The prime ideal spectrum of a tensor triangulated category

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Given a triangulated category, it is an interesting challenge to classify all thick subcategories. In my talk, I presented some recent work of Paul Balmer [1]. He defines a prime ideal spectrum for each tensor triangulated category and assigns to each object its support. This idea leads to a complete classification of all thick tensor ideals. The model for such a classification is Thomason's classification of thick tensor ideals for the category of perfect complexes on a scheme [3].

Balmer's classification provides an extremely elegant and conceptual explanation of various existing classifications. This includes the classification of thick tensor ideals for the category of perfect complexes on a scheme [Hopkins, Neeman, Thomason] and a similar classification for the stable category of representations of a finite group [Benson, Carlson, Rickard].

It turns out that Balmer's idea can be extended to obtain classifications of ideals in various settings. This should be relevant in representation theory, in particular when one studies the support varieties of representations.

The general set-up for the classification of ideals in terms of the prime ideal spectrum is the following: We consider an *ideal lattice*, that is, a partially ordered set $L = (L, \leq)$, together with an associative multiplication $L \times L \to L$, such that the following holds.

(L1) The poset L is a *complete lattice*, that is

$$\bigvee_{a \in A} a := \sup A \quad \text{and} \quad \bigwedge_{a \in A} a := \inf A$$

exist in L for every subset $A \subseteq L$.

(L2) The lattice L is *compactly generated*, that is, every element in L is the supremum of compact elements.

(L3) We have for all $a, b, c \in L$

$$a(b \vee c) = ab \vee ac \quad \text{and} \quad (a \vee b)c = ac \vee bc.$$

(L4) The product of two compact elements is again compact.

For example, the thick tensor ideals in a small tensor triangulated category form such an ideal lattice. The compact elements are precisely the finitely generated ideals.

Call $p \in L$ *prime* if $ab \leq p$ implies $a \leq p$ or $b \leq p$ for all $a, b \in L$. An element $q \in L$ is *semi-prime* if $aa \leq q$ implies $a \leq q$ for all $a \in L$. Let $\operatorname{Spec} L$ denote the set of all primes in L. For $a \in L$, let

$$U(a) = \{p \in \operatorname{Spec} L \mid a \leq p\} \quad \text{and} \quad \operatorname{supp}(a) = \{p \in \operatorname{Spec} L \mid a \not\leq p\}.$$

The subsets of $\operatorname{Spec} L$ of the form $U(a)$ for some compact $a \in L$ are closed under forming finite intersections and finite unions; they form the basis of a topology on $\operatorname{Spec} L$.

Theorem. *The assignments*

$$L \ni a \mapsto \operatorname{supp}(a) = \bigcup_{\substack{b \le a \\ b \textit{ compact}}} \operatorname{supp}(b) \quad \textit{and} \quad \operatorname{Spec} L \supseteq Y \mapsto \bigvee_{\substack{\operatorname{supp}(b) \subseteq Y \\ b \textit{ compact}}} b$$

induce mutually inverse and inclusion preserving bijections between

(1) *the set of all semi-prime elements in* L*, and*
(2) *the set of all subsets* $Y \subseteq \operatorname{Spec} L$ *of the form* $Y = \bigcup_{i \in \Omega} Y_i$ *with quasi-compact open complement* $\operatorname{Spec} L \setminus Y_i$ *for all* $i \in \Omega$.

To give an example, take a commutative noetherian ring R and let $L(R)$ denote the lattice of thick tensor ideals of the category of perfect complexes over R. Note that in this case all elements in $L(R)$ are semi-prime. Using the description of $L(R)$ due to Hopkins and Neeman, one can show that $\operatorname{Spec} L(R)$ is homeomorphic to the prime ideal spectrum of R, endowed with the usual Zariski topology.

This example, as well as many more, are beautifully explained in Balmer's work [1]. What seems to be new is the general approach via ideal lattices. It covers for instance equally well the related classification of Serre subcategories of the category of finitely generated modules over a commutative noetherian ring.

References

[1] P. Balmer: *The spectrum of prime ideals in tensor triangulated categories.* `arXiv:math.CT/0409360`.

[2] M. Hochster:*Prime ideal structure in commutative rings.* Trans. Amer. Math. Soc. **142** (1969), 43–60.

[3] R.W. Thomason: *The classification of triangulated subcategories.* Compositio Math. **105** (1997), 1–27.