Microscopy of simple representations

Markus Reineke

Let $Q = (Q_0, Q_1)$ be a finite quiver, and let $d \in \mathbf{N}Q_0$ be a dimension vector. A year ago I proved:

Theorem 1. *There exists a polynomial $a_d^Q(t) \in \mathbf{Z}[t]$ such that, for any finite field k, the evaluation $a_d^Q(|k|)$ equals the number of isomorphism classes of absolutely simple representations S of kQ of dimension vector d (i.e. $\overline{k} \otimes_k S$ is a simple representation of $\overline{k}Q$).*

Computer experiments show that the nature of these polynomials is rather mysterious. However, a special value has a simple interpretation:

Theorem 2. *If $\dim d > 1$, the polynomial $a_d^Q(t)$ has a zero at $t = 1$, and $\left.\frac{a_d^Q(t)}{t-1}\right|_{t=1}$ equals the number of cyclic equivalence classes of primitive cycles in Q of weight d.*

A cycle ω in Q is of weight d if it passes d_i times through each vertex $i \in Q_0$. It is called primitive if it is not a proper power of another cycle. The equivalence relation is cyclic rotation of paths.

The proof works as follows:

Step 1: Let $R_d(Q) = \bigoplus_{(\alpha:i\to j)\in Q_1} \mathrm{Hom}(\mathbf{C}^{d_i}, \mathbf{C}^{d_j})$ be the variety of complex representations of Q of dimension vector d, on which the algebraic group $G_d := \prod_{i\in Q_0} \mathrm{GL}_{d_i}(\mathbf{C})$ acts by base change. The projective space $\mathbf{P}R_d(Q)$ contains an open subset U corresponding to the simple representations, which admits a geometric quotient $\mathbf{P}\mathcal{M}_d(Q) := U/G_d$, a smooth, but non-projective complex variety. By Theorem 1 and some properties of ℓ-adic cohomology, the value $\left.\frac{a_d^Q(t)}{t-1}\right|_{t=1}$ equals the Euler characteristic in cohomology with compact support $\chi_c(\mathbf{P}\mathcal{M}_d(Q))$. This reduces the theorem to a topological statement.

Step 2: The Borel localization formula in equivariant cohomology gives the following: given a torus action on a complex variety, the Euler characteristic χ_c is preserved under passage to the fixed point set. Here we have an action of the torus $T_Q = (\mathbf{C}^*)^{Q_1}$ on $R_d(Q)$ by rescaling of arrows, which passes to an action on $\mathbf{P}\mathcal{M}_d(Q)$. It thus suffices to compute (the Euler characteristic of) $\mathbf{P}\mathcal{M}_d(Q)^{T_q}$.

Step 3: Given an indivisible vector $\lambda \in \mathbf{N}Q_1$, define a quiver Q_λ (an *almost universal abelian covering* of Q) with set of vertices $Q_0 \times \mathbf{Z}Q_1/\mathbf{Z}\lambda$ and arrows $(\alpha, \mu) : (i, \mu) \to (j, \mu + e_\alpha)$ for all $(\alpha : i \to j) \in Q_1$ and all $\mu \in \mathbf{Z}Q_1/\mathbf{Z}\lambda$. Given $d \in \mathbf{N}Q_0$, consider dimension vectors $\widetilde{d} \in \mathbf{N}(Q_\lambda)_0$ such that $\sum_\mu \widetilde{d}_{i,\mu} = d_i$ for all $i \in Q_0$.

Proposition. *The fixed point set $\mathbf{P}\mathcal{M}_d(Q)^{T_Q}$ is isomorphic to the disjoint union $\bigcup_{\lambda,\widetilde{d}} \mathbf{P}\mathcal{M}_{\widetilde{d}}(Q_\lambda)$ running over all λ and $\widetilde{d}$ as above.*

By additivity of Euler characteristic and Step 2, $\chi_c(\mathbf{P}\mathcal{M}_d(Q))$ equals the sum $\sum_{\lambda,\widetilde{d}} \chi_c(\mathbf{P}\mathcal{M}_{\widetilde{d}}(Q_\lambda))$.

Step 4: The theorem can now be proved by induction on $|Q_0|$, assuming in each step w.l.o.g. that $\mathrm{supp}(d) = Q$ and that Q is connected. The reduction process ends with quivers $\overline{Q}$ such that either $\mathbf{P}\mathcal{M}_d(\overline{Q}) = \emptyset$, or $\overline{Q}$ is an $\widetilde{A}_n$-quiver with cyclic orientation, and $d_i = 1$ for all $i \in I$, in which case $\mathbf{P}\mathcal{M}_d(Q)$ is a single point, thus of Euler characteristic 1. To count how many times this quiver is produced in the reduction process, its arrows have to be labelled (up to cyclic permutation) by arrows of the original quiver Q which form a primitive cycle. This proves Theorem 2.

This principle of proof may be called microscopy for two reasons: on the one hand, the iterated application of localization "zooms" into the moduli space $\mathbf{P}\mathcal{M}_d(Q)$ of simple representations. On the other hand, simples belonging to the fixed point set $\mathbf{P}\mathcal{M}_d(Q)^{T_Q}$ possess an inner structure (they lift to a simple representation of some Q_λ), so the proof also "looks at simples under a microscope".