An introduction to B. Toën's construction of derived Hall algebras

Bernhard Keller

The Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ of a finitary abelian category $\mathcal{A}$ is the free abelian group on the isomorphism classes of $\mathcal{A}$ endowed with the multiplication whose structure constants are given by the Hall numbers f_{XY}^{Z}, which count the number of subobjects of Z isomorphic to X and such that Z/X is isomorphic to Y, *cf.* [1]. Thanks to Ringel's famous theorem [6] [7], for each simply laced Dynkin diagram Δ, the *positive part* of the Drinfeld-Jimbo quantum group $U_q(\Delta)$ (*cf. e.g.* [4]) is obtained as the (generic, twisted) Ringel-Hall algebra of the abelian category of finite-dimensional representations of a quiver $\vec{\Delta}$ with underlying graph Δ. Since Ringel's discovery, it has been pointed out by several authors, *cf. e.g.* [3], that an extension of the construction of the Ringel-Hall algebra to the *derived category* of the representations of $\vec{\Delta}$ might yield the *whole* quantum group. However, if one tries to mimic the construction of $\mathcal{H}(\mathcal{A})$ for a triangulated category $\mathcal{T}$ by replacing short exact sequences by triangles, one obtains a multiplication which fails to be associative, *cf.* [2]. A solution to this problem has been proposed by Bertrand Toën in his recent preprint [8]. He obtains an explicit formula[1] for the structure constants ϕ_{XY}^{Z} of an associative multiplication on the rational vector space generated by the isomorphism classes of any triangulated category $\mathcal{T}$ which appears as the perfect derived category of a dg category T over a finite field all of whose Hom-complexes have homology of finite total dimension. The resulting $\mathbb{Q}$-algebra is the *derived Hall algebra*. Toën's formula for the structure constants reads as follows:

$$\phi_{XY}^{Z} = \sum_{f} |\operatorname{Aut}(f/Z)|^{-1} \prod_{i>0} |\operatorname{Ext}^{-i}(X,Z)|^{(-1)^i} |\operatorname{Ext}^{-i}(X,X)|^{(-1)^{i+1}},$$

where f ranges over the set of orbits of the group $\operatorname{Aut}(X)$ in the set of morphisms $f : X \to Z$ whose cone is isomorphic to Y, and $\operatorname{Aut}(f/Z)$ denotes the stabilizer of f under the action of $\operatorname{Aut}(X)$. Toën's proof of associativity is inspired by methods from the study of higher moduli spaces [11] [9] [10] and by the homotopy theoretic approach to K-theory [5]. It remains to be investigated if and how the derived Hall algebra of the category of representations of $\vec{\Delta}$ over a finite field is related to the quantum group $U_q(\Delta)$. In any case, it seems likely that Toën's construction will prove influential in the study of Ringel-Hall algebras.

References

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