Graded and Koszul categories

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(joint work with Øyvind Solberg)

1. Graded categories

Let K be a field, and let $\mathcal{C}$ be an additive K-category. We say $\mathcal{C}$ is *graded* if for each pair of objects, C and D we have a decomposition $\operatorname{Hom}_{\mathcal{C}}(C, D) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(C, D)_i$ as $\mathbb{Z}$-graded K-vector spaces and if f is in $\operatorname{Hom}_{\mathcal{C}}(C, C')_i$ and g is in $\operatorname{Hom}_{\mathcal{C}}(C', D)_j$, then gf is in $\operatorname{Hom}_{\mathcal{C}}(C, D)_{i+j}$. In particular, the identity maps are in degree zero.

Examples.
(1) *Let $\Lambda = \bigoplus_{i\geq 0} \Lambda_i$ be a positively graded K-algebra. Denote by $\operatorname{Gr}(\Lambda)_0$ the category of graded modules and degree zero maps, and by $\operatorname{Gr}(\Lambda)$ the category of graded modules and maps $\operatorname{Hom}_{\operatorname{Gr}(\Lambda)}(M, N) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr}(\Lambda)_0}(M, N[i])$. Then $\operatorname{Gr}(\Lambda)$ is a graded*
(2) *Let $\mathcal{C}$ be an additive K-category and denote by $\operatorname{rad}\mathcal{C}$ the radical of $\mathcal{C}$. Then the associated graded category $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ has the same objects as $\mathcal{C}$ and maps $\operatorname{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(C, D) = \bigoplus_{i\geq 0} \operatorname{rad}^i(C, D)/\operatorname{rad}^{i+1}(C, D)$.*
(3) *Let $\mathcal{C}$ be an abelian K-category with enough projective (injective) objects. The Yoneda or Ext-category $E(\mathcal{C})$ has the same objects as $\mathcal{C}$ and maps $\operatorname{Hom}_{E(\mathcal{C})}(A, B) = \bigoplus_{k\geq 0} \operatorname{Ext}^k_{\mathcal{C}}(A, B)$.*

2. Functors between graded K-categories

Let $\mathcal{C}$ and $\mathcal{D}$ be two graded K-categories. A contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ is a functor of graded categories if it is a functor such that it induces a degree zero homomorphism of K-vector spaces $F\colon \operatorname{Hom}_{\mathcal{C}}(C, D) \to \operatorname{Hom}_{\mathcal{D}}(F(C), F(D))$.

Example. *Let $\mathcal{C}$ be a graded K-category. For an object C in $\mathcal{C}$ the representable functors $\operatorname{Hom}_{\mathcal{C}}(C, -)\colon \mathcal{C} \to \operatorname{Gr}(K)$ and $\operatorname{Hom}_{\mathcal{C}}(-, C)\colon \mathcal{C}^{\mathrm{op}} \to \operatorname{Gr}(K)$ are functors of graded categories.*

Denote by $\operatorname{Gr}(\mathcal{C})_0$ the category with objects the functors of graded categories $F\colon \mathcal{C}^{\mathrm{op}} \to \operatorname{Gr}(K)$ and morphisms the natural transformations $\eta\colon F \to G$ with each $\eta_C\colon F(C) \to G(C)$ a degree zero morphism. This is an abelian category.

Let $\operatorname{Gr}(\mathcal{C})$ be the category with the same objects as $\operatorname{Gr}(\mathcal{C})_0$ and maps $\operatorname{Hom}_{\operatorname{Gr}(\mathcal{C})}(F, G) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr}(\mathcal{C})_0}(F, G[i])$. The category $\operatorname{Gr}(\mathcal{C})$ is a graded K-category.

3. Weakly Koszul and Koszul categories

Let $\mathcal{C}$ be a graded K-category. We say that $\mathcal{C}$ is *generated in degree zero and one*, if it is positively graded, that is: $\operatorname{Hom}_{\mathcal{C}}(C, D) = \bigoplus_{i\geq 0} \operatorname{Hom}_{\mathcal{C}}(C, D)_i$ and for any triple of objects A, B and C and for $i, j \geq 0$ the maps $\operatorname{Hom}_{\mathcal{C}}(A, C)_i \times \operatorname{Hom}_{\mathcal{C}}(C, B)_j \to \operatorname{Hom}_{\mathcal{C}}(A, B)_{i+j}$ given by $(f, g) \mapsto gf$ are onto.

