On the growth of the Coxeter transformations of derived-hereditary algebras

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(joint work with Helmut Lenzing)

For a finite dimensional k-algebra A of finite global dimension, the *Coxeter transformation* φ_A is an automorphism of the Grothendieck group $K_0(A)$. Moreover, for any complex $X^\bullet$ in the bounded derived category $D(A) := D^b(\operatorname{mod} A)$ of finite dimensional A-modules, we have $[X^\bullet]\varphi_A = [\tau_{D(A)} X^\bullet]$, where $\tau_{D(A)}$ is the automorphism of $D(A)$ given by the Auslander-Reiten translation. The characteristic polynomial $\chi_A(T)$ of φ_A, called the *Coxeter polynomial* and the corresponding spectral radius $\rho(\varphi_A) = \{\|\lambda\| \colon \lambda \in \operatorname{Spec} \varphi_A\}$ control the growth behavior of φ_A and hence of $\tau_{D(A)}$. Clearly, χ_A and $\rho(\varphi_A)$ are invariant under derived equivalences of the algebra A and provide natural links between the representation theory of finite dimensional algebras and other theories: the theory of Lie algebras, the theory of C^*-algebras, the spectral theory of graphs and the theory of knots and links, among other topics.

In the representation theory of algebras several cases have been extensively studied. For a hereditary algebra $A = k[\vec{\Delta}]$ associated to a finite quiver $\vec{\Delta}$ without oriented cycles, either Δ is Dynkin or affine and $\rho(\varphi_A) = 1$, or A is of wild type and $\rho(\varphi_A)$ is a simple root of the Coxeter polynomial; moreover, if A is wild for any non-preprojective indecomposable module X, the sequence of vectors $([\tau_A^n X])_{n\in\mathbb{N}}$ grows exponentially with ratio $\rho(\varphi_A)$, where τ_A denotes the Auslander-Reiten translation in the module category $\operatorname{mod} A$. Moreover, if A is wild, we have $\rho(\varphi_B) < \rho(\varphi_A)$ for any algebra $B = k[\vec{\Delta}']$ where Δ' is a proper full subgraph of Δ. For a canonical algebra $A = A(\mathbf{p}, \boldsymbol{\lambda})$, associated to a weight sequence $\mathbf{p} = (p_1, \ldots, p_t)$ of positive integers and a parameter sequence $\boldsymbol{\lambda} = (\lambda_3, \ldots, \lambda_t)$ of pairwise distinct non-zero elements from the base field k, the K-theory is well understood. In this case $\rho(\varphi_A) = 1$, even while A is a one-point extension $B[M]$ of the hereditary star $B = \mathbb{T}_{p_1,\ldots,p_t}$

$$\begin{tikzcd}
(1,1) \arrow[r] & (1,2) \arrow[r] & \cdots \arrow[r] & (1,p_1-1) \arrow[ddr] & \\
(2,1) \arrow[r] & (2,2) \arrow[r] & \cdots \arrow[r] & (2,p_2-1) \arrow[dr] & \\
 & & \vdots & & \bullet \\
(t,1) \arrow[r] & (t,2) \arrow[r] & \cdots \arrow[r] & (t,p_t-1) \arrow[ur] &
\end{tikzcd}$$

which has spectral radius $\rho(\varphi_B)$ arbitrarily large. In case A is wild, that is $\mathbb{T}_{p_1,\ldots,p_t}$ is not Dynkin or affine, the growth of τ_A is more complicated than in the hereditary case, since there are indecomposable A-modules X and Y for which $([\tau_A^n X])_n$ and

$([\tau_A^{-n} Y])_n$ grow exponentially with $\rho(\varphi_B)$ while $([\tau_A^{-n} X])_n$ and $([\tau_A^{n} Y])_n$ grow linearly.

We consider the case of an algebra A *derived equivalent to a hereditary algebra* $k[\vec{\Delta}]$. These algebras may be obtained by a finite sequence of tilting processes starting from $k[\vec{\Delta}]$. Moreover, if Δ is of Dynkin or affine type, the construction of A and its Auslander-Reiten quiver is well described. In general, φ_A is conjugate to $\varphi_{k[\vec{\Delta}]}$, hence if Δ is of wild type, $\rho(\varphi_A)$ is a simple root of $\chi_A(T)$. On the other hand, we show simple examples of derived hereditary algebras A and B, with B a full convex subcategory of A and $\rho(\varphi_B) > \rho(\varphi_A)$. We give conditions on a B-module M such that, for the one-point extension $A = B[M]$, the inequality $\rho(\varphi_B) \leq \rho(\varphi_A)$ is satisfied. Namely, we prove that such a module should be *derived-directing*, that is, $M = F(X)$ for X a direct sum of directing complexes in $D(k[\vec{\Delta}])$ and $F\colon D(k[\vec{\Delta}]) \to D(A)$ an equivalence of triangulated categories.

We describe all possible one-point extensions $B[M]$, of certain representation-finite algebras B derived equivalent to wild hereditary algebras, by an indecomposable B-module M. For modules which are not derived-directing, we find algebras $A = B[M]$ which are not derived canonical or derived tame or wild hereditary; nevertheless, the spectral radius $\rho(\varphi_A)$ of the Coxeter polynomial is 1, but not an eigenvalue of φ_A. This new class of algebras will be further studied.

References

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