Reducing cohomology by split pairs

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In order to compare cohomology in two abelian categories, and in particular to show non-vanishing of certain cohomology, the following situation is studied:

Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories. A pair (F, G) of additive functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ is a split pair of functors *(between $\mathcal{A}$ and $\mathcal{B}$) if the composition $F \circ G$ is an autoequivalence of the category $\mathcal{B}$. If the categories are equipped with exact structures, and if the two functors are exact with respect to these exact structures, the split pair is called an* exact split pair of functors *(between $\mathcal{A}$ and $\mathcal{B}$).*

An exact split pair on abelian level induces a split pair on derived level; hence cohomology can be compared.

Easy examples of exact split pairs (A, B) are:

Split quotients: B is a *split quotient* of A, if B is a subring of A (via an embedding ε sending the unit of B to that of A) and there exists a surjective homomorphism $\pi\colon A \to B$, such that the composition $\pi \circ \varepsilon$ is the identity on B.

Morita equivalences.

Corner rings eAe, provided Ae is projective over eAe.

A more general class of examples is the following:

Let A be a ring, e an idempotent, and B a split quotient of eAe (viewed as a subring of eAe). Then we call B a corner split quotient *if there is a left A- and right eAe-module S, which is projective as a right B-module (via the embedding of B into eAe) and which satisfies $eS \simeq B$ as left B-modules.*

Up to composition with certain Morita equivalences, every exact split pair between module categories is a corner split quotient.

Applications include a proof of some cases of the strong no loops conjecture, and results relating Brauer algebras with various symmetric groups in the context of [2].

References

[1] L. Diracca and S. König, *Cohomological reduction by split pairs.* Preprint (2005).

[2] R. Hartmann and R. Paget, *Young modules and filtration multiplicities for Brauer algebras.* Preprint (2005).