Definition 1. A functor F in $\mathrm{Gr}(\mathcal{C})_0$ is a *Koszul functor* if there exists an exact sequence of graded functors and degree zero maps

$$\begin{aligned}\cdots \to \mathrm{Hom}_{\mathcal{C}}(-,C_k)[-k] \to \cdots \to \mathrm{Hom}_{\mathcal{C}}(-,C_2)[-2] \\ \to \mathrm{Hom}_{\mathcal{C}}(-,C_1)[-1] \to \mathrm{Hom}_{\mathcal{C}}(-,C_0) \to F \to 0\end{aligned}$$

Definition 2. Let $\mathcal{C}$ be a Krull-Schmidt category, then the simple functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}\, K$ are of the form $S_C = \mathrm{Hom}_{\mathcal{C}}(-,C)/\operatorname{rad}(-,C)$ with C indecomposable.

Assume that $\mathcal{C}$ is graded and generated in degrees zero and one, then it is *Koszul* if every graded simple object $S_C\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Gr}(K)$ is Koszul.

Definition 3. Let $\mathcal{C}$ be a Krull-Schmidt K-category (not necessarily graded).

(1) A functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}\, K$ is *weakly Koszul* if it has a minimal projective resolution $\cdots \to P_k \to P_{k-1} \to \cdots \to P_1 \to P_0 \to F \to 0$ with P_i finitely generated and $\operatorname{rad}^{i+1}(P_j) \cap \Omega^{j+1}(G) = \operatorname{rad}^i(\Omega^{j+1}(G))$ for $j \geq 0$ and $i \geq 1$.

(2) If every simple functor in $\mathrm{mod}(\mathcal{C})$ is weakly Koszul, then $\mathcal{C}$ is *weakly Koszul*.

The results for weakly Koszul algebras obtained in [4, 5] extend to weakly Koszul categories.

4. Applications of Koszul categories to the representation theory of finite dimensional algebras

Let Λ be a finite dimensional K-algebra, and denote by $\operatorname{ind}\Lambda$ the category of indecomposable finitely generated modules. The category $\mathcal{A}_{\mathrm{gr}}(\operatorname{ind}\Lambda)$ has the same objects as $\operatorname{ind}\Lambda$ and maps $\mathrm{Hom}_{\mathcal{A}_{\mathrm{gr}}(\operatorname{ind}\Lambda)}(X,Y) = \bigoplus_{i\geq 0} \operatorname{rad}^i(X,Y)/\operatorname{rad}^{i+1}(X,Y)$.

The objects in $\operatorname{ind}\Lambda$ decompose as a disjoint union $\cup_{\sigma\in\Sigma}\mathcal{C}_\sigma$, where $\mathcal{C}_\sigma$ are Auslander-Reiten components. The category $\mathcal{A}_{\mathrm{gr}}(\operatorname{ind}\Lambda)$ is a disjoint union $\cup_{\sigma\in\Sigma}\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma)$ of categories. Hence, $\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\operatorname{ind}\Lambda)) = \prod_{\sigma\in\Sigma}\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma))$. The categories $\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\operatorname{ind}\Lambda))$ and each $\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma))$ have global dimension 2.

We obtain generalizations of results given by the first author and results related to the hereditary categories with Serre duality studied by D. Happel, H. Lenzing, I. Reiten and M. Van den Bergh.

