The K-theory of triangulated derivators

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It has been known for a long time that chain complexes are very useful, and it is a good idea to study categories of chain complexes. If $\mathcal{A}$ is an abelian category then the chain complexes in $\mathcal{A}$ are sequences

$$\cdots \xrightarrow{\partial} X^{i-1} \xrightarrow{\partial} X^i \xrightarrow{\partial} X^{i+1} \xrightarrow{\partial} \cdots$$

with $\partial\partial = 0$. It is an old idea to look at categories whose objects are the chain complexes.

It is not quite so clear what the morphisms in the category should be. There are two traditional choices. In the derived category $D(\mathcal{A})$ the morphisms between two chain complexes X and Y are composites of homotopy equivalence classes of chain maps and of inverses of homology isomorphisms. The derived category $D(\mathcal{A})$ satisfies a short list of properties, which are formulated as the axioms of a *triangulated category.* One can find an extensive treatment of this subject in, for example, [9].

Another very classical construction is to consider the category $C(\mathcal{A})$. The objects are still the chain complexes, but the morphisms are now the chain maps (not homotopy equivalence classes, and nothing formally inverted). It is customary to view $C(\mathcal{A})$ as a model category. There are at least three ways to give an axiomatic description of model categories: Quillen closed model categories [7], Waldhausen model categories [10] and the complicial biWaldhausen categories of Thomason [8]. In all of these we assume we are given a mapping cone functor. We also declare some morphisms to be special. Certain of the morphisms are the so–called *cofibrations,* while some others are declared to be *weak equivalences.* The combined data is assumed to satisfy a fairly long list of axioms. For us the important feature is that the category $C(\mathcal{A})$, with all the added structure that it carries by virtue of being a model category, carries the information needed to construct the derived category $D(\mathcal{A})$. Given any model category $\mathcal{C}$ there is an associated homotopy category $\mathrm{ho}\,\mathcal{C}$, and $\mathrm{ho}\,C(\mathcal{A})$ is just $D(\mathcal{A})$.

For various reasons people have, over the last decade, been led to consider constructions intermediate between model categories and triangulated categories. The constructions fall into two broad categories:

(1) dg–categories, or the more general A_∞ categories.
(2) Grothendieck derivators.

I will say almost nothing about (1). The basic idea of a dg–category (or of the more general A_∞ categories) is to consider the morphisms in $C(\mathcal{A})$ not as groups, but as chain complexes of abelian groups. For any two objects of $C(\mathcal{A})$, that is for any two chain complexes X and Y in $\mathcal{A}$, we construct a natural chain complex $\mathrm{Hom}(X, Y)$ of abelian groups, whose 0th homology is the usual group of morphisms up to homotopy. The axioms of dg–categories (or A_∞ categories) encapsulate the properties this construction has. The literature is enormous; for a sample, the reader is referred to [1] and [6].

A completely different way to obtain a construction, intermediate between model categories and triangulated categories, is (2) above; it goes by the name *Grothendieck derivator.* The idea is to consider not just the derived category of $\mathcal{A}$, but the derived categories of all functor categories $\mathrm{Hom}(I^{\mathrm{op}}, \mathcal{A})$.

Suppose I is a small category, and $\mathcal{A}$ is any (fixed) abelian category. Then the category $\mathrm{Hom}(I^{\mathrm{op}}, \mathcal{A})$ is naturally an abelian category. We can form the derived category of $\mathrm{Hom}(I^{\mathrm{op}}, \mathcal{A})$, that is

$$\mathbb{D}(I) = D\big(\mathrm{Hom}(I^{\mathrm{op}}, \mathcal{A})\big).$$

If $F : I \longrightarrow J$ is a functor of small categories, we get an induced functor of triangulated categories

$$\mathbb{D}(F) : \mathbb{D}(J) \longrightarrow \mathbb{D}(I).$$

If F, G are two functors $F, G : I \longrightarrow J$ and $\phi : F \Longrightarrow G$ is a natural transformation, then we deduce a natural transformation

