Quantum affine $\mathfrak{gl}_n$ via Ringel-Hall algebras

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Ringel-Hall algebras were introduced in [6] and provide a generalisation of the classical Hall algebra of a discrete valuation ring with finite residue field to an arbitrary finitary ring. It was later shown in [4] that in the case of a hereditary algebra, the (twisted, generic) composition algebra (informally, the subalgebra of the Ringel-Hall algebra generated by the simple modules) realises the quantum group of the same type. In particular, this isomorphism identifies the simple modules with the Chevalley generators.

For an affine Lie algebra, Drinfeld gave a 'new realisation' of the quantised enveloping algebra by quantising the loop-algebra construction of the Lie algebra [2]. An explicit isomorphism between the two presentations was given by Beck [1] in the untwisted case, but the question of understanding the Drinfeld generators in terms of Ringel-Hall algebras remained open.

In the talk, we solved this problem for the affine Lie algebra $\widehat{\mathfrak{sl}}_n$ using the Ringel-Hall algebra of the cyclic quiver with n vertices. In fact, we extended the result to include $\widehat{\mathfrak{gl}}_n$ and thus proved a conjecture of Schiffmann [9].

Let C_n be the cyclic quiver with vertices $1, \dots, n$ and arrows $i \to i+1 \bmod n$. The (generic) composition algebra $\mathcal{C}_v(C_n)$ was originally studied in [7], and then Schiffmann [8] proved that the whole Ringel-Hall algebra $\mathcal{H}_v(C_n)$ consists of the composition algebra together with a central polynomial subalgebra $\mathcal{Z}_n$ on countably many generators. In particular, since the composition algebra is isomorphic to the affine quantum group of type $\widehat{\mathfrak{sl}}_n$, this result showed that the whole Ringel-Hall algebra is isomorphic to the quantum group of type $\widehat{\mathfrak{gl}}_n$.

Explicit generators for this central subalgebra were subsequently given in [5], where it was also shown that this is in fact the whole of the centre of the Ringel-Hall algebra. Furthermore, a Hopf algebra monomorphism was given from Macdonald's ring of symmetric functions to the centre $\Psi_n : \Lambda \to \mathcal{Z}_n$.

Let g_{ir} for $i = 1, \dots, n$ and $r > 0$ be the Heisenberg generators for Drinfeld's new realisation of $\mathcal{U}_v(\widehat{\mathfrak{gl}}_n)$ (see for example [3]). Then the isomorphism between the quantum group and the Ringel-Hall algebra sends $v^{ir} g_{ir}$ to the element $-\pi_{ir} + v^r \pi_{i-1r}$, where $\frac{r}{[r]}\pi_{ir}$ is the image of the r-th power sum function under the composition $\Lambda \xrightarrow{\Psi_i} \mathcal{Z}_i \subset \mathcal{H}_v(C_i) \hookrightarrow \mathcal{H}_v(C_n)$. (Here we have used the natural embedding of the Hall algebras arising from the embedding of the module categories $\mathrm{mod} C_i \hookrightarrow \mathrm{mod} C_n$ which identifies the first $i-1$ simple modules.)

We remark that this Hopf algebra isomorphism restricts to Green's isomorphism $\mathcal{U}_v^+(\widehat{\mathfrak{sl}}_n) \to \mathcal{C}_v(C_n)$ (after using Beck's isomorphism, suitably normalised). Moreover, the natural 'upper left corner' embeddings on the quantum group side, as described in [3], correspond to the natural embeddings of Ringel-Hall algebras mentioned above.

References

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