Theorem 1. (a) *The category* $\operatorname{ind}\Lambda$ *is weakly Koszul.*

(b) *The category* $\mathcal{A}_{\mathrm{gr}}(\operatorname{ind}\Lambda)$ *is Koszul, in particular each* $\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma)$ *is Koszul.*

(c) *Denote by* $\mathrm{Fin}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma))$ *the full subcategory of* $\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma))$ *of all functors whose minimal projective resolutions consist of finitely generated projective functors. Then for each* F *in* $\mathrm{Fin}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma))$ *there exists a subfunctor* G *of* F *such that some shift* $G[i]$ *is Koszul and* F/G *is of finite length.*

(d) *Any simple* S_C *with* C *indecomposable non-projective satisfies the Gorenstein condition, that is;*

(i) $\mathrm{Hom}(S_C, \mathrm{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma)}(-,X[n])) = 0$ *for all* X *and* n.

(ii) $\mathrm{Ext}^1_{\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma))}(S_C, \mathrm{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma)}(-,X[n])) = 0$ *for all* X *and* n.

(iii) $\mathrm{Ext}^2_{\mathrm{Gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma))}(S_C, \mathrm{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma)}(-,X[n])) = \overline{S}_{\tau C}[n+2](X)$, *where* $\overline{S}_{\tau C} = \mathrm{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C}_\sigma)}(\tau C,-)/\operatorname{rad}(\tau C,-)$ *and* τC *is the Auslander-Reiten translation of* C.

Theorem 2. *Let $\mathcal{C}$ be a regular Auslander-Reiten component of a finite dimensional algebra Λ and $E(S(\mathcal{C}))$ the associated* Ext*-category. Then the following statements are true.*

(a) *$E(S(\mathcal{C}))$ is a Frobenius category of radical cube zero.*

(b) *The categories $E(S(\mathcal{C}))/\operatorname{soc} E(S(\mathcal{C}))$ and $\mathcal{C}^{\mathrm{op}}/\operatorname{rad}^2$ are equivalent and $\operatorname{Gr}(\mathcal{C}^{\mathrm{op}}/\operatorname{rad}^2)$ is stably equivalent to $\operatorname{Gr}(\mathcal{S})$, where $\operatorname{Gr}(\mathcal{S})$ decomposes as a product of sections $\operatorname{Gr}(\mathcal{S}) = \prod_j \operatorname{Gr}(\mathcal{S}_j) \times \operatorname{Gr}(\mathcal{S}_j^{\mathrm{op}})$ and each $\mathcal{S}_j$ is a hereditary category such that $\mathcal{S}_j$ and $\mathcal{S}_i$ have the same quiver Q but $\mathcal{S}_j$ and $\mathcal{S}_j^{\mathrm{op}}$ have opposite quivers.*

(c) *If the quiver Q of $\mathcal{S}_j$ is finite, then $\mathcal{S}_j$ is of infinite representation type.*

Theorem 3. *Let $\mathcal{C}$ be a regular Auslander-Reiten component of a finite dimensional algebra Λ. Assume the quiver Q of the sections $\mathcal{S}_j$ of $E(S(\mathcal{C}))$ is infinite and is not of type $\mathbb{A}_\infty$, $\mathbb{D}_\infty$, or $\mathbb{A}_\infty^\infty$.*

(a) *Then any finitely presented functor F in $\operatorname{gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$ is either of finite length or it has infinite Gelfand-Krillov dimension.*

(b) *The category of finitely presented functors $\operatorname{gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$ is not noetherian.*

(c) *If $E(S(\mathcal{C}))$ has sections of type $\mathbb{A}_\infty$, $\mathbb{D}_\infty$ or $\mathbb{A}_\infty^\infty$, then it is noetherian of Gelfand-Krillov dimension 2.*

Our last theorem is the following.

Theorem 4. *Let $\mathcal{C}$ be a regular Auslander-Reiten component of a finite dimensional algebra Λ. Assume that $E(S(\mathcal{C}))$ has sections of type $\mathbb{A}_\infty$, $\mathbb{D}_\infty$ or $\mathbb{A}_\infty^\infty$. Then the quotient category of the finitely presented functors modulo the functors of finite length, $\operatorname{Qgr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$, is hereditary and noetherian with Serre duality.*

If the sections of $E(S(\mathcal{C}))$ are not infinite of type $\mathbb{A}_\infty$, $\mathbb{D}_\infty$ or $\mathbb{A}_\infty^\infty$, then $\operatorname{Qgr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$ is not noetherian.

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