$$\mathbb{D}(\phi) : \mathbb{D}(G) \Longrightarrow \mathbb{D}(F).$$

This data assembles to give a 2–functor from the category $\mathcal{C}at$ of small categories to the category $\mathcal{T}ri$ of triangulated categories. Since this 2–functor is contravariant, we denote it

$$\mathbb{D} : \mathcal{C}at^{\mathrm{op}} \longrightarrow \mathcal{T}ri.$$

The idea of derivators is to encapsulate the extra structure of the 2–functors $\mathbb{D} : \mathcal{C}at^{\mathrm{op}} \longrightarrow \mathcal{T}ri$ which arise as $D\big(\mathrm{Hom}(I^{\mathrm{op}}, \mathcal{A})\big)$. For example, they have the useful property that for any functor $F : I \longrightarrow J$ of small categories, the induced functor $\mathbb{D}(F) : \mathbb{D}(J) \longrightarrow \mathbb{D}(I)$ has both a right and a left adjoint.

The first attempt to describe this was made by Heller [4]. Independently, but a little later, there is Keller's PhD thesis [5], and the manuscript by Grothendieck [3]. Still later there is the work of Franke [2], which cites Heller and Keller. Heller, Keller and Franke should undoubtedly receive recognition for their independent contributions. But in the last few years the name that has become attached to these is "Grothendieck derivators", possibly because the manuscript which Grothendieck wrote was so massive.

In the late 1990s Maltsiniotis took it upon himself to edit Grothendieck's manuscript and publish it. The work is still ongoing, with contributions by Cisinski and Keller. Much more can be found on Maltsiniotis' web page

`http://www.math.jussieu.fr/~maltsin`

In the process of editing the manuscript Maltsiniotis has done a great deal of work. In particular he defined for every derivator $\mathbb{D}$ a K–theory $K(\mathbb{D})$. And he formulated three conjectures about the K–theory of triangulated derivators (see pages 6–8 of the manuscript *La K–théorie d'un dérivateur triangulé* on Maltsiniotis' web page, as above). We wish to report on recent progress regarding Conjecture 3.

Conjecture 3 of Maltsiniotis says that additivity should hold for derivator K–theory. One way to formalise the conjecture is the following: A derivator $\mathbb{D}$ is a functor from small categories to triangulated categories. Given a derivator $\mathbb{D}$ we can define a new derivator $\mathbb{D}'$ by the rule

$$\mathbb{D}'(I) = \mathbb{D}(\mathbf{1} \times I),$$

where $\mathbf{1}$ is the category

$$\mathbf{1} \quad = \quad \cdot \longrightarrow \cdot$$

There are two inclusions of the one–point, terminal category into $\mathbf{1}$. These induce two inclusions of I into $\mathbf{1} \times I$, and hence two maps

$$\mathbb{D}(I) \xleftarrow{\pi_0} \mathbb{D}(\mathbf{1} \times I) \xrightarrow{\pi_1} \mathbb{D}(I)$$

As we let I vary, these give two natural transformations, π_0 and π_1, from $\mathbb{D}'$ to $\mathbb{D}$. They induce two maps in K–theory

$$K(\mathbb{D}) \xleftarrow{K(\pi_0)} K(\mathbb{D}') \xrightarrow{K(\pi_1)} K(\mathbb{D}).$$

Conjecture 3 of Maltsiniotis, the "additivity conjecture", asserts that the map

$$K(\mathbb{D}') \xrightarrow{\begin{pmatrix} K(\pi_0) \\ K(\pi_1) \end{pmatrix}} K(\mathbb{D}) \times K(\mathbb{D})$$

is an isomorphism.

Very recently Garkusha proved the conjecture in the special case where the derivator comes from a biWaldhausen complicial model. Garkusha's paper should appear soon in *Mathematische Zeitschrift.* A little later Cisinski, Keller, Maltsiniotis and I found a proof that works for a general derivator.

References

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