Infinitesimal deformations of derived categories

Bernhard Keller

(joint work with Christof Geiß)

According to Kontsevich-Soibelman [3, section 2.1], cf. also [1], the shifted Hochschild complex C(A, A)[1] of a differential graded algebra A over a field of characteristic 0 is the 'moduli space of A_{∞} -categories'. We propose to interpret this statement to the effect that the differential graded Lie algebra C(A, A)[1] should control the deformations of the derived Morita class [8] [2] [9] of A, or, in more sloppy terms, the deformations of the derived category $\mathcal{D}A$. In particular, one expects a canonical bijection between the second Hochschild cohomology $\operatorname{HH}^{2}(A, A)$ and the equivalence classes of infinitesimal deformations of $\mathcal{D}A$. We show that such a bijection does indeed exist in many cases, notably if A itself has right bounded homology. In the general case, we obtain a bijection between the equivalence classes of Morita deformations of A and the 2-cocycles which act nilpotently in the graded endomorphism ring of each perfect object over A. Our proof starts from the observation that a Hochschild 2-cocycle c naturally gives rise to a deformation $A_c[\varepsilon]$ of A in the category of curved A_{∞} -algebras and that the (flat) derived category of $A_c[\varepsilon]$ admits a compact generator: the lift to $A_c[\varepsilon]$ of the cone over the graded endomorphism of the free module A induced by c. The links of these results with Lowen-Van den Bergh's deformation theory for abelian categories [6] [7] [5] [4] remain to be elucidated.

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