Maximal orthogonal subcategories of triangulated categories satisfying Serre duality

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1. Motivation

The classical Auslander correspondence gives a bijection between the set of Morita-equivalence classes of representation-finite finite-dimensional algebras Λ and that of finite-dimensional algebras Γ with $\operatorname{gl.dim}\Gamma \leq 2$ and $\operatorname{dom.dim}\Gamma \geq 2$. Our motivation comes from a higher dimensional generalization [5] of the Auslander correspondence in Theorem 1.2.

Definition 1.1. Let $\mathbf{T}$ be a triangulated category (resp. a full subcategory of abelian category) and $n \geq 0$. For a functorially finite full subcategory $\mathbf{C}$ of $\mathbf{T}$, put

$$\begin{aligned} \mathbf{C}^{\perp_n} &:= \{X \in \mathbf{T} \mid \operatorname{Ext}^i(\mathbf{C}, X) = 0 \text{ for any } i \ (0 < i \leq n)\}, \\ {}^{\perp_n}\mathbf{C} &:= \{X \in \mathbf{T} \mid \operatorname{Ext}^i(X, \mathbf{C}) = 0 \text{ for any } i \ (0 < i \leq n)\}. \end{aligned}$$

We call $\mathbf{C}$ a *maximal n-orthogonal subcategory* of $\mathbf{T}$ if $\mathbf{C} = \mathbf{C}^{\perp_n} = {}^{\perp_n}\mathbf{C}$ holds [4].

By definition, $\mathbf{T}$ is a unique maximal 0-orthogonal subcategory of $\mathbf{T}$.

Theorem 1.2. *For any $n \geq 1$, there exists a bijection between the set of equivalence classes of maximal $(n-1)$-orthogonal subcategories $\mathbf{C}$ of $\operatorname{mod}\Lambda$ with additive generators M and finite-dimensional algebras Λ, and the set of Morita-equivalence classes of finite-dimensional algebras Γ with $\operatorname{gl.dim}\Gamma \leq n+1$ and $\operatorname{dom.dim}\Gamma \geq n+1$. It is given by $\mathbf{C} \mapsto \Gamma := \operatorname{End}_\Lambda(M)$.*

Important examples of maximal orthogonal subcategories appear in the work of Buan-Marsh-Reineke-Reiten-Todorov on cluster categories [1], that of Geiß-Leclerc-Schröer on preprojective algebras [3], and in considerations of invariant subrings of finite subgroups G of $\mathrm{GL}_d(k)$ (see [4]). Let us find some kind of higher dimensional analogy of Auslander-Reiten theory by considering maximal orthogonal subcategories.

2. Triangulated categories

In this section, let $\mathbf{T}$ be a triangulated category with a Serre functor S, and $\mathbf{C}$ a maximal $(n-1)$-orthogonal subcategory of $\mathbf{T}$.

Theorem 2.1 ([6]). (1) *$S_n := S \circ [-n]$ gives an autoequivalence of $\mathbf{C}$.*

(2) *$\mathbf{C}$ has "Auslander-Reiten $(n+2)$-angles", i.e. any $X \in \mathbf{C}$ has a complex*

$$S_nX \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X$$

which is obtained by glueing triangles $X_{i+1} \longrightarrow C_i \xrightarrow{f_i} X_i \longrightarrow X_{i+1}[1]$, $0 \leq i < n$, with $X_0 = X$, $X_n = S_nX$, $C_i \in \mathbf{C}$ and the following sequences *are exact.*

$$\mathbf{C}(-,S_nX) \xrightarrow{\cdot f_n} \mathbf{C}(-,C_{n-1}) \xrightarrow{\cdot f_{n-1}} \cdots \xrightarrow{\cdot f_1} \mathbf{C}(-,C_0) \xrightarrow{\cdot f_0} J_{\mathbf{C}}(-,X) \longrightarrow 0$$
$$\mathbf{C}(X,-) \xrightarrow{f_0\cdot} \mathbf{C}(C_0,-) \xrightarrow{f_1\cdot} \cdots \xrightarrow{f_{n-1}\cdot} \mathbf{C}(C_{n-1},-) \xrightarrow{f_n\cdot} J_{\mathbf{C}}(S_nX,-) \longrightarrow 0$$

It is quite interesting to study the relationship among all maximal $(n-1)$-orthogonal subcategories of $\mathbf{T}$. In the rest of this section, assume that $\mathbf{T}$ is *n-Calabi-Yau*, i.e. $S_n = 1$. For example, if Λ is a d-dimensional symmetric order, then $\underline{\mathrm{CM}}\Lambda$ is $(d-1)$-Calabi-Yau.

