Algebras derived equivalent to self-injective algebras

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This talk describes some work from the recent PhD thesis of my student Salah Al-Nofayee. [1]

Recall that two algebras A and B over a field k are said to be *derived equivalent* if the derived categories $D(A-\mathrm{Mod})$ and $D(B-\mathrm{Mod})$ of the module categories of A and B are equivalent as triangulated categories.

Many properties are preserved under derived equivalence. Here is one example that we proved some time ago.

Theorem 1 ([2], Corollary 5.3). *A finite-dimensional algebra derived equivalent to a symmetric algebra is itself symmetric.*

In fact, there is a rather more satisfactory proof than the one that appears there, as symmetric algebras can be characterized by properties of their derived categories.

Theorem 2 ([3], Corollary 3.2). *A finite dimensional algebra A is symmetric if and only if the vector spaces* $\mathrm{Hom}(P,M)$ *and* $\mathrm{Hom}(M,P)$ *are naturally dual whenever M and P are objects of* $D(A-\mathrm{Mod})$ *such that M is isomorphic to a bounded complex of finitely generated modules and P is perfect (i.e., isomorphic to a bounded complex of finitely generated projective modules).*

For some time, the corresponding statement for self-injective algebras has been open. Recently, in his PhD thesis, it was proved by Salah Al-Nofayee.

Theorem 3 (Al-Nofayee, [1]). *A finite-dimensional algebra derived equivalent to a self-injective algebra is itself self-injective.*

The proof uses a result of Saorín and Zimmermann-Huisgen on rigidity of tilting complexes [4], stating that for a given finite sequence $\{P_i, i \in \mathbb{Z}\}$ of finitely generated projective modules for a finite dimensional algebra, there are, up to isomorphism, only a finite number of tilting complexes of the form

$$\cdots \to P_1 \to P_0 \to P_{-1} \to \ldots .$$

In fact, Saorín and Zimmermann-Huisgen prove this for algebras over an algebraically closed field, but it is easy to deduce from this the statement for general fields.

Using this result, one can show that if T is a tilting complex for a self-injective algebra A, then there is some power ν_A^t of the Nakayama functor $\nu_A = ? \otimes_A DA$ for which

$$\nu_A^t(T) \cong T.$$

This implies that if B is derived equivalent to A, and therefore isomorphic to the endomorphism algebra of some tilting complex for A, then some power $(\mathbf{L}\nu_B)^t$ of the left derived functor of the Nakayama functor takes projective modules to projective modules. From this one can prove by reverse induction on t that this is true for *all* powers of $\mathbf{L}\nu_B$, and in particular, an injective cogenerator $\nu_B(B) = DB$ is projective, and so B is self-injective.

Unfortunately there seems to be no simple property of the derived category that characterizes the self-injective algebras, as there is in the case of symmetric algebras.

With the help of this theorem, Al-Nofayee also generalized a theorem [3, Theorem 5.1] that characterizes the sets of objects in the derived category of a symmetric algebra that correspond to the simple modules under some derived equivalence. For a self-injective algebra, his necessary and sufficient conditions for such a set $\{X_1, \ldots, X_n\}$ of objects of $D^b(A - \mathrm{mod})$ are:

- $\mathrm{Hom}(X_i, X_j[t]) = 0$ for all $1 \leq i, j \leq n$ and $t < 0$.
- $\mathrm{Hom}(X_i, X_j) = 0$ for $i \neq j$, and $\mathrm{End}(X_i)$ is a division ring for every i.
- $X_1, \ldots, X_n$ generate $D^b(A - \mathrm{mod})$ as a triangulated category.
- The set $\{X_1, \ldots, X_n\}$ is closed (up to isomorphism) under the Nakayama functor ν_A.

The last condition is automatic for symmetric algebras, since then the Nakayama functor is isomorphic to the identity functor, but for non-symmetric algebras there are simple examples that show that the first three conditions are not sufficient.

References

[1] S. Al-Nofayee, *Derived equivalence for self-injective algebras and t-structures*, PhD thesis, University of Bristol (2004).

[2] J. Rickard, *Derived equivalences as derived functors.* J. London Math. Soc. (2), **43**(1) (1991), 37–48.

[3] J. Rickard, *Equivalences of derived categories for symmetric algebras.* J. Algebra, **257** (2002), 460–481.

[4] B. Zimmermann-Huisgen and M. Saorín. *Geometry of chain complexes and outer automorphisms under derived equivalence.* Trans. Amer. Math. Soc., **353** (2001), 4757–4777.