λ_q, q not a root of unity in K, provides a counterexample to Happel's question.

If q is a root of unity in K, then the Hochschild cohomology ring for Λ_q, in all characteristics, is also completely described in [2]. In particular, it follows that in this case, $\mathrm{HH}^n(\Lambda_q) \neq 0$ for an infinite number of n's. On the other hand, by appropriate choices of the root of unity q, it is shown that there can be arbitrarily large gaps where the Hochschild cohomology vanishes.

The story of the Hochschild homology groups of the Λ_q is different. Y. Han [4] shows that for any q, $\mathrm{HH}_n(\Lambda_q) \neq 0$ for all n.

The main technique to describe the Hochschild cohomology rings is to find a minimal projective Λ_q^e-resolution of Λ_q. It turns out that the basic requirement in finding such resolutions is that Λ_q is a Koszul algebra for $q \neq 0$. The techniques we employ generalize to arbitrary Koszul algebras and can be found in [3].

References

[1] L. Avramov and S. Iyengar, *Gaps in Hochschild cohomology imply smoothness for commutative algebras*, `arXiv:math.AC/0412259`.

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[3] R. Buchweitz, E. L. Green, N. Snashall, Ø. Solberg, *Comultiplicative structures for Koszul algebras*, preprint 2004.

[4] Y. Han, *Hochschild (co)homology dimension*, `arXiv:math.RA/0408402`.

[5] D. Happel, *Hochschild cohomology of finite-dimensional algebras*, Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988), 108-126, Lecture Notes in Math., 1404, Springer, Berlin, 1989.