Definition 2.2. Assume that $\mathbf{C}$ *satisfies the strict no-loop condition*, i.e. for any $X \in \operatorname{ind}\mathbf{C}$, $X \notin \operatorname{add}\bigoplus_{i=1}^{n-1} C_i$ holds in Theorem 2.1, (2). Define a full subcategory $\mu_{X,i}(\mathbf{C})$ of $\mathbf{T}$ by

$$\operatorname{ind}\mu_{X,i}(\mathbf{C}) := (\operatorname{ind}\mathbf{C}\backslash\{X\}) \cup \{X_i\} \qquad (X \in \operatorname{ind}\mathbf{C},\ i \in \mathbf{Z}/n\mathbf{Z})$$

where X_i is the term of the triangle in Theorem 2.1, (2). This can be regarded as a higher dimensional generalization of the Fomin-Zelevinsky mutation in [1] and [3].

Theorem 2.3 ([6]). *Assume that $\mathbf{C}$ satisfies the strict no-loop condition. For any $X \in \operatorname{ind}\mathbf{C}$, $\{\mu_{X,i}(\mathbf{C}) \mid i \in \mathbf{Z}/n\mathbf{Z}\}$ is the set of all maximal $(n-1)$-orthogonal subcategories of $\mathbf{T}$ containing $\operatorname{ind}\mathbf{C}\backslash\{X\}$.*

2.4. It is an interesting question when transitivity holds in $\mathbf{T}$, i.e. the set of all maximal $(n-1)$-orthogonal subcategories of $\mathbf{T}$ is transitive under the action of mutations defined in Definition 2.2. It is known that transitivity holds for cluster categories $\mathbf{T}$ [1], and $\mathbf{T} = \mathrm{CM}\,\Lambda$ for the Veronese subring Λ of degree 3 of $k[[x,y,z]]$ (see [8]).

3. Derived equivalence

It is suggestive to relate our question in 2.4 to Van den Bergh's generalization [7] of the Bondal-Orlov conjecture [2] in algebraic geometry, which asserts that *all (commutative or non-commutative) crepant resolutions of a normal Gorenstein domain have the same derived category.* Let us generalize the concept of Van den Bergh's non-commutative crepant resolutions [7] of commutative normal Gorenstein domains to our situation.

3.1. Let Λ be an R-order which is an isolated singularity. We call $M \in \mathrm{CM}\,\Lambda$ a *NCC resolution* of Λ if $\Lambda \oplus \operatorname{Hom}_R(\Lambda, R) \in \operatorname{add} M$ and $\Gamma := \operatorname{End}_\Lambda(M)$ is an R-order with $\operatorname{gl.dim}\Gamma = d$. We have the remarkable relationship below between NCC resolutions and maximal $(d-2)$-orthogonal subcategories [5].

Proposition. *Let $d \geq 2$. Then $M \in \mathrm{CM}\,\Lambda$ is a NCC resolution of Λ if and only if $\operatorname{add} M$ is a maximal $(d-2)$-orthogonal subcategory of $\mathrm{CM}\,\Lambda$.*

3.2. We conjecture that *the endomorphism rings $\operatorname{End}_\Lambda(M)$ are derived equivalent for all maximal $(n-1)$-orthogonal subcategories $\operatorname{add} M$ of $\mathrm{CM}\,\Lambda$.* This is an analogy of the Bondal-Orlov and Van den Bergh conjecture by 3.1, and true for $n = 2$.

Theorem ([5]). *Let* $\mathbf{C}_i = \operatorname{add} M_i$ *be a maximal* 1*-orthogonal subcategory of* $\operatorname{CM}\Lambda$ *and* $\Gamma_i := \operatorname{End}_\Lambda(M_i)$*,* $i = 1, 2$*. Then* Γ_1 *and* Γ_2 *are derived equivalent.*

Corollary 3.3 ([5, 6]). *All NCC resolutions of* Λ *are derived equivalent if*

(1) $d \leq 3$*, or*
(2) Λ *is a symmetric order and transitivity holds in* $\underline{\mathrm{CM}}\Lambda$ (2.4).

References

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