Cluster tilting I

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1. Tilting Theory

For an introduction to tilting theory, see e.g. [1]. The initial motivation was provided by Gabriel's Theorem [11], which states that the path algebra $H = kQ$ of a quiver Q over an algebraically closed field k has finite representation type if and only if Q is a *Dynkin quiver* of type A, D or E. Bernstein, Gelfand and Ponomarev [3] found an alternative proof employing so-called *reflection functors*, which relate the representation theory of a quiver with that of a second quiver in which all arrows incident with a fixed source or sink of the original quiver have been reversed.

These reflection functors can be realised as Hom-functors; see [2]. If S is a projective noninjective simple module corresponding to the source or sink, then the reflection functor is realised in the form $\mathrm{Hom}(T, -)$, where T is the direct sum of $\tau^{-1}S$ and the indecomposable projective modules not isomorphic to S.

2. Cluster-tilting theory

A key result in the work on cluster-tilted algebras [4, 6] has been the generalisation of APR-tilting theory to arbitrary vertices of Q. Let S be the simple module associated to a vertex i in Q. Then it is shown in [4] that there is an algebra B with simple module S' such that $\frac{\mathrm{mod}\, H}{\mathrm{add}(S)} \simeq \frac{\mathrm{mod}\, B}{\mathrm{add}(S')}$, where $\mathrm{add}(M)$ denotes the additive subcategory generated by a module M. In fact, this result, suitably adapted, holds more generally for a large family of algebras known as the *cluster-tilted algebras*. This theory was inspired by recent development of the theory of *cluster algebras* (see [10]).

We define the *cluster category* $\mathcal{C} = \mathcal{C}_H$ as the quotient of the bounded derived category of its module category by the autoequivalence $F = [1]\tau^{-1}$ (see [6]), where [1] denotes the shift. Keller [12] has shown that $\mathcal{C}$ is naturally triangulated. A combinatorial/geometric definition in type A_n has been given in [8]. Cluster categories are also studied in [7, 9, 5, 13, 14]. We also remark that the cluster category is Calabi-Yau of dimension 2. This category can be regarded as an extension of the usual module category in which any almost complete cluster-tilting object has precisely two complements. An object T in $\mathcal{C}$ is labelled a *(cluster-) tilting object* if $\mathrm{Ext}^1_{\mathcal{C}}(T, T) = 0$ and T has a maximal number of nonisomorphic indecomposable direct summands. Any tilting module over H can be regarded as a tilting object in $\mathcal{C}$.

A *cluster-tilted algebra* is an algebra of the form $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ where T is a tilting object in $\mathcal{C}$; it is easy to see that H itself is cluster-tilted. Suppose that $\overline{T}$ is an almost complete tilting object in $\mathcal{C}$. Then in [6] it is shown that there are precisely two ways in which $\overline{T}$ can be completed to a tilting object, giving rise to tilting objects $T = \overline{T} \oplus M$ and $T' = \overline{T} \oplus M'$. Let $A = \mathrm{End}(T)^{\mathrm{op}}$ and $B = \mathrm{End}(T)^{opp}$. In [4] it is shown that, in this situation,

$$\frac{\mathrm{mod}\, A}{\mathrm{add}(S)} \simeq \frac{\mathrm{mod}\, B}{\mathrm{add}(S')},$$

where S and S' are certain simple modules over A and B respectively. Thus it is natural to define B as an algebra "cluster-tilted" from A at the vertex corresponding to M. APR-tilting is a special case of this construction. For example, the quiver of the algebra cluster-tilted from the path algebra of the quiver in Figure 1(a) at the vertex 2 is shown in Figure 1(b), with relations given by $ab = bc = ca = 0$.

FIGURE 1. Cluster-tilting in type A_3

REFERENCES

[1] I. Assem. *Tilting theory — an introduction*, Topics in algebra, Part 1 (Warsaw, 1988), 127–180, Banach Center Publ., 26, Part 1, PWN, Warsaw, 1990.

[2] M. Auslander, M. Platzeck and I. Reiten. *Coxeter functors without diagrams*, Trans. AMS. **250** (1979), 1–46.

[3] I. N. Bernstein, I. M. Gelfand and V. A. Ponomarev. *Coxeter functors and Gabriel's theorem*, Uspehi Mat. Nauk 28, no. 2 **170** (1973), 19–33. English Translation in Russian Math. Surveys **28** (1973), no. 2, 17–32.

[4] A. Buan, R. J. Marsh and I. Reiten. *Generalised APR-tilting*, Preprint `arXiv:math.RT/0402054`, February 2004, to appear in Trans. Amer. Math. Soc.

[5] A. Buan, R. J. Marsh and I. Reiten. *Cluster mutation via quiver representations*, Preprint `arXiv:math.RT/0412077`, 2004.

[6] A. Buan, R. J. Marsh, M. Reineke, I. Reiten, G. Todorov. *Tilting theory and cluster combinatorics*, Preprint `arXiv:math.RT/0402054`, February 2004.

[7] P. Caldero and F. Chapoton. *Cluster algebras from cluster categories*, Preprint `arXiv:math.RT/0410187`, 2004.

[8] P. Caldero, F. Chapoton and R. Schiffler. *Quivers with relations arising from clusters (A_n case)*, Preprint `arXiv:math.RT/0401316`, 2004, to appear in Trans. Amer. Math. Soc.

[9] P. Caldero, F. Chapoton and R. Schiffler. *Quivers with relations and cluster tilted algebras*, Preprint `arXiv:math.RT/0411238`, 2004.

[10] S. Fomin and A. Zelevinsky. *Cluster algebras I*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[11] P. Gabriel. *Unzerlegbare Darstellungen. I*, Manuscripta Math. **6** (1972), 71–103; correction, ibid. **6** (1972), 309.

[12] B. Keller. *Triangulated orbit categories of derived categories of hereditary algebras*, Preprint, 2003.

[13] B. Zhu. *BGP reflection functors and cluster combinatorics*, Preprint, 2004.

[14] B. Zhu. *Tilting theory in cluster categories of hereditary abelian categories*, Preprint, 2004.